Numerical Dynamic Programming and Applications

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Outline

- Introduction of Numerical Dynamic Programming
- Advances in Numerical Dynamic Programming
  - Shape-preserving Approximation
  - Hermite Approximation
  - Parallelization
- Applications
  - Dynamic Portfolio Optimization
  - Dynamic and Stochastic Integration of Climate and Economy

Introduction of Dynamic Programming

- Finite horizon and/or non-stationary dynamic programming problems
- Value function:

$$V_t(x_t, \theta_t) = \max_{a_s \in \mathcal{D}(x_s, \theta_s, s)} \sum_{s=t}^{T-1} \beta^{s-t} \mathbb{E}\left\{u_s(x_s, a_s, \theta_s)\right\} + \beta^{T-t} \mathbb{E}\left\{V_T(x_T, \theta_T)\right\}$$

- Bellman equation:

$$\begin{aligned} V_t(x, \theta) = \max_{a \in \mathcal{D}(x, \theta, t)} \quad & u_t(x, a) + \beta \mathbb{E}\left\{V_{t+1}(x^+, \theta^+) \mid x, \theta, a\right\}, \\ \text{s.t.} \quad & x^+ = g_t(x, \theta, a, \omega), \\ & \theta^+ = h_t(\theta, \epsilon), \end{aligned}$$

Three Numerical Parts in DP

- Approximation of Value Functions
  - (Multidimensional) Chebyshev polynomails
- Numerical Integration
  - Gauss-Hermite quadrature
- Optimization
  - NPSOL

Typical Application I

- Optimal growth problem:

$$\begin{aligned} V_0(k_0) = \max_{c,l} \quad & \sum_{t=0}^{T-1} \beta^t u(c_t, l_t) + \beta^T V_T(k_T), \\ \text{s.t.} \quad & k_{t+1} = F(k_t, l_t) - c_t, \quad 0 \le t < T, \\ & \underline{k} \le k_t \le \bar{k}, \quad 1 \le t \le T, \\ & c_t,\ l_t \ge \epsilon, \quad 0 \le t < T, \end{aligned}$$

- Bellman equation:

$$\begin{aligned} V_t(k) \quad = \quad \max_{c,l} \quad & u(c, l) + \beta V_{t+1}(k^+), \\ \text{s.t.} \quad & k^+ = F(k, l) - c, \\ & \underline{k} \le k^+ \le \bar{k}, \quad c, l \ge \epsilon, \end{aligned}$$

Typical Application II

- Multi-stage portfolio optimization problem:

$$V_0(W_0) = \max_{S_t, 0 \le t < T} \mathbb{E}\{u(W_T)\},$$

- Wealth transition:

$$W_{t+1} = R_f(W_t - e^\top S_t) + R^\top S_t,$$

- Bellman equation:

$$\begin{aligned} V_t(W) &= \max_{B,S} \mathbb{E}\{V_{t+1}(R_f B + R^\top S)\}, \\ &\quad \text{s.t.} \quad B + e^\top S = W, \end{aligned}$$

Typical Application III

Multi-country optimal growth problem:

$$
\begin{aligned}
V_0(k_0,\theta_0) \quad = \quad & \max_{k_t, I_t, c_t, l_t} \; \mathbb{E}\left\{ \sum_{t=0}^{T-1} \beta^t u(c_t, l_t) + \beta^T V_T(k_T, \theta_T) \right\}, \\
\text{s.t.} \quad & k_{t+1,j} = (1-\delta)k_{t,j} + I_{t,j}, \quad j = 1, \ldots, d, \\
& \Gamma_{t,j} = \frac{\zeta}{2} k_{t,j} \left(\frac{I_{t,j}}{k_{t,j}} - \delta \right)^2, \quad j = 1, \ldots, d, \\
& \sum_{j=1}^{d} \left(c_{t,j} + I_{t,j} - \delta k_{t,j} \right) = \sum_{j=1}^{d} \left(f(k_{t,j}, l_{t,j}, \theta_t) - \Gamma_{t,j} \right), \\
& \theta_{t+1} = g(\theta_t, \epsilon_t).
\end{aligned}
$$

Numerical Dynamic Programming

Value function iteration method for solving finite-horizon and/or non-stationary dynamic programming problems.

