Tackling Multiplicity of Equilibria with Gröbner Bases

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Motivation

Multiplicity of equilibria is a serious threat to predictions and sensitivity analysis in economic models

Sufficient conditions for uniqueness sometimes exist but are often too restrictive

Uniqueness of equilibrium in policy analysis is often just assumed

Algorithms for solving applied models do not search for more than one equilibrium

Prevalence of multiplicity in "realistically calibrated" models is largely unknown

Problem at Hand

Economic equilibrium characterized as a solution of a system of polynomial equations

$$f(x) = 0 \quad \text{where} \quad x \in \mathbb{R}^n$$

Additional condition $x_i > 0$ for some or all variables

Find all equilibria

Outline

Polynomials

Monomial in $x_1, x_2, \ldots, x_n$: $\quad x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

Exponents $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$

Polynomial f in the n variables $x_1, x_2, \ldots, x_n$ is a linear combination of finitely many monomials with coefficients in a field $\mathbb{K}$

$$f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{K}, \quad S \subset \mathbb{Z}_+^n \text{ finite}$$

Examples of $\mathbb{K}$: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Polynomial Ideals

Polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ = set of all polynomials in $x = (x_1, \ldots, x_n)$ with coefficients in some field $\mathbb{K}$

$I \subset \mathbb{K}[x]$ is an ideal,

- if $f, g \in I$, then $f + g \in I$
- if $f \in I$ and $h \in \mathbb{K}[x]$, then $hf \in I$

Ideal generated by $f_1, \ldots, f_k$,

$$I = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[x]\} = \langle f_1, \ldots, f_k \rangle$$

Polynomials $f_1, \ldots, f_k$ are basis of I

Complex Varieties

Set of common complex zeros of $f_1, \ldots, f_k \in \mathbb{K}[x]$

$$V(f_1, f_2, \ldots, f_k) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_k(x) = 0\}$$

$V(f_1, f_2, \ldots, f_k)$ complex variety defined by $f_1, f_2, \ldots, f_k$

Study of polynomial equations on algebraically closed fields

Field $\mathbb{R}$ is not algebraically closed, but $\mathbb{C}$ is

For an ideal $I = \langle f_1, \ldots, f_k \rangle = \langle g_1, \ldots, g_l \rangle$

$$V(I) = V(f_1, f_2, \ldots, f_k) = V(g_1, g_2, \ldots, g_l).$$

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$$V(I) = V(f_1, f_2, \ldots, f_k) = V(g_1, g_2, \ldots, g_l).$$

Simple Version of the Shape Lemma

$V(f_1, f_2, \ldots, f_n) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_n(x) = 0\}$
is zero-dimensional and has d complex roots

No multiple roots

All roots have distinct value for last coordinate x_n

Then:

$V(f_1, f_2, \ldots, f_n) = V(G)$ where

$$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

Polynomial r has degree d, polynomials v_i have degrees less than d

On the Assumptions

$x_1^2 - x_2 = 0,\ x_2 - 4 = 0$ has solutions $(2, 4), (-2, 4)$

No polynomial $x_1 - v_1(x_2)$ can yield 2 and -2 for $x_2 = 4$

After reordering of variables the shape lemma holds
$x_2 - 4 = 0, x_1^2 - 4 = 0$

$x_1^2 + x_2 - 1 = 0,\ x_2^2 - 1 = 0$, sol's $(\sqrt{2}, -1), (-\sqrt{2}, -1), (0, 1)$

Solution $(0, 1)$ has multiplicity 2
No linear term in x_1 of form $x_1 - v_1(x_2)$ can yield multiplicity

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Satisfying the Assumptions

No multiple roots

Add additional variable and equation

$$1 - t \det[D_x f(x)] = 0$$

All roots have distinct value for last coordinate

Add equation and new last variable

$$x_{n+1} - \sum_{l=1}^{n} \alpha_l x_l = 0$$

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Buchberger's Algorithm

Example of Gröbner basis

$$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

Buchberger's algorithm allows calculation of Gröbner bases

If all coefficients of $f_1, \ldots, f_n$ are rational then the polynomials r, $v_1, v_2, \ldots, v_{n-1}$ have rational coefficients and can be computed exactly

Software $\mathrm{SINGULAR}$: implementation of Buchberger's algorithm

Real Solutions

Univariate polynomial $r(x_n) = \sum_{i=0}^{d} a_i \; x_n^i$ of degree d

Fundamental Theorem of Algebra

Polynomial $r(x_n)$ has d complex roots.

