Existence, Uniqueness, and Computational Theory for Time Consistent Equilibria: A Hyperbolic Discounting Example

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Introduction

- Dynamic Inconsistency Problems
  - Frequently arise in monopoly problems, government policy problems, and hyperbolic discounting.
  - Multiple equilibria generally exist even after we focus on feedback equilibria (a.k.a., Markov perfect eqm.)
  - Continuity is a frequent selection criterion
  - Numerical solution is difficult
- This paper
  - Uses a hyperbolic discounting problem as an example; techniques obviously apply to many other models
  - Uses asymptotic methods to examine existence and uniqueness
  - Proves a sufficient condition for local determinacy and differentiability of equilibrium manifold
  - Uses results to develop stable numerical methods
  - Demonstrates that nonlocal extrapolations perform well
  - Real title of this paper is "Existence and Uniqueness Theory for Singularly Perturbed Nonlinear Differential Composition Equations", but let's not tell anyone

Related Literature

- Multiplicity of Nash equilibria
  - Reputation based on past actions
  - Algorithms: Judd-Yeltekin-Conklin (2003), Sleet-Yeltekin (2002)
- Multiple feedback Nash equilibria abound!
  - Stokey (1981) - expectations of future durable goods monopolist behavior given cumulative stock
  - Fudenberg-Tirole (1983) - state is capacity, each firm chooses investment.
  - Krusell-Smith (2003) - hyperbolic discounting - state is capital stock, decision is consumption

- Computation of feedback equilibria
  - Continuity of strategies is often implicitly assumed
    * Flexible polynomial approximation methods in ag econ, IO, and public finance - Wright-Williams (1984), Kotlikoff-Shoven-Spivak (1988), Rui-Miranda (1996), Ha-Sibert (1997), Vedenov-Miranda (2001), Doraszelski (2003)
    * Linear approx. methods in macro - Krusell, Rios-Rull, and Quadrini (1997) (inferior to KKS, according to K and R-R)
  - Existence problem
    * No general existence theorem for continuous, pure strategy feedback equilibrium
    * Laibson and coauthors discretize problem and sometimes get apparently discontinuous solutions
  - Multiplicity problem in computational methods
    * Finding all numerical solutions is difficult
    * KKS procedure has multiple solutions
  - Multiple multiplicity problems!

Growth with Hyperbolic Discounting

- We examine a simple growth problem.
  - output: $f(k)$
  - savings rule: $k_{t+1} = h(k_t)$
  - hyperbolic discounting: $(u_t \equiv u(c_t))$

$$
\begin{aligned}
U_0 &= u_0 + \beta(\delta u_1 + \delta^2 u_2 + \delta^3 u_3 + \cdots) \\
&\vdots \\
U_t &= u_t + \beta(\delta u_{t+1} + \delta^2 u_{t+2} + \cdots).
\end{aligned}
$$

- Equilibrium definition
  - Let $h(k)$ be a stationary decision rule (a restriction of stationary feedback eqm. concept)
  - Let $V(k)$ be the discounted value of utility to me of consumption *after today* if *tomorrow's* capital stock is k. Hence

$$V(k) = u_1 + \delta u_2 + \delta^2 u_3 + \cdots$$

    and

$$U_0 = u_0 + \beta\delta V(k_1)$$

  - A *continuously differentiable feedback Nash equilibrium* is a pair of functions $V(k)$ (C^2) and $h(k)$ (C^1) that satisfy the value function equation

$$V(k) = u(f(k) - h(k)) + \delta V(h(k)), \tag{1}$$

    and the first-order condition

$$u'(f(k) - h(k)) = \beta\delta V'(h(k)), \tag{2}$$

    plus global optimality
  - Standard definition from time consistency literature - Phelps, Pollak, Goldman, etc.

- Generalized Euler Equation:
  - KKS (2002) show that you can reduce the two-equation equilibrium system (1, 2) in $V(k)$ and $h(k)$ to one equation in $h(k)$

$$\begin{aligned} 0 &= u'(f(k) - h(k)) - \beta\delta u'(f(h(k)) - h(h(k))) \\ &\quad \times \left(f'(h(k)) + \left(\frac{1}{\beta} - 1\right) h'(h(k)) \right) \end{aligned}$$

  - $h'(h(k))$ looks familiar - "conjectural variation"
  - Similar transformation for differential games - Rincon-Zapatero (1998)
  - The GEE formulation has no added mathematical content
    - Useful for exposition
    - Bad for computation - has a triple composition term

– We shall rewrite the GEE as

$$\begin{aligned} 0 &= u'(f(k) - h(k)) - \left(\frac{\delta}{1+\varepsilon}\right) u'(f(h(k)) - h(h(k))) \\ &\qquad \times (f'(h(k)) + \varepsilon h'(h(k))) \\ &\equiv G(k, h(k), h(h(k)), \varepsilon h'(h(k)), \varepsilon) \end{aligned}$$

where

$$\varepsilon = \frac{1}{\beta} - 1$$

– ε is a parameter measuring deviation from exponential discounting.

– $\varepsilon = 0$ is the normal exponential case with a unique solution

Polynomial Approximations

- Standard method in public finance, agricultural economics, and IO literatures
  - Examples: Wright-Williams (1984), Kotlikoff-Shoven-Spivak (1988), Rui - Miranda (1996), Ha-Sibert (1997), Judd (1998), Vedenov-Miranda (2001), Doraszelski (2003)
  - Assume approximation $\hat{h}(k) = \sum_{i=0}^{n} a_i k^i$
  - Identify coefficients, a, by solving projection equations

$$0 = \int_I G\left(k, \hat{h}(k), \hat{h}\left(\hat{h}(k)\right), \varepsilon \hat{h}'\left(\hat{h}(k)\right), \varepsilon\right) \phi_j(k)\, dk, \quad j = 0, .., n$$

- Apply projection method (see Judd, 1992) to KKS problem
  - Successful in Rui-Miranda, Ha-Sibert, Vedenov-Miranda, and Doraszelski
  - Consider example from KKS

$$\begin{aligned} u(c) &= -\ln c,\ f(k) = \frac{144}{342}k^{\alpha} + .9k \\ \delta &= .95,\ \beta \in \{1, .95, .9, .85, .8\} \end{aligned}$$

  - Use Chebyshev collocation with degree 31 polynomial
  - With standard linear initial guess we find solution in 3 seconds
  - Maximum normalized Euler equation error on $[.25, 1.75]$ is 10^{-13}
  - Steady states:

Table 1: Steady State Capital Stock from Projection Method

β :	1.00	.950	.900	.850	.800
steady state k :	1.00	.904	.809	.716	.625

- Problems: Are there multiple equilibria? multiple numerical solutions?