- Initialization. Choose the approximation grid, $X = \{x_i : 1 \leq i \leq m\}$, and choose functional form for $\hat{V}(x; b)$. Let $\hat{V}(x; b^T) = V_T(x)$. Iterate through steps1 and 2 over $t = T - 1, ..., 1, 0$.
- Step 1. Maximization step: Compute

$$v_i = \max_{a_i \in \mathcal{D}(x_i, t)} u_t(x_i, a_i) + \beta \mathbb{E}\{\hat{V}(x_i^+; \mathbf{b}^{t+1})\},$$

  for each $x_i \in X$, $1 \leq i \leq m$.
- Step 2. Fitting step: Using the appropriate approximation method, compute the b^t such that $\hat{V}(x; b^t)$ approximates (x_i, v_i) data.

Computational Challenges

- Smooth function approximation is important for high-dimensional problems:
  - It can avoid the curse of dimensionality
  - Fast Newton-type optimization solvers can be applied
- Monotonicity and concavity of value functions may be NOT preserved by smooth function approximation
  - Difficult for optimization solvers to find global maximizers
- High-dimensional problems require many approximation nodes
  - Efficient usage of all possible information (such as slopes of value functions) can improve much
  - Parallelization can also be very efficient

Approximation

- Chebyshev polynomial approximation

$$\hat{V}(x;\mathbf{b}) = \sum_{j=0}^{n} b_j \mathcal{T}_j(Z(x)),$$

  - Chebyshev polynomial basis: $\mathcal{T}_j(z) = \cos(j \cos^{-1}(z))$
  - Normalization: $Z(x) = \frac{2x - x_{\mathbf{min}} - x_{\mathbf{max}}}{x_{\mathbf{max}} - x_{\mathbf{min}}}$
- Multidimensional Chebyshev polynomial approximation
  - Complete polynomial approximation:

$$\hat{V}_n(x;\mathbf{b}) = \sum_{0 \le |\alpha| \le n} b_\alpha \mathcal{T}_\alpha(Z(x)),$$

  - $\mathcal{T}_\alpha(z)$ denote the product $\mathcal{T}_{\alpha_1}(z_1) \cdots \mathcal{T}_{\alpha_d}(z_d)$

Shape-preserving Chebyshev Interpolation

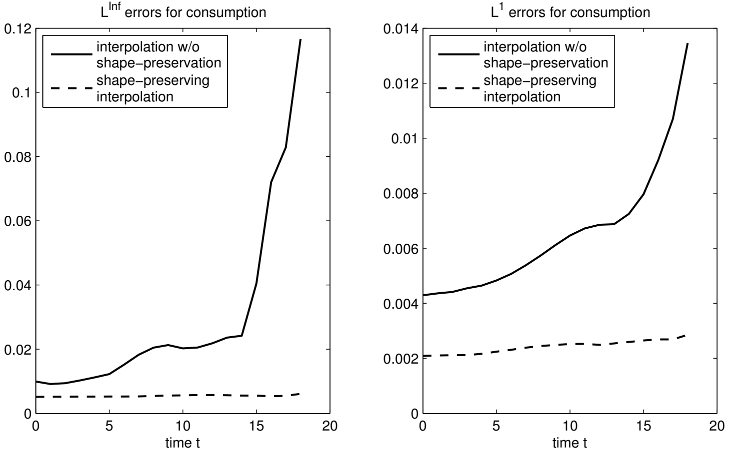
- LP problem to find coefficients

$$
\begin{aligned}
\min_{b_j, b_j^+, b_j^-} \quad & \sum_{j=0}^{m-1} (b_j^+ + b_j^-) + \sum_{j=m}^{n} (j+1-m)^2 (b_j^+ + b_j^-), \\
\text{s.t.} \quad & \sum_{j=0}^{n} b_j \mathcal{T}_j' (y_{i'}) > 0, \qquad i' = 1, \ldots, m', \\
& \sum_{j=0}^{n} b_j \mathcal{T}_j'' (y_{i'}) < 0, \qquad i' = 1, \ldots, m', \\
& \sum_{j=0}^{n} b_j \mathcal{T}_j (z_i) = v_i, \quad i = 1, \ldots, m, \\
& b_j - \hat{b}_j = b_j^+ - b_j^-, \quad j = 0, \ldots, m-1, \\
& b_j = b_j^+ - b_j^-, \quad j = m, \ldots, n, \\
& b_j^+, \ b_j^- \geq 0, \quad j = 1, \ldots, n,
\end{aligned}
$$

