Constraint Qualification Examples

We will look at a simple economics example showing how constraint qualifications are important in economics.

Time and money budget problem

Maximize utility, u(x, y), subject to two constraints - money and time.

money price of x is 10
money price of y is 1
money budget is 21

time price of x is 1
time price of y is 8
time budget is 10

Our utility function will be x y, and it will be our objective, obj.
c1 and c2 are the two constraints, both c1 and c2 must be nonpositive

```
obj =  x y
c1 = 10 x + y - 21
c2 = x + 8 y - 10
```

```
x y
-21 + 10 x + y
-10 + x + 8 y
```

Form Lagrangian

```
lag = obj + λ1 c1 + λ2 c2
```

```
x y + (-21 + 10 x + y) λ1 + (-10 + x + 8 y) λ2
```

The first-order conditions (KKT conditions) are

```
foc1 = D[lag, x]
```

```
y + 10 λ1 + λ2
```

```
foc2 = D[lag, y]
```

```
x + λ1 + 8 λ2
```

We know solution is at x=2 and y=1. Substitute this into first-order conditions

```
{foc1, foc2} /. x → 2 /. y → 1
```

$\{1 + 10\,\lambda 1 + \lambda 2,\ 2 + \lambda 1 + 8\,\lambda 2\}$

```
Solve[% == 0, {λ1, λ2}]
```

$\left\{\left\{\lambda 1 \to -\frac{6}{79},\ \lambda 2 \to -\frac{19}{79}\right\}\right\}$

We have unique multipliers, both positive, as predicted by KKT

Time, money, and weight problem

Maximize utility subject to three constraints:
Same problem as above (limited time and money) but add constraint that x + y cannot exceed 3. Suppose that each unit of x and y weigh one kilogram, and that your sack can carry only 3 kilograms of purchases from the store to home.

Note that solution is still x=2 and y=1.
We have the same problem as before except now we have one more constraint, that c3 = x + y - 3 cannot be positive

```
obj = - x y;
c1 = 10 x + y - 21;
c2 = x + 8 y - 10;
c3 = x + y - 3;
```

Lagrangian

```
lag = obj + λ1 c1 + λ2 c2 + λ3 c3
```

$-x\,y + (-21 + 10\,x + y)\,\lambda 1 + (-10 + x + 8\,y)\,\lambda 2 + (-3 + x + y)\,\lambda 3$

KKT conditions

```
foc1 = D[lag, x]
```

$-y + 10\,\lambda 1 + \lambda 2 + \lambda 3$

```
foc2 = D[lag, y]
```

$-x + \lambda 1 + 8\,\lambda 2 + \lambda 3$

```
{foc1, foc2} /. x → 1 /. y → 1
```

$\{-1 + 10\,\lambda 1 + \lambda 2 + \lambda 3,\ -1 + \lambda 1 + 8\,\lambda 2 + \lambda 3\}$

```
Solve[% == 0, {λ1, λ2, λ3}]
```

Solve::svars :
Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{\lambda 1 \to \frac{7}{79} - \frac{7\,\lambda 3}{79},\ \lambda 2 \to \frac{9}{79} - \frac{9\,\lambda 3}{79}\right\}\right\}$$

The set of constraints is bounded since λ's must be nonnegative, but the multipliers are not unique.

LICQ

If X is a solution to a constrained optimization problem AND if the number of active constraints exceed the number of first-order conditions, then there cannot be unique shadow prices.
Furthermore, most algorithms will fail to converge to a solution