Finding All Equilibria in Static and Dynamic Games

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Motivation

Many economic models (may) have multiple equilibria

Equilibrium conditions are often nonlinear systems of equations

Algorithms for finding all solutions to polynomial systems exist

– Economics: GAMBIT ?

– Groebner Bases

– Resultants, Multiresultants

– Homotopy Methods

What, if anything, works for interesting economic models?

Solving Interesting Systems

Homotopy Methods: long history of applications in economics

Currently the only methods working for models of moderate size

Application of these methods requires work

Application in this paper:

Discrete-time stochastic games with a finite number of states

– Wide range of applications

– Active area of computational economics

– Examples of equilibrium multiplicity

– Polynomial equilibrium equations

Overview of this Talk

- Polynomial Systems of Equations

- Homotopy Method in Complex Space

- Structural Properties

- Static Game: Bertrand Price Competition

- Dynamic Game: Patent Race

- Dynamic Game: Learning Curve

Polynomial Systems of Equations

Complex polynomial system: $f(z) = 0$

$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, $f : \mathbb{C}^n \to \mathbb{C}^n$

Equation i:

$$f_i(z_1, z_2, \ldots, z_n) = \sum_{j=1}^{m_i} \left(a_{ij} \prod_{k=1}^{n} z_k^{d_{ijk}} \right) = 0,$$

$d_{ijk} \in \mathbb{N}_0$: degree of the kth variable in the jth term of equation i

Degree of f_i: $d_i = \max_{j=1,\ldots,m_i} \sum_{k=1}^{n} d_{ijk}$

Total degree of f: $d = \prod_{i=1}^{n} d_i$

Number of Solutions

Bezout's Theorem Polynomial system $f(z) = 0$ has at most d isolated solutions

"Generic" polynomial systems have exactly d distinct isolated solutions

(Garcia and Li (1980))

Example:

$$\begin{aligned} f_1(z_1, z_2) &= \quad z_1 z_2 - z_1 - z_2 + 1 &= 0 \\ f_2(z_1, z_2) &= (z_1)^2 z_2 - z_1 (z_2)^2 + 1 &= 0 \end{aligned}$$

$d_1 = 2$, $d_2 = 3$, so $d = 2 \times 3 = 6$

Homotopy Approach

Construct "easy" polynomial system $g(z) = 0$

$g_i(z) = c_i(z_i)^{d_i} - b_i$ for $b_i, c_i \in \mathbb{C} - \{0\}$

$g_i(z) = 0$ has d_i isolated solutions

$g(z) = 0$ has $d = \prod_{i=1}^{n} d_i$ isolated solutions

Homotopy function

$$H(z,t) = (1-t)g(z) + tf(z) \text{ with } t \in [0,1)$$

$H(z,0) = g(z) = 0$ has d known isolated solutions

$H(z,1) = f(z) = 0$ is system of interest

Theorem for Homotopy Approach

Morgan (1986):

For almost all parameters $b \in \mathbb{C}^n$ and $c \in \mathbb{C}^n$, the following properties hold.

1. The preimage $H^{-1}(0)$ consists of d smooth paths.
2. Each path either diverges to infinity or converges to a solution of $f(z) = 0$ as t approaches 1.
3. Each isolated solution of $f(z) = 0$ has a path converging to it.
4. Paths are monotonically increasing in t (Cauchy-Riemann equations).

Example revisited

$$
\begin{aligned}
f_1(z_1, z_2) &= \quad z_1 z_2 - z_1 - z_2 + 1 \quad = 0 \\
f_2(z_1, z_2) &= (z_1)^2 z_2 - z_1 (z_2)^2 + 1 = 0
\end{aligned}
$$

$d = 6$, so 6 paths must be tracked

Two real and two complex solutions

$$\left(1, \frac{1}{2}(1 \pm \sqrt{5})\right) \text{ and } \left(\frac{1}{2}(1 \pm i\sqrt{3}), 1\right)$$

