# Using Mathematica to solve simple optimal tax problems

July 26, 2013

# Overview

# Policy Optimization in a Life-Cycle Model





Type Space

$$\Rightarrow \theta \in \Theta$$

Policy Space

$$\Rightarrow \kappa \in K$$

Consumer Utility Function

$$\Rightarrow U(x|\theta,\kappa) = \int_0^T e^{-\rho t} u(x_t|\theta,\kappa)$$

Policy-Maker Revenue Function

$$\Rightarrow V(x|\theta,\kappa) = \int_0^T e^{-rt} \mathcal{T}(x_t|\theta,\kappa)$$

# Simple Life-Cycle Model



We assume:

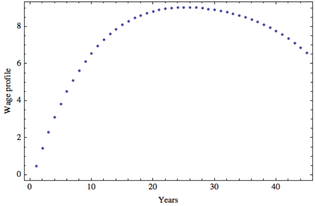
- Homogeneous preferences. $\theta$ is the same for everyone.
- A separable CRRA utility function

$$u(c_t, l_t|\gamma, \eta) = \frac{c_t^{1-\gamma}}{1-\gamma} - \frac{l_t^{1+\eta}}{1+\eta}$$

- A T-year discrete period.

$$U(c, l|\gamma, \eta) = \sum_{t=1}^{T} \beta^{t-1} u(c_t, l_t|\gamma, \eta)$$

# Simple Life-Cycle Model




- Exogenous wage profile w(t)=4th degree polynomial [Hubbard, Judd, 1986].
- A retirement age $T_{ret}$ at which $\ell_t = 0$
- A borrowing constraint $a_{min}$
- A policy space of two dimensions.
  1. $\tau_k$ Capital Tax Rate
  2. $\tau_\ell$ Labor Tax Rate

# Purpose of this Work





- Use Multi-objective optimization to get a broad picture of optimal tax policy in a simple life-cycle model.
- Create a portable environment capable of evaluating parameters specified by user.

# Implementation: Multi-Objective Optimization



1. $$U^*(c^*, \ell^* | \gamma, \eta) = \max_{c,\ell \in B(\mathbf{c},\mathbf{l})} \sum_{t=1}^{T} \beta^{t-1} \frac{c_t^{1-\gamma}}{1-\gamma} - \frac{\ell_t^{1+\eta}}{1+\eta}$$

2. $$V(c, l) = \sum_{t=1}^{T} \frac{1}{1+r}^{t-1} \mathcal{T}\left(w_t \ell_t, ra_t | \tau_l, \tau_k\right)$$

$$\text{s.t.} \qquad (1+r)a_t + w_t\ell_t \geq c_t + \mathcal{T}(w_t\ell_t, ra_t|\tau_l, \tau_k) + a_{t+1}$$

$$c_t > 0, \ell_t \geq 0 \qquad \forall t \in \{1 \dots T\}$$

# Objective function and constraints

$$\max_{c,\ell \in B(\mathbf{c},\mathbf{l})} \sum_{t=1}^{T} \beta^{t-1} \frac{c_t^{1-\gamma}}{1-\gamma} - \frac{\ell_t^{1+\eta}}{1+\eta}$$

s.t.

$$Ra_t + w_t\ell_t \geq c_t + \mathcal{T}(w_t\ell_t, ra_t | \tau_l, \tau_k) + a_{t+1}$$

$$c_t > 0, \ell_t \geq 0$$

$$a_1 = 0, a_{T+1} \geq 0$$

$$a_t \geq a_{min} \qquad \forall t \in \{1 \ldots T\}$$

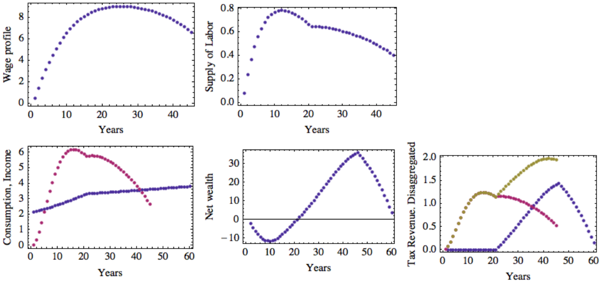
Since consumer is allowed to borrow, the Tax function $\mathcal{T}$ must be represented like this:

$$\mathcal{T}(w_t \ell_t, ra_t | \tau_l, \tau_k) = \begin{cases} \tau_l w_t \ell_t + \tau_k ra_t & a_t > 0 \\ \tau_l w_t \ell_t & \text{otherwise} \end{cases}$$