- y: shape nodes; z: approximation nodes

Application in Example I

Figure: Errors of numerical dynamic programming with Chebyshev interpolation with/without shape-preservation for growth problems



Hermite Value Function Iteration

- Envelope Theorem: If

$$\begin{aligned} H(x) &= \max_a \; f(x,a) \\ & \quad \text{s.t.} \;\; g(x,a) = 0, \\ & \qquad\quad\; h(x,a) \geq 0, \end{aligned}$$

then

$$\frac{\partial H(x)}{\partial x_j} = \frac{\partial f}{\partial x_j}(x, a^*(x)) + \lambda^*(x)^\top \frac{\partial g}{\partial x_j}(x, a^*(x)) + \mu^*(x)^\top \frac{\partial h}{\partial x_j}(x, a^*(x))$$

Get Slopes Easily

- Equivalent formulation:

$$
\begin{aligned}
H(x) \quad = \quad & \max_{a,y} \; f(y, a) \\
& \text{s.t.} \;\; g(y, a) = 0, \\
& \qquad h(y, a) \geq 0, \\
& \qquad x_j - y_j = 0, \quad j = 1, \ldots, d,
\end{aligned}
$$

- Get slope of H easily:

$$\frac{\partial H(x)}{\partial x_j} = \tau_j^*(x),$$

  - $\tau_j^*(x)$: *the shadow price of the trivial constraint* $x_j - y_j = 0$

Multidimensional Hermite Approximation

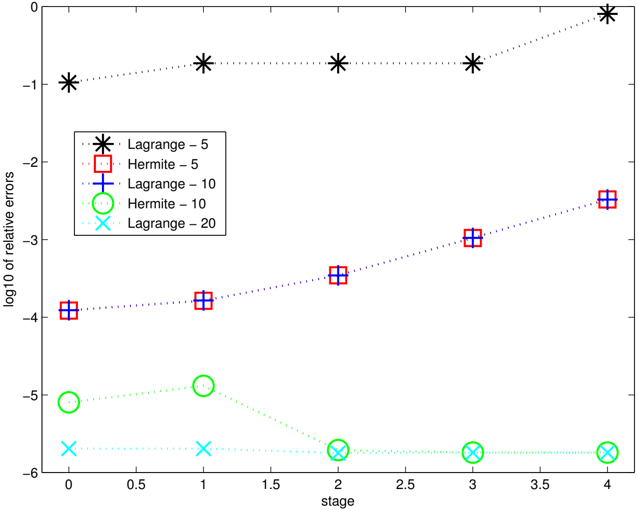
- Least-square problem

$$\min_{\mathbf{b}} \quad \sum_{i=1}^{N}\left(v_i - \sum_{0\leq|\alpha|\leq n} b_\alpha \mathcal{T}_\alpha\left(x^i\right)\right)^2 + \sum_{i=1}^{N}\sum_{j=1}^{d}\left(s_j^i - \sum_{0\leq|\alpha|\leq n} b_\alpha \frac{\partial}{\partial x_j}\mathcal{T}_\alpha\left(x^i\right)\right)^2$$

- Hermite data $\{(x^i, v_i, s^i) : i = 1, \ldots, N\}$:
  - $v_i = V(x^i)$,
  - $s_j^i = \frac{\partial}{\partial x_j} V(x^i)$