Two paths diverge to infinity

Diverging Paths

Sequence of points (z^s, t^s), $s = 1, 2, \ldots,$ on a diverging path

$t^s \longrightarrow 1$ and so $\|z^s\| \longrightarrow \infty$

Sequence $z^s/\|z^s\|$ has a limit point $\bar{z} \neq 0$

$$0 = \frac{H_i(z^s, t^s)}{\|z^s\|^{d_i}} = \frac{(1-t)g_i(z^s) + t f_i(z^s)}{\|z^s\|^{d_i}} \longrightarrow f_i^0(\bar{z})$$

$f_i^0(z)$ is the homogeneous part of f_i, the terms of f_i with maximal degree d_i

$\bar{z}$ with $f_i^0(\bar{z}) = 0$ and $\bar{z}_i = 1$ for some i: “solution at infinity”

Example cont'd

$$
\begin{aligned}
f_1(z_1, z_2) &= z_1 z_2 - z_1 - z_2 + 1 &= 0 \\
f_2(z_1, z_2) &= (z_1)^2 z_2 - z_1 (z_2)^2 + 1 &= 0
\end{aligned}
$$

Homogeneous part

$$
\begin{aligned}
f_1(z_1, z_2) &= z_1 z_2 &= 0 \\
f_2(z_1, z_2) &= (z_1)^2 z_2 - z_1 (z_2)^2 &= 0
\end{aligned}
$$

Two solutions at infinity: (1,0) and (0,1)

Real solutions, complex solutions, solutions at infinity

Two Difficulties

Homotopy approach is very intuitive, but has significant drawbacks

1. Number of finite solutions is usually much smaller than Bezout number d
   - Bezout number grows exponentially in the number of nonlinear equations
   - Most paths diverge

2. Paths diverging to infinity are a nuisance
   - Of no economic interest
   - Large computational effort
   - Require decision to truncate
   - Risk of truncating very long but converging path

Homogenization

Homogenization $\hat{f}_i(z_0, z_1, \ldots, z_n)$ of the polynomial $f_i(z_1, \ldots, z_n)$ of degree d_i

$$\hat{f}_i(z_0, z_1, \ldots, z_n) = z_0^{d_i} f_i(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}).$$

Transformed system $\hat{f}(\hat{z}) = 0$ where $\hat{z} = (z_0, z_1, \ldots, z_n)$

n equations in $n+1$ unknowns

Example cont'd: homogenized system

$$\begin{array}{rclcl} \hat{f}_1(z_0, z_1, z_2) & = & z_1 z_2 - z_0 z_1 - z_0 z_2 + (z_0)^2 & = & 0 \\ \hat{f}_2(z_0, z_1, z_2) & = & (z_1)^2 z_2 - z_1 (z_2)^2 + (z_0)^3 & = & 0 \end{array}$$

Solutions of Homogenized System

If $\hat{f}(\hat{z}) = 0$ then $\hat{f}(c\hat{z}) = 0$ for all $c \in \mathbb{C}$

Solutions are complex lines through the origin in $\mathbb{C}^{n+1}$

Relationship between solutions of

$$f(z) = 0, \; z \in \mathbb{C}^n \quad \text{and} \quad \hat{f}(\hat{z}) = 0, \hat{z} \in \mathbb{C}^{n+1}, \; z_0 \neq 0$$

If $f(z) = 0$ then for $\hat{z} = (1, z)$, $\hat{f}(\hat{z}) = 0$

If $\hat{f}(\hat{z}) = 0$ for some $\hat{z} = (z_0, z)$ then $f(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}) = 0$

Solutions at Infinity

If $z_0 = 0$, so that $\hat{z} = (0, z)$, then

$\hat{f}_i(\hat{z}) = f_i^0(z_1, \ldots, z_n)$, the homogeneous part of f_i (only terms of degree d_i)

Solutions $[\hat{z}]$ with $z_0 = 0$ of $\hat{f}(\hat{z}) = 0$ are the solutions at infinity !