Bounds on the number of (positive) real roots exist

Descartes's Rule of Signs

The number of positive reals roots of a polynomial is at most the number of sign changes in its coefficient sequence

Sturm's Theorem gives exact number of real zeros

in a given interval

Summary: Solving Polynomial Systems

Objective: find all solutions to $f(x) = 0$ with $x, f(x) \in \mathbb{R}^n$

View the system in complex space, $f(x) = 0$ with $x, f(x) \in \mathbb{C}^n$

$$V(f_1, f_2, \dots, f_n) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \dots = f_n(x) = 0\}$$

Apply Buchberger's algorithm to find Gröbner basis G

If Shape Lemma holds, then $V(f) = V(G)$ for a G of the shape

$$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \dots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

Apply Sturm's Theorem to r to find number of real solutions

Find approximation of all (complex) solutions by solving $r(x_n) = 0$

SINGULAR

$$x - yz^3 - 2z^3 + 1 = -x + yz - 3z + 4 = x + yz^9 = 0$$

SINGULAR code

```
ring R=0,(x,y,z),lp;
ideal I=(
x-y*z**3-2*z**3+1,
-x+y*z-3*z+4,
x+y*z**9);
ideal G=groebner(I);
```

```
> G;
G[1]=2z11+3z9-5z8+5z3-4z2-1
G[2]=2y+18z10+25z8-45z7-5z6+5z5-5z4+5z3+40z2-31z-6
G[3]=2x-2z9-5z7+5z6-5z5+5z4-5z3+5z2+1
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```

Parameterized Shape Lemma

Let $E \subset \mathbb{R}^m$ be an open set of parameters and let $f_1, \ldots, f_n \in \mathbb{K}[e_1, \ldots, e_m; x_1, \ldots, x_n]$ with $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$ and $(x_1, \ldots, x_n) \in \mathbb{C}^n$. Suppose that for each $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_m) \in E$ the Jacobian matrix $D_x f(\bar{e}; x)$ has full rank n whenever $f(\bar{e}; x) = 0$ and all d solutions have a distinct last coordinate x_n.

Then there exist $r, v_1, \ldots, v_{n-1} \in \mathbb{K}[e; x_n]$ and $w_1, \ldots, w_{n-1} \in \mathbb{K}[e]$ such that for generic $\bar{e}$,

$$\{x \in \mathbb{C}^n : f_1(\bar{e}; x) = \ldots = f_n(\bar{e}; x) = 0\} =$$

$$\{x \in \mathbb{C}^n : w_1(\bar{e})x_1 = v_1(\bar{e}; x_n), \ldots, w_{n-1}(\bar{e})x_{n-1} = v_{n-1}(\bar{e}; x_n);$$

$$r(\bar{e}; x_n) = 0\}\,.$$

The degree of r in x_n is d, the degrees of $v_1, \ldots, v_{n-1}$ in x_n are at most $d - 1$.

SINGULAR with PARAMETERS

$$x - yz^3 - 2z^3 + 1 = -x + yz - 3z + 4 = ex + yz^9 = 0$$

SINGULAR code

```
ring R=(0,e),(x,y,z),lp;
ideal I=(
x-y*z**3-2*z**3+1,
-x+y*z-3*z+4,
e*x+y*z**9);
ideal G=groebner(I);

