Computing Equilibria of Repeated And Dynamic Games

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Introduction

- Repeated and dynamic games have been used to model dynamic interactions in:
    - Industrial organization,
    - Principal-agent contracts,
    - Social insurance problems,
    - Political economy games,
    - Macroeconomic policy-making.

Introduction

- These problems are difficult to analyze unless severe simplifying assumptions are made:
    - Equilibrium selection
    - Functional form (cost, technology, preferences)
    - Size of discounting

Goal

- Examine *entire set* of pure-strategy equilibrium values in repeated and dynamic games
- Propose a general algorithm for computation that can handle
    - large state spaces,
    - flexible functional forms,
    - any discounting,
    - flexible informational assumptions.

Approach

- Computational method based on Abreu-Pearce-Stacchetti (APS) (1986,1990) set-valued techniques for repeated games.
- APS show that set of equilibrium payoffs a fixed point of an operator similar to Bellman operator in DP.
- APS method not directly implementable on a computer. Requires approximation of arbitrary sets.
- Our method allows for
  - parsimonious representation of sets/correspondences on a computer
  - preserves monotonicity of underlying operator.

Contributions

- Develop a general algorithm that
    - computes pure-strategy equilibrium value sets of repeated and dynamic games,
    - provides upper and lower bounds for equilibrium values and hence computational error bounds,
    - computes equilibrium strategies.
- Based on: Judd-Yeltekin-Conklin (2003), Sleet and Yeltekin(2003), Yeltekin-Judd (2011)

REPEATED GAMES

Stage Game

- A_i – player i's action space, $i = 1, \cdots, N$
- $A = \times_{i=1}^{N} A_i$ – action profiles
- $\Pi_i(a)$ – Player i payoff, $i = 1, \cdots, N$
- Maximal and minimal payoffs

$$\underline{\Pi}_i \equiv \min_{a \in A} \Pi_i(a), \quad \bar{\Pi}_i \equiv \max_{a \in A} \Pi_i(a)$$

Supergame G^∞

- Action space: A^∞
- h_t: t-period history: $\{a_s\}_{s=0}^{t-1}$ with $a_s \in A$
- Set of t-period histories: H_t
- Preferences:

$$w_i(a^\infty) = \frac{1-\delta}{\delta} E_0 \Sigma_{t=1}^{\infty} \delta^t \Pi_i(a_t).$$

- Strategies: $\{\sigma_{i,t}\}_{t=0}^{\infty}$ with $\sigma_{i,t} : H_t \to A_i$.
- Subgame Perfect Equilibrium Payoffs

$$V^* \subset \mathcal{W} = \times_{i=1}^{N} [\underline{\Pi}_i \,,\ \overline{\Pi}_i]$$

Example 1: Prisoner's Dilemma

- Static game: player 1 (2) chooses row (column)

	Left	Right
Up	$4,\ 4$	$0,\ 6$
Down	$6,\ 0$	$2,\ 2$

- Static Nash equilibrium
  - (Down, Right) with payoff $(2, 2)$
- Suppose δ is close to 1
- G^∞ includes (Up, Left) forever with payoff $(4, 4)$
  - Rational if all believe a deviation causes permanent reversion to (Down, Right)
  - This is just one of many equilibria.

Static Equilibrium

- Static game

$b_{11},\ c_{11}$	$b_{12},\ c_{12}$
$b_{21},\ c_{21}$	$b_{22},\ c_{22}$

b_{ij} (c_{ij}) is player 1's (2's) return if player 1 (2) plays i (j).