Application in Example II

Figure: Errors of H-VFI or L-VFI for Dynamic Portfolio Optimization



Accuracy and Running Times

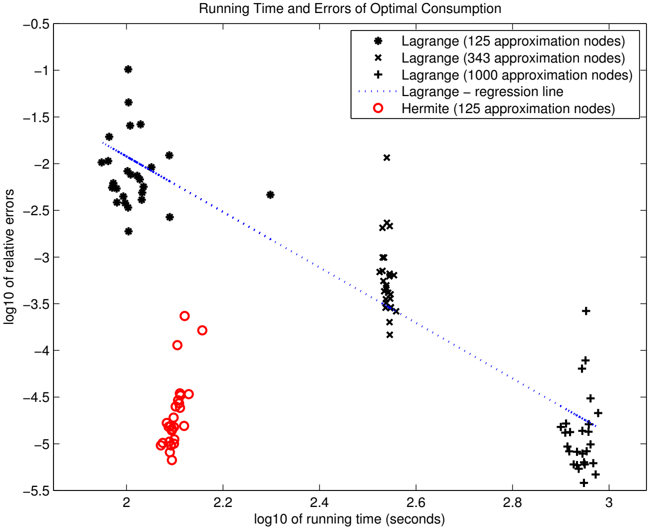
Table: Relative Errors and Running Times of L-VFI or H-VFI for Dynamic Portfolio Optimization

m	L-VFI error	H-VFI error	L-VFI time	H-VFI time
5	0.8	0.00327	9 seconds	10 seconds
10	0.00328	1.3×10^{-5}	12 seconds	17 seconds
20	2.0×10^{-6}		33 seconds	

- To reach the same accuracy of H-VFI, for one-dimensional problems, L-VFI needs
  - twice as many nodes
  - twice as much time

Application in Example III (Three Countries)

Figure: L-VFI vs H-VFI for Three-Country Optimal Growth Problems



Application in Example III (Six Countries)

Table: H-VFI vs L-VFI for Six-Dimensional Stochastic Problems

	error of c_0^*		error of l_0^*		time (hour)	
m	L-VFI	H-VFI	L-VFI	H-VFI	L-VFI	H-VFI
3	$3.8(-2)$	$3.6(-3)$	$5.4(-2)$	$5.2(-3)$	0.3	0.67
5	$5.5(-3)$		$8.2(-3)$		8.74	
6	$3.1(-3)$		$4.5(-3)$		36.6	

Note: $a(k)$ means $a \times 10^k$.

- To reach the same accuracy of H-VFI, for six-dimensional problems, L-VFI needs
  - 64 times as many nodes (6^6 nodes vs 3^6 nodes)
  - 55 times as much time (36.6 hours vs 0.67 hours)

Parallel Resources

- Multi-Core Local Machine (OpenMPI)
- Supercomputers
  - Beagle: 17,424 cores, 150 Teraflops (10^{12} FLOPS)
  - Cray Titan in Oak Ridge: 299,008 cores, 17.59 Petaflops (10^{15} FLOPS)
  - Around 2018: 1 Exaflops (10^{18} FLOPS)
- Distributed Parallelization (Grid Computing)
  - Condor
  - Cloud

Parallel Computing

- High Performance Computing
  - One application, many cores
  - focus on speed of floating-point operations (flops)
  - distributed memory and/or shared memory (MPI and/or OpenMP)
  - Graphics processing unit (GPU computing)
- High Throughput Computing: Many applications at the same time (Grid computing)
- Data-intensive Computing: Focus on speed of I/O operations

Parallelization in Dynamic Programming

- Parallelization in Maximization step in NDP: Compute

$$v_i = \max_{a_i \in \mathcal{D}(x_i, t)} u_t(x_i, a_i) + \beta \mathbb{E}\{\hat{V}(x_i^+; \mathbf{b}^{t+1})\},$$

  for each $x_i \in X_t$, $1 \leq i \leq m_t$.
- Master-Worker system: Master processor, Worker processors.
  - Workers solve the independent maximization problems
  - Master distributes tasks, collects results, does the fitting step

Parallel DP Algorithm for Master

Initialization. Set up $\hat{V}(x, \theta; \mathbf{b}^T)$, for all $\theta \in \Theta = \{\theta_j = (\theta_{j1}, \ldots, \theta_{jk}) : 1 \leq j \leq N\}$. Choose the approximation nodes, $X_t = \{x_i^t = (x_{i1}^t, \ldots, x_{id}^t) : 1 \leq i \leq m_t\}$. Iterate through steps1 and 2 over $t = T - 1, ..., 1, 0$.