G[1];
G[1]=2*z11+3*z9-5*z8+(5e)*z3+(-4e)*z2+(-e)
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```
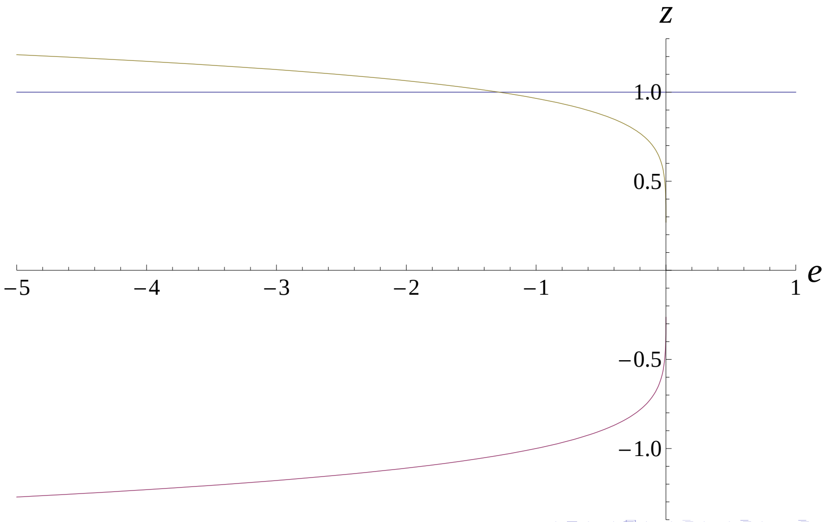

Real Solutions to `G[1]=0`



Trouble in Paradise

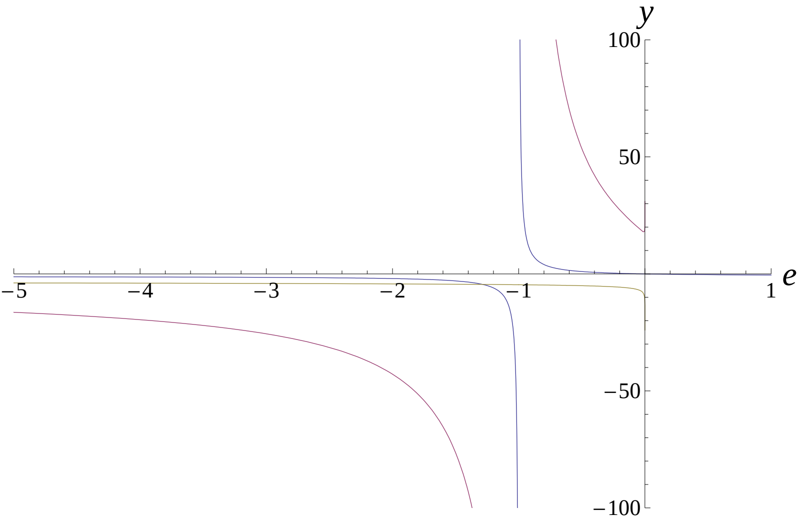
```
G[1]=2*z11+3*z9-5*z8+(5e)*z3+(-4e)*z2+(-e)

G[2]=(-e2-e)*y+(-8e-10)*z10+(-10e-15)*z8+(20e+25)*z7
     +(5e)*z6+(-5e)*z5+(5e)*z4+(-5e)*z3
     +(-20e2-20e)*z2+(16e2+15e)*z+(3e2+3e)

G[3]=(-e-1)*x+2*z9+5*z7-5*z6+5*z5-5*z4+5*z3-5*z2-1
```