Recursive Formulation

- Each SPE payoff vector is supported by
  - profile of actions consistent with Nash today
  - continuation payoffs that are SPE payoffs
- Each stage of subgame perfect equilibrium of G^{∞} is a static equilibrium to some one-shot game A, augmented by values from δV^*:

$\delta^* b_{11} + \delta u_{11},\ \delta^* c_{11} + \delta w_{11}$	$\delta^* b_{12} + \delta u_{12},\ \delta^* c_{12} + \delta w_{12}$
$\delta^* b_{21} + \delta u_{21},\ \delta^* c_{21} + \delta w_{21}$	$\delta^* b_{22} + \delta u_{22},\ \delta^* c_{22} + \delta w_{22}$

$\delta^* = 1 - \delta$

Steps: Computing the Equilibrium Value Set

1. Define an operator that maps today's equilibrium values to tomorrow's.
2. Show operator is monotone and equilibrium payoff set is its largest fixed point. [Requires some work. We use Tarski's FP theorem.]
3. Define approximation for operator and sets that
    - Represent sets parsimoniously on computer
    - Preserve monotonicity of operator
4. Define appropriately chosen initial set, apply operator until convergence.

Step 1: Operator

$B^* : \mathcal{P} \to \mathcal{P}$.

- Let $\mathcal{W} \in \mathcal{P}$.

$$B^*(\mathcal{W}) = \cup_{(a,w)}\{(1-\delta)\Pi(a) + \delta w\}$$

subject to:

$$w \in \mathcal{W}$$

and for each $\forall i \in N$, $\forall \tilde{a} \in A_i$

$$(1-\delta)\Pi_i(a) + \delta w_i \geq (1-\delta)\Pi_i(\tilde{a}, a_{-i}) + \delta \underline{w}_i\}$$

where $\underline{w}_i = \min\{w_i | w \in \mathcal{W}\}$.

Step 2: Self-generation

A set $\mathcal{W}$ is self-generating if :

$$\mathcal{W} \subseteq B^*(\mathcal{W})$$

An extension of the arguments in APS establishes the following:

- Any self-generating set is contained within V^*,
- V^* itself is self-generating.

Step 2: Factorization

$b \in B^*(\mathcal{W})$ if there is an action profile a and continuation payoff $w \in \mathcal{W}$, s.t

- b is value of playing a today and receiving continuation value w ,
- for each i, player i will choose to play a_i
- punishment value drawn from set $\mathcal{W}$.

Step 2: Properties of B^*

- Monotonicity: B^* is monotone in the set inclusion ordering:

$$\text{If } \mathcal{W}_1 \subseteq \mathcal{W}_2, \text{ then } B^*(\mathcal{W}_1) \subseteq B^*(\mathcal{W}_2)$$

- Compactness: B^* preserves compactness.
- Implications:
  1) V^* is the maximal fixed point of the mapping B^*;
  2) V^* can be obtained by repeatedly applying B^* to any set that contains V^*.

Step 3: Approximation

- V^* is not necessarily a convex set
  - We need to approximate both V^* and the correspondence $B^*(W)$
  - As a first step, use public randomization to convexify the equilibrium value set.

Step 3: Public randomization

- Public lottery with support contained in $\mathcal{W}$.
- Public lottery specifies continuation values for the next period
  - Lottery determines Nash equilibrium for next period.
  - Strategies now condition on histories of actions and lottery outcomes.
- Modified operator:

$$B(W) = B(co(\mathcal{W})) = co(B^*(co(\mathcal{W}))),$$

  where $W = co(\mathcal{W})$
- V equilibrium value set of supergame with public randomization.
- B is monotone and V is the largest fixed point of B.

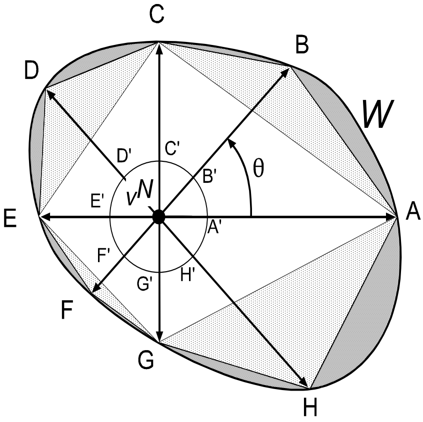
Step B: Approximations

- Modified operator B preserves monotonicity and compactness.
- Produces a sequence of convex sets that converge to equilibrium.
- Two approximations:
    - outer approximation
    - inner approximation

Piecewise-Linear **Inner** Approximation

- Suppose we have M points $Z = \{(x_1, y_1), ..., (x_M, y_M)\}$ on the boundary of a convex set W.
- The convex hull of Z, $co(Z)$, is contained in W and has a piecewise linear boundary.
- Since $co(Z) \subseteq W$, we will call $co(Z)$ the inner approximation to W generated by Z.