Example cont'd: homogenized system

$$\begin{aligned} \hat{f}_1(z_0, z_1, z_2) &= z_1 z_2 - z_0 z_1 - z_0 z_2 + (z_0)^2 &= 0 \\ \hat{f}_2(z_0, z_1, z_2) &= (z_1)^2 z_2 - z_1 (z_2)^2 + (z_0)^3 &= 0 \end{aligned}$$

$$\begin{aligned} f_1^0(z_1, z_2) &= z_1 z_2 &= 0 \\ f_2^0(z_1, z_2) &= (z_1)^2 z_2 - z_1 (z_2)^2 &= 0 \end{aligned}$$

Projective Transformation

Define a new linear function, coefficients $\xi_i \in \mathbb{C} \setminus \{0\}$

$$u(z_0, z_1, \ldots, z_n) = \xi_0 z_0 + \xi_1 z_1 + \ldots + \xi_n z_n$$

Projective transformation $F(z_0, z_1, \ldots, z_n)$ of the polynomial $f(z_1, \ldots, z_n)$ is

$$\begin{aligned} F_0(z_0, z_1, \ldots, z_n) &= u(z_0, z_1, \ldots, z_n) - 1 \\ F_i(z_0, z_1, \ldots, z_n) &= \hat{f}_i(z_0, z_1, \ldots, z_n), \qquad i = 1, \ldots, n \end{aligned}$$

$F(\hat{z}) = 0$, system of $n + 1$ equations in $n + 1$ variables

Same degree d as the polynomial $f(z)$

No Diverging Paths

Theorem [Morgan, 1986]

If the system $\hat{f}(\hat{z}) = 0$ has only a finite number of solutions in CP^n, then for almost all $\xi \in \mathbb{C}^n$ the system $F(\hat{z}) = 0$ has exactly d solutions (counting multiplicities) in $\mathbb{C}^{n+1}$ and so no solutions at infinity

Now use original homotopy on new system F

Reduction in the Number of Paths

Idea: exploit special structure of polynomial system to eliminate extraneous paths

m-homogeneity: generalization of homogenization of $f(z)$

Partition the set of variables $z_1, \ldots, z_n$ into m subsets

Homogenize $f(z)$ with respect to the variables in each subset

m-homogeneous Bezout Theorem: number of isolated solutions is at most $B < d$

Start system with same m-homogeneous structure: fewer paths

Summary

Homotopy methods for finding all solutions of systems of polynomial equations

Smooth paths, parameter t increases along each path

Isolated solutions, diverging paths

Projective transformation: compactification of paths

Can reduce number of paths by initial analysis: m-homogeneity

Lots of other improvements possible, active field of research

Polynomial Systems of Equations

Algorithms for finding all solutions

Drexler (1977), Garcia and Zangwill (1979), Morgan (1986),

Morgan and Sommese (1987), Verschelde and Cools (1993), Morgan et al. (1995),

Sturmfels (2002)

Publicly available software (among others)

POLSYS_PLP (Wise et al. (2000)) based on HOMPACK90 (Watson et al. (1997))

PHCpack (Verschelde (1997)) written in Ada

Feasible to solve problems of moderate size

Bertrand Price Competition

Two firms x and y producing goods x and y, resp., prices $p_x,\ p_y$

Three types of customers with demand functions:

$$Dx1 = A - p_x, \quad Dy1 = 0; \quad Dx3 = 0, \quad Dy3 = A - p_y$$

$$Dx2 = np_x^{-\sigma}\left(p_x^{1-\sigma} + p_y^{1-\sigma}\right)^{\frac{\gamma-\sigma}{-1+\sigma}}, \quad Dy2 = np_y^{-\sigma}\left(p_x^{1-\sigma} + p_y^{1-\sigma}\right)^{\frac{\gamma-\sigma}{-1+\sigma}}$$