Step 1. Separate the maximization step into N tasks, one task per $\theta_j \in \Theta = \{\theta_j = (\theta_{j1}, \ldots, \theta_{jk}) : 1 \leq j \leq N\}$. Each task contains the parameters $\mathbf{b}^{t+1}$, and a given θ_j. Then send these tasks to the workers.

Step 2. Wait until all tasks are done by the workers. Then collect the parameters $\mathbf{b}_j^t$ from the workers, for $1 \leq j \leq N$.

Parallel DP Algorithm for Workers

Step 1. Receive the parameters $\mathbf{b}^{t+1}$ for one specific θ_j from the master.

Step 2. For θ_j, compute

$$v_{ij} = \max_{a_{ij} \in \mathcal{D}(x_i, \theta_j, t)} u(x_i, \theta_j, a_{ij}) + \beta \mathbb{E}\{\hat{V}(x_i^+, \theta_j^+; \mathbf{b}^{t+1})\},$$

for each $x_i \in X_t$, $1 \leq i \leq m_t$.

Step 3. Using an appropriate approximation method, compute the $\mathbf{b}_j^t$, such that $\hat{V}(x, \theta_j; \mathbf{b}_j^t)$ approximates $\{(x_{ij}, v_{ij})$: $1 \leq i \leq m_t\}$.

Step 4. Send $\mathbf{b}_j^t$ to the master.

Parallelization Results for Example III

- Multi-country optimal growth problem:

$$\begin{aligned} V_t(k,\theta) \quad = \quad & \max_{c,l,I} \quad u(c,l) + \beta \mathbb{E}\left\{ V_{t+1}(k^+,\theta^+) \mid \theta \right\}, \\ & \text{s.t.} \quad k_j^+ = (1-\delta)k_j + I_j + \epsilon_j, \quad j = 1,\ldots,d, \\ & \qquad \Gamma_j = \frac{\zeta}{2} k_j \left(\frac{I_j}{k_j} - \delta \right)^2, \quad j = 1,\ldots,d, \\ & \qquad \sum_{j=1}^{d} (c_j + I_j - \delta k_j) = \sum_{j=1}^{d} (f(k_j,l_j,\theta_j) - \Gamma_j), \\ & \qquad \theta^+ = g(\theta,\xi_t), \end{aligned}$$

- Four-dimensional k (continuous)
- Four-dimensional θ (discrete with 7 values per country)
- Four-dimensional ϵ (discrete with 3 values per country)

Results for Example III

- 2401 tasks per value function iteration
- 2401 optimization problems per task

Table: Statistics of parallel dynamic programming under HTCondor-MW for the growth problem

Wall clock time for three VFIs	8.28 hours
Total time workers were assigned	16.9 days
Average wall clock time per task	199 seconds
Number of (different) workers	50
Overall Parallel Performance	98.6%

Parallel Efficiency for Example III

Table: Parallel efficiency for various number of worker processors

# Worker processors	Parallel efficiency	Average task wall clock time (seconds)	Total wall clock time (hours)
50	98.6%	199	8.28
100	97%	185	3.89
200	91.8%	186	2.26

PART II:

NEW APPLICATIONS

Portfolio with Transaction Costs

- Multi-stage Portfolio Optimization Problem:

$$
\begin{aligned}
V_0(W_0, x_0) = \max_{\delta_t} \quad & \mathbb{E}\{u(W_T)\} \\
\text{s.t.} \quad & W_{t+1} = \mathbf{e}^\top X_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t)W_t), \\
& X_{t+1,i} = R_i(x_{t,i} + \delta_{t,i})W_t, \\
& y_t = \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\
& x_{t+1,i} = X_{t+1,i}/W_{t+1}, \\
& t = 0, \ldots, T-1; \quad i = 1, \ldots, k,
\end{aligned}
$$

- τ: proportional transaction costs
- $\delta_{t,i} > 0$ means buying, and $\delta_{t,i} < 0$ means selling