Inner approximation

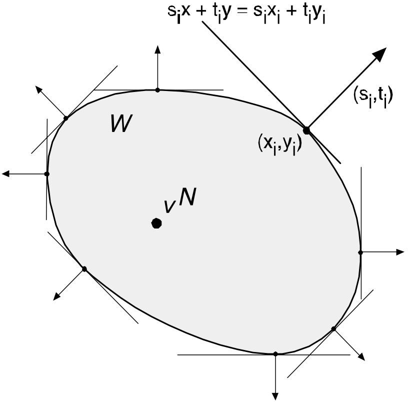


Inner approximations

Piecewise-Linear **Outer** Approximation

- Suppose we have
  - M points $Z = \{(x_1, y_1), ..., (x_M, y_M)\}$ on the boundary of W, and
  - corresponding set of subgradients, $R = \{(s_1, t_1), ..., (s_M, t_M)\}$;
- Therefore,
  - the plane $s_i x + t_i y = s_i x_i + t_i y_i$ is tangent to W at (x_i, y_i), and
  - the vector (s_i, t_i) with base at (x_i, y_i) points away from W.

Outer approximation



A convex set and supporting hyperplanes

Key Properties of Approximations

Definition

Let $B^I(W)$ be an inner approximation of $B(W)$ and $B^O(W)$ be an outer approximation of $B(W)$; that is $B^I(W) \subseteq B(W) \subseteq B^O(W)$.

Lemma

Next, for any $B^I(W)$ and $B^O(W)$, (i) $W \subseteq W'$ implies $B^I(W) \subseteq B^I(W')$, and (ii) $W \subseteq W'$ implies $B^O(W) \subseteq B^O(W')$.

Step 4: Initial Guesses and Convergence

Proposition

Suppose $B^O(\cdot)$ is an outer monotone approximation of $B(\cdot)$. Then the maximal fixed point of B^O contains V. More precisely, if $W \supseteq B^O(W) \supseteq V$, then $B^O(W) \supseteq B^O(B^O(W)) \supseteq \cdots \supseteq V$.

Lemma

$W \supseteq B^O(W) \supseteq V$.

Step 4: Initial Guesses and Convergence

Proposition

Suppose $B^I(\cdot)$ is an inner monotone approximation of $B(\cdot)$. Then the maximal fixed point of B^I is contained in V. More precisely, if $W \subseteq B^I(W) \subseteq V$, then $B^I(W) \subseteq B^I(B^I(W)) \subseteq \cdots \subseteq V$.

Lemma

$W \subseteq B^I(W) \subseteq V$.

Fixed Point

These results together with the monotonicity of the B operator, implies the following theorem.

Theorem

Let V be the equilibrium value set. Then (i) if $W_0 \supseteq V$ then $B^O(W_0) \supseteq B^O(B^O(W_0)) \supseteq \cdots \supseteq V$, and (ii) if $W_0 \subset B^I(W_0)$ then $B^I(W_0) \subset B^I(B^I(W_0)) \subseteq \cdots \subseteq V$. Furthermore, any fixed point of B^I is contained in the maximal fixed point of B, which in turn is contained in the maximal fixed point of B^O.

Monotone Inner Hyperplane Approximation

Input: Points $Z = \{z_1, \cdots, z_M\}$ such that $W = co(Z)$.

Step 1 Find extremal points of $B(W)$:

For each search subgradient $h_\ell \in H,\ \ell = 1, .., L$.