# Some individual profiles obtained

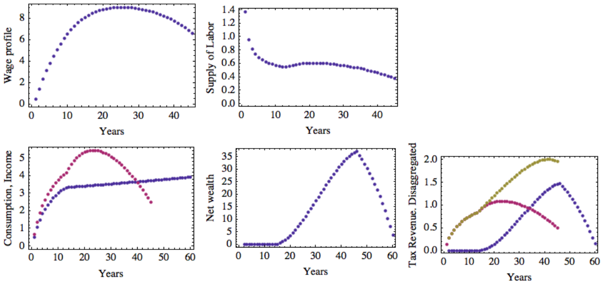


Figure : Individual profile with: $\gamma = 2, \eta = 1$ Subject to: $\tau_k = 0.40, \tau_l = 0.20$. Borrowing Constraint$=-20$



# Some individual profiles obtained

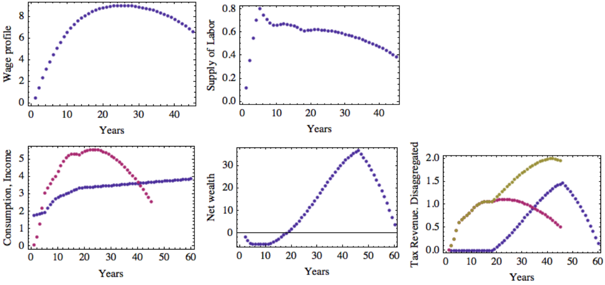
Figure : Individual profile with: $\gamma = 2, \eta = 1$ Subject to: $\tau_k = 0.40, \tau_l = 0.20$. Borrowing Constraint=0




We can see that the same consumer with a same tax rate can change its profile sharply due to a borrowing constraint.

# Some individual profiles obtained

Figure : Individual profile with: $\gamma = 2, \eta = 1$ Subject to: $\tau_k = 0.40, \tau_l = 0.20$. Borrowing Constraint$=-5$



# Building Data Base



Data Base was built with different elasticities and borrowing constraints

$\gamma = \frac{1}{2}, 1, 2, 4$

$\eta = 1, 2, 5, 10$

$a_{min} = 0, -10, -20, -\infty$

For each one of these, we create our grid of tax policies $\tau_\ell, \tau_\ell$.

# Data Base in Mathematica



In Mathematica tax policies $\tau_\ell, \tau_\ell$ were evaluated with a 5% grid interval, giving us a total of 25600 calculations.

| **Run** | **Single** | **Parallel** |
|---|---|---|
| Simple Run | 7h20m* | – |
| Warm Start | 6h28m* | 2h56m |
| Warm Start "smart" cycle | 6h | 2h38m |

Table : Run Times in Mathematica

# Data Base in Ipopt



In C++ through Ipopt the tax policies were evaluated with a 1% grid interval, giving us a total of 640,000 calculations.

| **Run** | **Single** |
|---|---|
| Simple Run | – |
| Warm Start (primals) | 4h |
| Warm Start (primals & duals) | 3h00m |
| Warm Start (p & d) + "smart" cycle | 2h40m |

Table : Run Times in Ipopt

-The linear solver used for Ipopt was MUMPS, in order to avoid the need for a srictly academic license.

