Dynamic Programming with Piecewise Linear Interpolation

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April 20, 2011

0.1 Piecewise Linear Interpolation

If Lagrange data $\{(x_i, v_i) : i = 1, \ldots, m\}$ is given, then its piecewise linear interpolation is

$$\hat{V}(x) = b_{j,0} + b_{j,1}x, \quad \text{if } \ x \in [x_j, x_{j+1}],$$

where

$$\begin{aligned} b_{j,1} &= \frac{v_{j+1} - v_j}{x_{j+1} - x_j}, \\ b_{j,0} &= v_i - b_{j,1}x_i, \end{aligned}$$

for $j = 1, \ldots, m-1$.

In the maximization step of numerical DP algorithms, one could directly solve the maximization problem

$$v_i^t = \max_{a_i \in \mathcal{D}(x_i,t)} \quad u_t(x_i, a_i) + \beta \; \hat{V}\left(x_i^+; \mathbf{b}^{t+1}\right)$$

where

$$x^+ = g(x, a)$$

Problem: $\hat{V}(x; \mathbf{b}^{t+1})$ is not differentiable, making it difficult to solve the optimization problem for a.

0.2 Min-Function Approach

The differentiability problem is solved as follows:

$$\begin{aligned} v_i^t = & \max_{a_i \in \mathcal{D}(x_i,t), y_{ik}, x_{ik}^+} & u_t(x_i, a_i) + \beta\ y_i \\ & \text{s.t.} & x_i^+ = g(x_i, a) \\ & & y_i \leq b_{j,0}^{(t+1)} + b_{j,1}^{(t+1)} x_i^+, \quad 1 \leq j < m \end{aligned}$$

The objective function is smooth and inequality constraints are linear and sparse so that we can apply fast Newton type optimization algorithms to solve this problem if g is also smooth. Although this new model adds $(m-1)n$ linear inequality constraints, few of them will be active, such that optimization softwares can still solve the new model quickly.

Moreover, this way does not need to find the interval where x_{ik}^+ locates, while the naive way has to.

0.3 Convex-Set Approach

Both previous methods need to calculate coefficients; this is very complicated for multi-dimensional piecewise linear interpolation.

The following method has no need to compute the coefficients explicitly:

$$
\begin{aligned}
v_i^t = \max_{a_i \in \mathcal{D}(x_i,t), \mu_j \geq 0, y_i, x_i^+} \quad & u_t(x_i, a_i) + \beta\, y_i, \\
\text{s.t.} \quad & x_i^+ = g(x_i, a), \\
& x_i^+ \leq \sum_{j=1}^{m} \mu_j x_j^{(t+1)} \\
& y_i = \sum_{j=1}^{m} \mu_j v_j^{(t+1)} \\
& \sum_{j=1}^{m} \mu_j = 1
\end{aligned}
\tag{1}
$$