(1) For each $a \in A$, solve the linear program

$$\begin{array}{rcl} c_\ell(a) & = & \max_w \ h_\ell \cdot [(1-\delta)\Pi(a) + \delta w] \\ & \text{(i)} & w \in W \\ & \text{(ii)} & (1-\delta)\Pi^i(a) + \delta w_i \geq \\ & & \qquad (1-\delta)\Pi_i^*(a_{-i}) + \delta \underline{w}_i,\ i = 1, .., N \end{array} \tag{1}$$

Let $w_\ell(a)$ be a w value which solves (1).

Monotone Inner Hyperplane Approximation cont'd

(2) Find best action profile $a \in A$ and continuation value:

$$
\begin{aligned}
a_\ell^* &= \arg\max \{c_\ell(a) | a \in A\} \\
z_\ell^+ &= (1-\delta)\Pi(a_\ell^*) + \delta w_\ell(a_\ell^*)
\end{aligned}
$$

Step 2 Collect set of vertices $Z^+ = \{z_\ell^+ | \ell = 1, ..., L\}$, and define $W^+ = co(Z^+)$.

The Outer Approximation, Hyperplane Algorithm

Outer approximation: Same as inner approximation except record normals and continuation values z_ℓ^+

Outer vs. Inner Approximations

- Any point within the inner approximation is an equilibrium
  - Can construct an equilibrium strategy from V.
  - There exist multiple such strategies

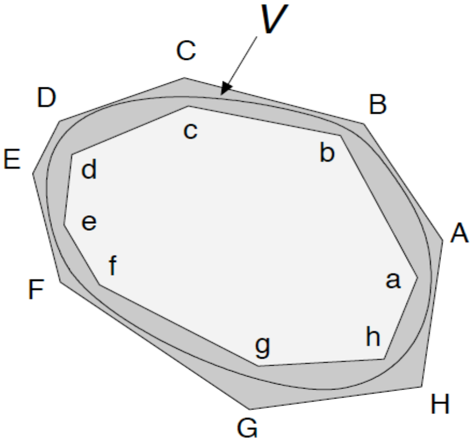
The Outer Approximation, Hyperplane Algorithm

- No point outside of outer approximation can be an equilibrium
  - Can demonstrate certain equilibrium payoffs and actions are not possible
  - E.g., can prove that joint profit maximization is not possible

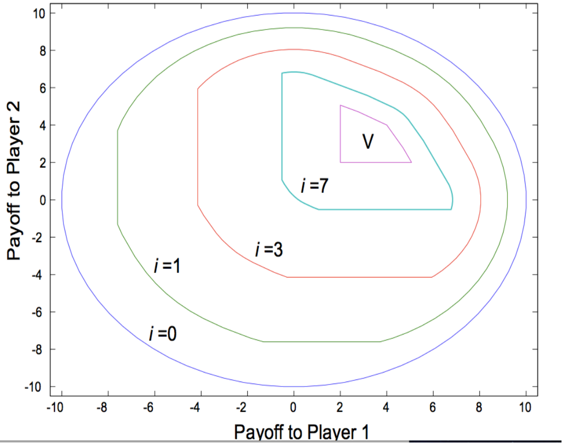
Error Bounds

- Difference between inner and outer approximations is approximation error
- Computations actually constitute a proof that something is in or out of equilibrium payoff set - not just an approximation.
- Difference is small in many examples.