For $e = 0$ and $e = -1$ the Gröbner basis does not have shape form

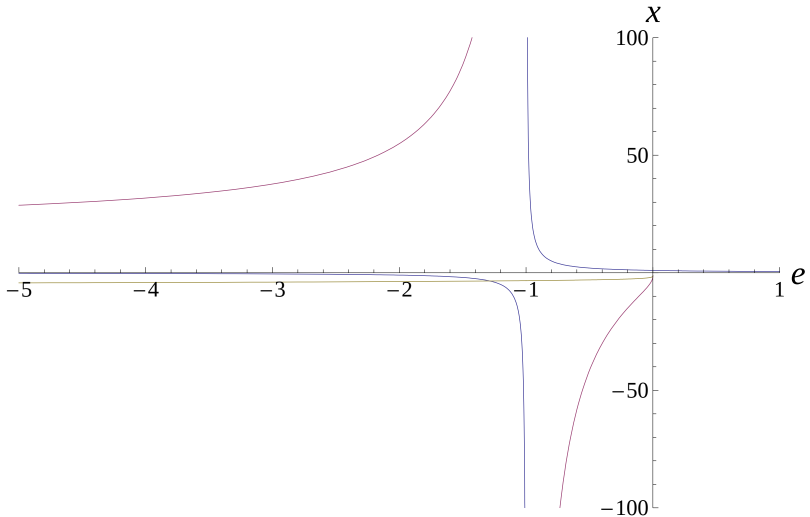
Must check solutions for all variables

Must solve original system for fixed value of e

Real Solutions to `G[2]=0`



Real Solutions to `G[3]=0`



Special Solutions

Gröbner basis for $e = 0$

```
G[1]=2z3+3z-5
G[2]=y
G[3]=x+3z-4
```

One real solution: $z = 1, y = 0, x = 1$

Gröbner basis for $e = -1$

```
G[1]=2z9+5z7-5z6+5z5-5z4+5z3-5z2-1
G[2]=33y+320z8+10z7+790z6-765z5+740z4-715z3+690z2-665z-94
G[3]=33x+10z8-10z7+35z6-60z5+85z4-110z3+135z2+5z+28
```

One real solution: $z = 0.965, y = -4.64, x = -3.37$

Detecting Multiplicity in Parameter Space

If along a path in parameter space the number of real solutions changes, then there must be a critical point

Search for critical points

$$\begin{aligned} r(e;x_n) &= 0 \\ \partial_{x_n} r(e;x_n) &= 0 \end{aligned}$$

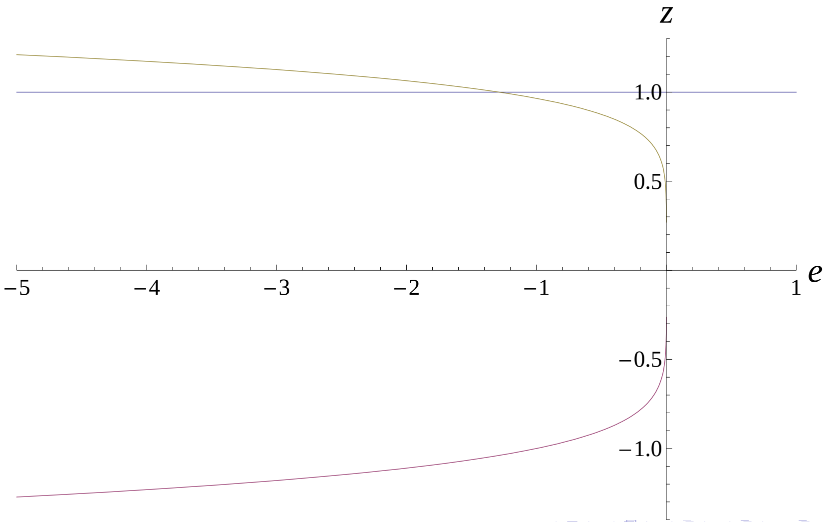
Easily possible for one parameter

Example

```
ring R=0,(e,z),lp;
ideal I=(
2*z**11+3*z**9-5*z**8+5*e*z**3-4*e*z**2-e,
11*2*z**10+9*3*z**8-8*5*z**7+3*5*e*z**2-2*4*e*z);
ideal G=groebner(I);
solve (G);
```