Bellman Equation

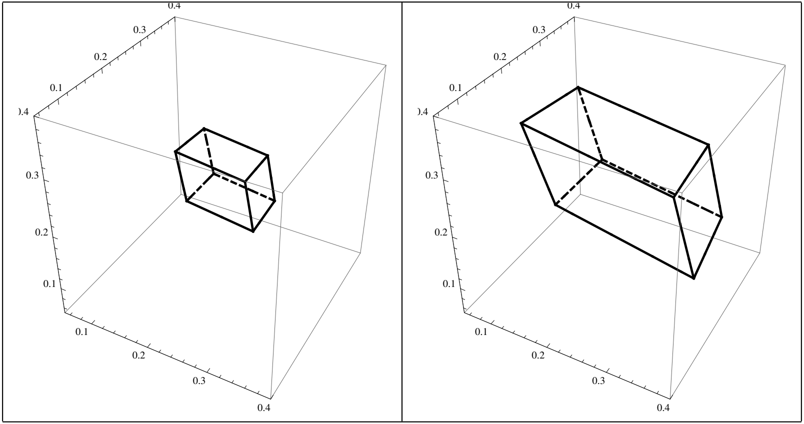
- Bellman equation

$$V_t(W_t, x_t) = \max_{\delta_t} \quad \mathbb{E}\left\{V_{t+1}(W_{t+1}, x_{t+1})\right\},$$

where

$$\begin{aligned}
y_t &\equiv \mathbf{e}^\top(\delta_t + \tau|\delta_t|), \\
X_{t+1,i} &\equiv R_i(x_{t,i} + \delta_{t,i})W_t, \\
W_{t+1} &\equiv \mathbf{e}^\top X_{t+1} + R_f(1 - \mathbf{e}^\top x_t - y_t)W_t, \\
x_{t+1,i} &\equiv X_{t+1,i}/W_{t+1},
\end{aligned}$$

No-trade regions



$t = 0$	$t = 5$

Parallelization of Dynamic Portfolio

- Number of Value Function Iterations: 6
- Number of optimization problems in one VFI: 15625
- Number of quadrature points for the integration in the objective function for one optimization problem: 15625

	Num of Jobs in one VFI	Wall Clock Time	Total CPU Time	Parallel Efficiency
96 cores	625	1.27 hours	4.7 days	92.3%
480 cores	3125	16 minutes	4.9 days	92%
Condor MW 100 workers	3125	1.3 hours	4.7 days	89%

New Application II: Climate Change Analysis

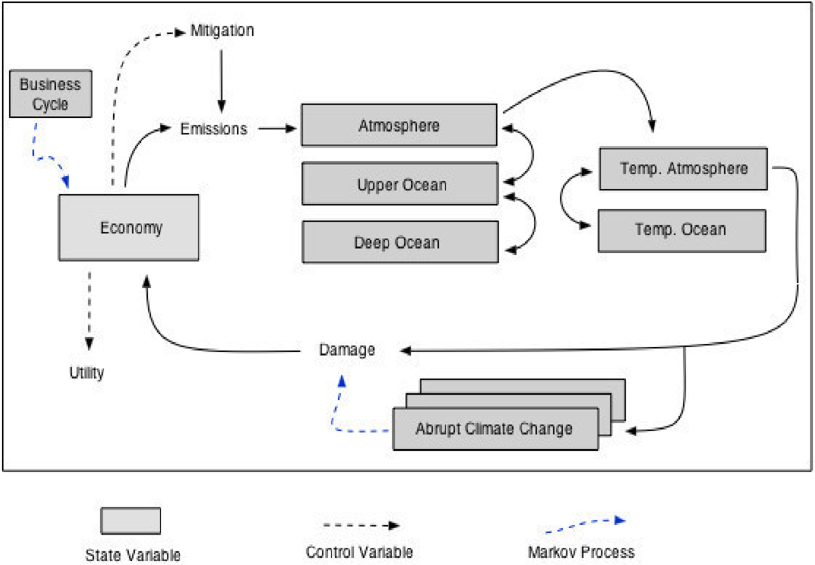
Question: What can and should be the response to rising CO2 concentrations?