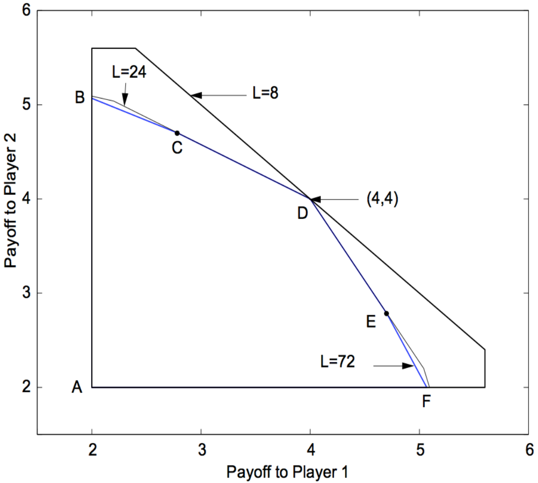
ErrorBounds



Convergence: Repeated Prisoner's Dilemma



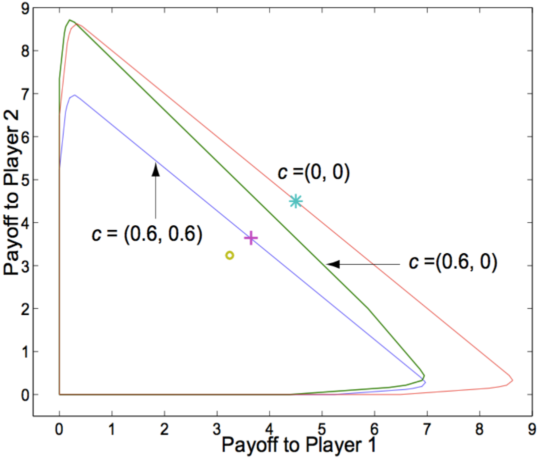
Hyperplanes: Repeated Prisoner's Dilemma



Example 2: Repeated Cournot Duopoly

- Firm i sales: q_i
- Firm i unit cost: $c_i = 0.6$
- Demand: $p = \max\{6 - q_1 - q_2, 0\}$
- Profit: $\Pi_i(q_1, q_2) = q_i(p - c_i)$
- Nash Eqm. Payoff of Stage Game: $(3.24, 3.24)$
- Shared Monopoly Payoff : $(3.64, 3.64)$

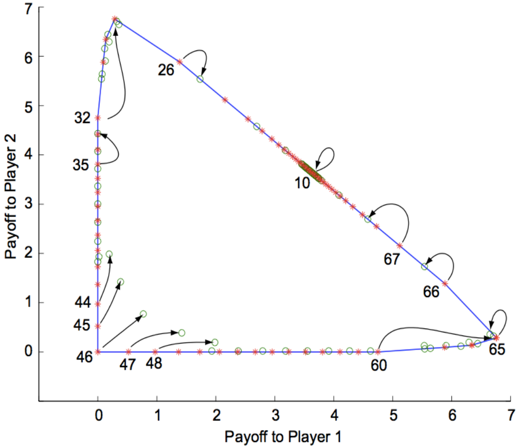
Repeated Cournot



Example 2: Repeated Cournot Duopoly

- Set of eqm payoffs quite large.
- Shared monopoly profits ($+$ and $\star$) are achievable (for $\delta = 0.8$)
- When costs are positive, threats far worse than reversion to Nash.

Strategies: Repeated Cournot



Strategies: Repeated Cournot

Actions, promises, and threats on the boundary of V, $c = 0.6$

ℓ	$(v_1(\ell), v_2(\ell))$		$(w_1(\ell), w_2(\ell))$		(q_1, q_2)		$\Pi(q_1, q_2)$	
2	3.97	3.30	3.75	3.52	1.7	0.9	4.8	2.4
8	3.71	3.57	3.72	3.55	1.3	1.3	3.6	3.6
10	3.64	3.64	3.64	3.64	1.3	1.3	3.6	3.6
27	0.29	6.76	0.36	6.65	0.0	3.0	0.0	7.1
46	0.00	0.00	0.77	0.77	5.1	5.1	-3.0	-3.0
60	4.75	0.00	6.71	0.32	5.1	2.1	-3.0	-1.3

Example 2: Repeated Cournot Duopoly

- Unlike APS's imperfect monitoring example, eqm. paths are not bang-bang.
- Continuation of worst eqm is not worst. Movement towards cooperation?
- Shared Monopoly: Markov and stationary.
- Low profits today for Firm i are supported by higher continuation values.

Next Meeting

- Dynamic Games
- Using algorithm to find endogenous state spaces.
- Extensions to planner+continuum of agents.
- Examples from applications in IO , Macro.