Two real solutions, $e = 0$ and $e = -\frac{9}{7} \approx -1.28571$

Real Solutions to `G[1]=0`



OLG Model

Double-ended infinity model

Discrete time, $t \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Representative agent born at t, lives for $N \geq 2$ periods

Endowment e_a depends solely on age $a = 1, \ldots, N$

$$U_t(c) = \sum_{a=1}^{N} u(c_a(t + a - 1))$$

Consumption $c_a(t + a - 1)$ of agent born at t in period $t + a - 1$

Equilibrium in OLG

$\left(p(t), (\bar{c}_a(t))_{a=1}^{N}\right)_{t\in\mathbb{Z}}$ such that

(1) $\sum_{a=1}^{N} (\bar{c}_a(t) - e_a) = 0$

(2) $(\bar{c}_1(t), \ldots, \bar{c}_N(t+N-1))$ solves

$$\max_{c(t),\ldots,c(t+N-1)} \quad U_t(c(t), \ldots, c(t+N-1))$$
$$\text{s. t.} \quad \sum_{a=1}^{N} p(t+a-1)\,(c(t+a-1) - e_a) = 0$$

Steady state

$$\frac{p_{t+1}}{p_t} = q > 0 \quad \text{and} \quad \bar{c}_a(t) = c_a$$

Equilibrium Equations

Utility $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ yields polynomial equilibrium equations

$$\begin{aligned} c_{a+1}^{\sigma} q - c_a^{\sigma} &= 0, \quad a = 1, \ldots, N-1 \\ \sum_{a=1}^{N} q^{a-1}(c_a - e_a) &= 0 \\ \sum_{a=1}^{N} (c_a - e_a) &= 0 \end{aligned}$$

Unique monetary steady state $q = 1$

Odd number of real steady states with $q \neq 1$

Equations for SINGULAR

Change of variable to reduce degree, $w = q^{1/\sigma}$

$$
\begin{aligned}
c_{a+1} w - c_a &= 0, \quad a = 1, \ldots, N-1, \\
\sum_{a=1}^{N} w^{\sigma(a-1)} (c_a - e_a) &= 0, \\
\sum_{a=1}^{N} c_a - e_a &= 0.
\end{aligned}
$$

SINGULAR

SINGULAR code for $N = 3$ and $\sigma = 2$

```
int n = 4;
ring R= (0,e,f,g,b),x(1..n),lp;
ideal I =(
-(f+x(2))*x(4)+(e+x(1)),
-(g+x(3))*x(4)+(f+x(2)),
x(1)+x(2)*x(4)**2+x(3)*x(4)**4,
x(1)+x(2)+x(3));
ideal G=groebner(I);
```

Uniqueness of Real Steady State

```
G[1]=(-g)*x(4)**4+(e+g)*x(4)**2+(-e)
```

Always two sign changes, even for larger N

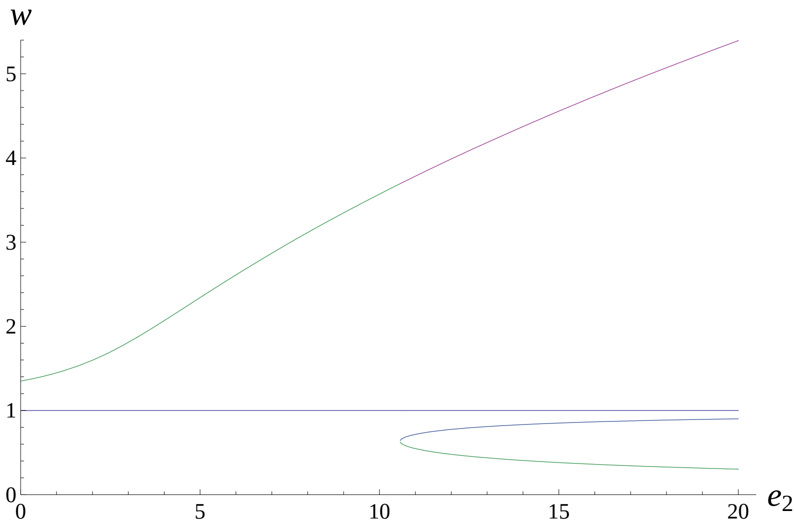
For $\sigma = 2$ unique real steady state for all N