# Visualization

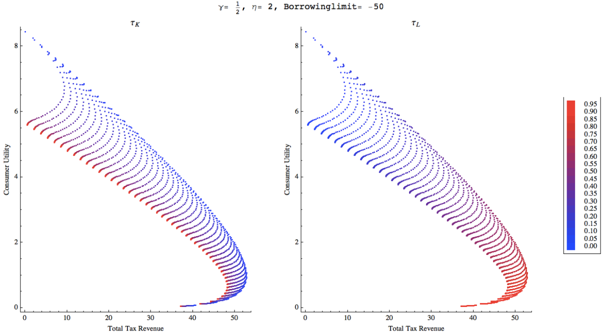
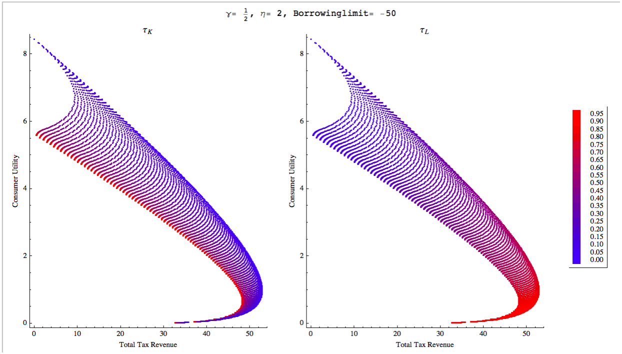



Figure : Tax Interval: 5x5

# Visualization



Figure : Tax Interval: 1x1



# Pareto Frontier

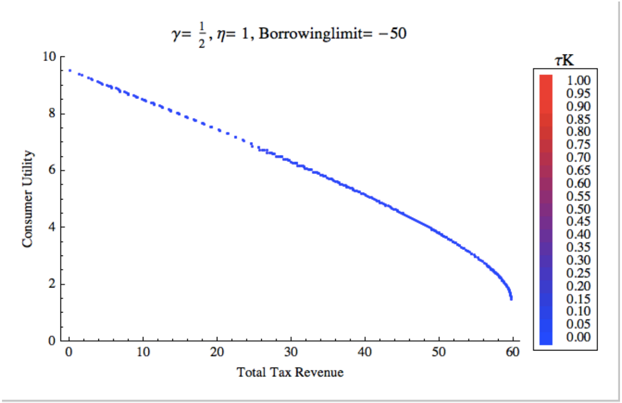


Calculation was made through Mathematica and an interactive tool of results can be constructed.

# Pareto Frontier

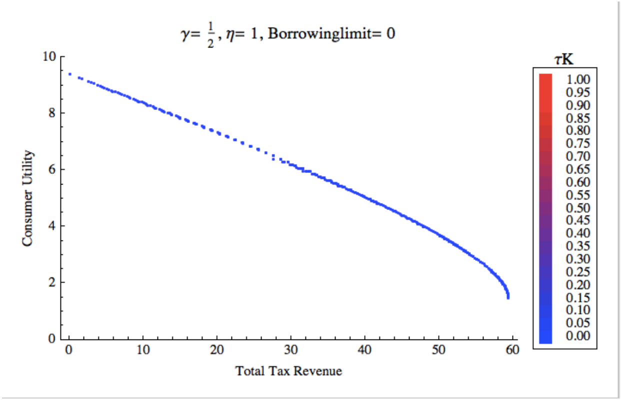






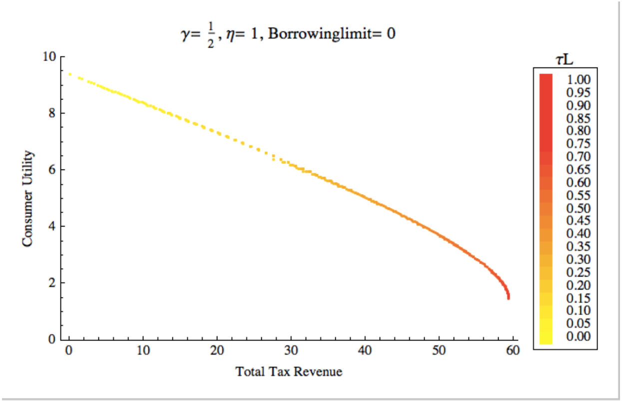
# Pareto Frontier





# Pareto Frontier





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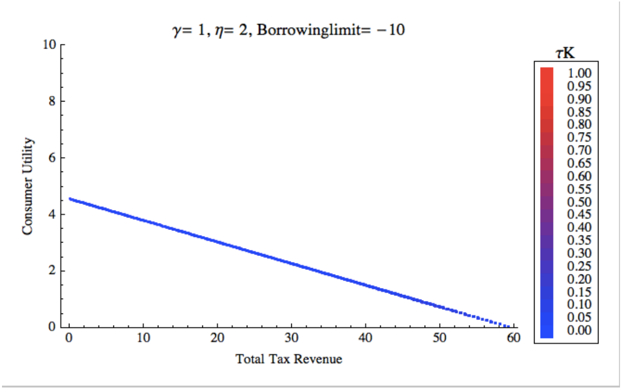