- Analytical tools in the literature: IAMs (Integrated Assessment Models)
  - Two components: economic model and climate model
  - Interaction is often limited: Economy emits CO2 which affects world average temperature which affects economic productivity.
- Existing IAMs cannot study dynamic decision-making in an evolving and uncertain world
  - Most are deterministic; economic actors know with certainty the consequences of their actions and the alternatives
  - Most are myopic; standard reason is computational feasibility

Nordhaus' DICE: The Prototypical Model

- DICE2007 was the only dynamic economic model used by the US Interagency Working Group on the Cost of Carbon
- Economic system
  - gross output: $Y_t \equiv f(k_t, t) = A_t k_t^{\alpha} l_t^{1-\alpha}$
  - damage factor: $\Omega_t \equiv 1/\left(1 + \pi_1 T_t^{\mathrm{AT}} + \pi_2 (T_t^{\mathrm{AT}})^2\right)$
  - emission control cost: $\Lambda_t \equiv \psi_t^{1-\theta_2} \theta_{1,t} \mu_t^{\theta_2}$, where μ_t is policy choice
  - output net of damages and emission control: $\Omega_t (1 - \Lambda_t) Y_t$
- Climate system
  - Carbon mass: $\mathbf{M}_t = (M_t^{\mathrm{AT}}, M_t^{\mathrm{UP}}, M_t^{\mathrm{LO}})^{\top}$
  - Temperature: $\mathbf{T}_t = (T_t^{\mathrm{AT}}, T_t^{\mathrm{LO}})^{\top}$
  - Carbon emission: $E_t = \sigma_t (1 - \mu_t) Y_t + E_t^{\mathrm{Land}}$
  - Radiative forcing: $F_t = \eta \log_2 \left(\left(M_t^{\mathrm{AT}} + M_{t+1}^{\mathrm{AT}} \right) / \left(2 M_0^{\mathrm{AT}} \right) \right) + F_t^{\mathrm{EX}}$

Figure: DSICE Framework



Uncertainty and Risk

All agree that uncertainty needs to be a central part of any IAM analysis
Multiple forms of uncertainty

- Risk: productivity shocks, taste shocks, uncertain technological advances, weather shocks
- Parameter uncertainty: policymakers do not know parameters that characterize the economic and/or climate systems
- Model uncertainty: policymakers do not know the proper model or the stochastic processes

Abrupt, Stochastic, and Irreversible Climate Change

Question: What is the optimal carbon tax when faced with abrupt and irreversible climate change?

- Common assumption in IAMs: damages depend only on contemporaneous temperature
- Our criticism: this cannot analyze the permanent and irreversible damages from tipping points
- We show that
  - Abrupt climate change can be modeled stochastically
  - The policy response to the threat of tipping points is very different from the policy response to standard damage representations.

Tipping point

- A tipping point is where temperature causes a big event with permanent damage
- The time of tipping is a Poisson process, and probability of a tipping point occurring at t equals the hazard rate $h_t(T_t^{\mathrm{AT}})$
- Examples:
  - Thermohaline circulation collapse
  - Extreme catastrophe (Weitzman (2009)): small probability (hazard rate is 0.1% at 2100) but big deduction of production (20% damage)

Cai-Judd-Lontzek DSICE Model

DSICE (**D**ynamic **S**tochastic **I**ntegrated Model of **C**limate and **E**conomy)

$$
\begin{aligned}
DSICE &= DICE2007 \\
&+ \text{stochastic damage factor} \\
&+ \text{stochastic production function} \\
&+ \text{flexible period length}
\end{aligned}
$$

DSICE: new features

- Economic system: $Y_t \equiv f(k_t, \zeta_t, t) = \zeta_t A_t k_t^{\alpha} l_t^{1-\alpha}$ where $\zeta_{t+1} = g^{\zeta}(\zeta_t, \omega_t^{\zeta})$ is an AR(1) process for the productivity state ζ
- Climate system: $\Omega_t \equiv (1 - J_t)/\left(1 + \pi_1 T_t^{AT} + \pi_2 (T_t^{AT})^2\right)$ where $J_{t+1} = g^{J}(J_t, \omega_t^{J})$ is a Markov process for the damage factor state J