Larger Risk-aversion

$\sigma = 3$ and $N = 3$

$$r(w) = e_3 w^6 - (e_1 + e_2 + e_3) w^4 + (e_1 + 2e_2 + e_3) w^3 - (e_1 + e_2 + e_3) w^2 + e_1$$

Four sign changes


w
5
4
3
2
1
0
0
5
10
15
20
e_2

Simple Arrow-Debreu Economy

Two types of agents, two commodities

Utility functions

$$u^1(c_1, c_2) = -\frac{64}{2} c_1^{-2} - \frac{1}{2} c_2^{-2}, \quad u^2(c_1, c_2) = -\frac{1}{2} c_1^{-2} - \frac{64}{2} c_2^{-2}$$

Parameterized individual endowments

$$e^1 = (1 - e, e), \quad e^2 = (e, 1 - e)$$

Implementation

SINGULAR code

```
int n = 3;
ring R= (0,e),(x(1),x(2),q),lp;
ideal I =(
-4*(e+x(2))*q+(1-e+x(1)),
-(1-e-x(2))*q+4*(e-x(1)),
x(1)+x(2)*q**3);
ideal G=groebner(I);
```

Gröbner Basis

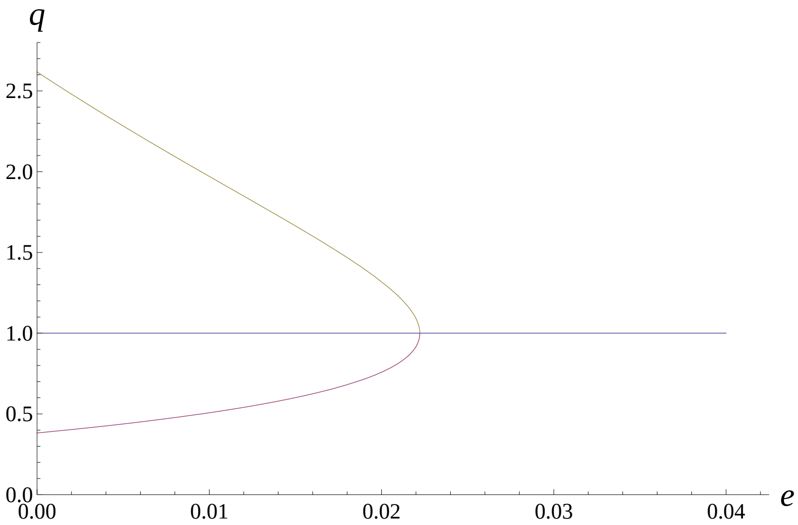
Shape Lemma holds

```
G[1]=(-15*e-1)*q**3+4*q**2-4*q+(15*e+1)
G[2]=(-225*e-15)*x(2)+(60*e+4)*q**2-16*q+
     (-225*e**2-30*e+15)
G[3]=15*x(1)+4*q+(-15*e-1)
```

Well-defined for $e > 0$

`G[1]` has always three solutions

$$1, \frac{3-15e-\sqrt{5}\sqrt{1-42e-135e^2}}{2(1+15e)}, \frac{3-15e+\sqrt{5}\sqrt{1-42e-135e^2}}{2(1+15e)}$$


q
2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.01
0.02
0.03
0.04
e

Outlook

Systems of polynomial equations are ubiquitous in economics

Methods from algebraic geometry are widely applicable

We have already computed multiple

- equilibria in GE models with complete or incomplete markets
- stationary equilibria (steady states) in OLG model
- equilibria in infinite-horizon models with complete markets
- Nash equilibria in strategic market games
- perfect Bayesian equilibria in a game with cheap talk