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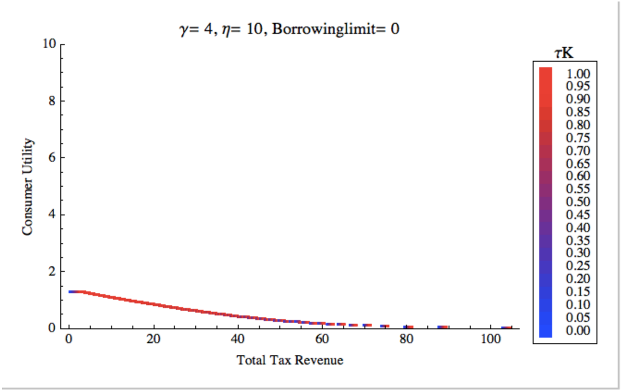

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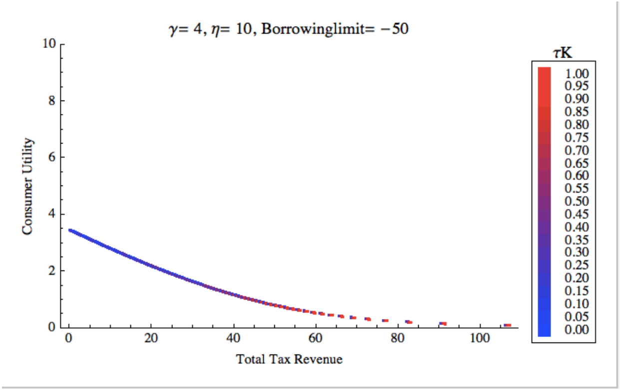
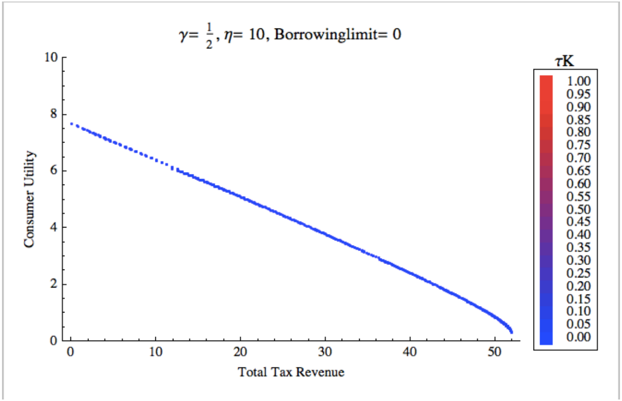


# Pareto Frontier





It is only within a corner of our state descriptors where a high capital tax rate could be an optimal point. This is due to the combination of two factors:

- A low intertemporal elasticity of substition for leisure and consumption makes consumer unwilling (almost indiferent) to shift their consumption and leisure patterns due to a capital tax rate boost.
- The same borrowing contraing prevented the consumer to ease his burden by any negative asset holding. Thus, consumer is also indifferent to shift their savings around time.

# Results





- We could see that the Pareto frontier is composed of policies involving nearly-zero capital tax rates
- With strong borrowing constraints, high capital tax rates may lay over the Pareto frontier. However, a low capital tax rate option is always available with similar revenue.
- In the case that we have a combination of low intertemporal elasticities of consumption and leisure, the borrowing constraint does make a capital tax rate an optimal condition. We basically leave the individual.

# References

R. Glenn Hubbard, Kenneth L. Judd (1986)
Liquidity Constraints, Fiscal Policy, and Consumption
*Brookings Papers on Economic Activity* 1986(1), 1-59