DP model of DSICE

- DP model for DSICE

$$
\begin{array}{rcl}
V_t(k, \mathbf{M}, \mathbf{T}, \zeta, J) & = & \max_{c,\mu} \quad u_t(c) + \beta \mathbb{E}[V_{t+1}(k^+, \mathbf{M}^+, \mathbf{T}^+, \zeta^+, J^+)] \\
\text{s.t.} \quad k^+ & = & (1-\delta)k + \Omega_t(1-\Lambda_t)Y_t - c, \\
\mathbf{M}^+ & = & \Phi^M \mathbf{M} + (E_t, 0, 0)^\top, \\
\mathbf{T}^+ & = & \Phi^T \mathbf{T} + (\xi_1 F_t, 0)^\top, \\
\zeta^+ & = & g^\zeta(\zeta, \omega^\zeta), \\
J^+ & = & g^J(J, \omega^J)
\end{array}
$$

- One year (or one quarter of a year) time steps over 600 years
- Seven continuous states: $k, \mathbf{M}, \mathbf{T}, \zeta$
- one discrete state: J

Epstein-Zin Preference

- Epstein-Zin preference

$$
\begin{aligned}
U_t\left(k, \mathbf{M}, \mathbf{T}, J\right) \quad = \quad & \max_{c,\mu} \Bigg\{ (1-\beta)\, u(c_t, l_t) + \\
& \beta \left[\mathbb{E}\left\{ \left(U_{t+1}\left(k^+, \mathbf{M}^+, \mathbf{T}^+, J^+\right)\right)^{1-\gamma} \right\} \right]^{\frac{1-\psi}{1-\gamma}} \Bigg\}^{\frac{1}{1-\psi}}
\end{aligned}
$$

  - ψ: the inverse of the intertemporal elasticity of substitution
  - γ: the risk aversion parameter

Standardized DP model:

$$
\begin{aligned}
V_t(k, \mathbf{M}, \mathbf{T}, J) = \max_{c,\mu} \quad & u(c_t, l_t) + \frac{\beta}{1-\psi} \times \\
& \left[\mathbb{E}\left\{ \left((1-\psi)\, V_{t+1}\left(k^+, \mathbf{M}^+, \mathbf{T}^+, J^+\right)\right)^{\frac{1-\gamma}{1-\psi}} \right\} \right]^{\frac{1-\psi}{1-\gamma}}
\end{aligned}
$$

Accuracy Test and Running Times

- Relative errors and running times for the deterministic problem for accuracy test

degree	k	M^{AT}	T^{AT}	c	μ	Time
4	6.4(−4)	5.8(−5)	6.1(−5)	1.8(−4)	1.7(−4)	7.8 minutes
6	2.5(−5)	9.4(−7)	1.0(−6)	2.6(−5)	9.5(−6)	2.2 hours

- Running times for various cases of DSICE

	Step Size h	Num of Nodes	Time
One Tipping Point	1 year	31,250	16 minutes
One Economic Shock & Three Tipping Points	1 year	625,000	15.7 hours
Parallel DSICE with One Economic Shock & Three Tipping Points across 112 cores	1 year	625,000	11.75 minutes (total CPU time: 20.9 hours)

Big Increase of Carbon Tax

	ψ	γ	Carbon tax
DICE	2	2	\$37
DSICE, tipping of 2.5% damage	2	10	\$54
DSICE, tipping of 5% damage	2	2	\$69
DSICE, tipping of 5% damage	2	10	\$75
DSICE, tipping of 5% damage	2	20	\$83
DSICE, disaster case	2	10	\$124

Parallelization of DSICE

- Larger-size DSICE:
  - Number of stages of multi-stage process could be large to represent a continuous tipping process
  - The transition system of carbon concentration and temperature are random
  - Many parameters, such as climate sensitivity, are estimated with errors (treated as random)
- One economic shock and six other continuous random variables
- Number of optimization problems in one VFI: 78,125
- Number of quadrature points for the integration in the objective function for one optimization problem: 78,125

Num of Cores	Num of Jobs in one VFI	Wall Clock Time of 1 VFI	Total CPU Time of 1 VFI	Parallel Efficiency
7,824	15,625	70 seconds	6.2 days	93%