Total Demand $Dx = Dx1 + Dx2 + Dx3$

Unit cost m, thus profit $R_x = (p_x - m)$

Necessary optimality condition $MR_x = MR_y = 0$

First-Order Conditions

$\sigma = 3;\ \gamma = 2;\ n = 2700;\ m = 1;\ A = 50$

First-order conditions for the two firms

$$MR_x \;=\; 50 - p_x + (p_x - 1)\left(-1 + \frac{2700}{p_x^6\left(p_x^{-2} + p_y^{-2}\right)^{3/2}} - \frac{8100}{p_x^4\sqrt{p_x^{-2} + p_y^{-2}}}\right) + \frac{2700}{p_x^3\sqrt{p_x^{-2} + \cdots}}$$

$$MR_y \;=\; 50 - p_y + (p_y - 1)\left(-1 + \frac{2700}{p_y^6\left(p_x^{-2} + p_y^{-2}\right)^{3/2}} - \frac{8100}{p_y^4\sqrt{p_x^{-2} + p_y^{-2}}}\right) + \frac{2700}{p_y^3\sqrt{p_x^{-2} + \cdots}}$$

Polynomial equations ?

Polynomial System

Define $Z = \sqrt{p_x^{-2} + p_y^{-2}}$, which yields a polynomial equation

$$0 = -p_x^2 - p_y^2 + Z^2 p_x^2 p_y^2$$

Substitute Z into denominator of MR_x and MR_y

$$0 = -2700 + 2700p_x + 8100Z^2p_x^2 - 5400Z^2p_x^3 + 51Z^3p_x^6 - 2Z^3p_x^7$$

$$0 = -2700 + 2700p_y + 8100Z^2p_y^2 - 5400Z^2p_y^3 + 51Z^3p_y^6 - 2Z^3p_y^7$$

Bezout number $d = 6 \cdot 10 \cdot 10 = 600$

Solutions

POLSYS_PLP: 3-homogeneous Bezout number $B = 182 < d = 600$

Total of 18 real solutions, 9 with negative values, 9 positive real solutions

Running time less than 12 sec

p_x	p_y
1.75653	1.75653
8.07580	8.07580
22.98653	22.98653
2.03619	5.63058
5.63058	2.03619
2.16820	25.15680
25.15680	2.16820
7.69768	24.25903
24.25903	7.69768

Global Optimality

Second-order conditions eliminate 5 positive real solutions

p_x	p_y
1.75653	1.75653
22.98653	22.98653
2.16820	25.15680
25.15680	2.16820

Global optimality: Is $p_x = 1.75653$ globally optimal given $p_y = 1.75653$?

Another system of polynomial equations

$$0 = 0.32410568484991703 p_x^2 + 1 - Z^2 p_x^2$$

$$0 = -2700 + 2700 p_x + 8100 Z^2 p_x^2 - 5400 Z^2 p_x^3 + 51 Z^3 p_x^6 - 2 Z^3 p_x^7$$

Equilibria

POLSYS_PLP: 2-homogeneous Bezout number $B = 20 < d = 40$

Total of 14 finite solutions, 8 complex, 6 real solutions ($<$ 1.5 sec)

Solution $p_x = 25.2234$ leads to higher profit than $p_x = 1.75653$

Thus, $(p_x, p_y) = (1.75653, 1.75653)$ not an equilibrium

Two asymmetric equilibria

p_x	p_y
2.16820	25.15680
25.15680	2.16820

Patent Race between Two Firms

N innovation stages

Firms start race at stage 0

Period t innovation stages: $(x_{1,t}, x_{2,t})$ where $x_{i,t} \in X \equiv \{0, ..., N\}\,, i = 1, 2$

Period t investment: $a_{i,t} \in A = [0, \bar{A}] \subset \mathbb{R}_+$, $i = 1, 2$

Cost of investment: $C_i(a) = c_i a^\eta,\ \eta \in \mathbb{N},\ c_i > 0,\ i = 1, 2$

Independent and stochastic innovation technologies

Transition from period to period: $x_{i,t+1} = x_{i,t}$ or $x_{i,t+1} = x_{i,t} + 1$

Markov process (depends on investment levels)

Transition from State to State

Firm i's state evolves according to

$$x_{i,t+1} = \begin{cases} x_{i,t}, & \text{with probability } \; p(x_{i,t}|a_{i,t}, x_{i,t}) \\ x_{i,t}+1, & \text{with probability } \; p(x_{i,t}+1|a_{i,t}, x_{i,t}) \end{cases}$$

Distribution over next period's states (polynomial specification!)

$$\begin{aligned} p(x|a,x) &= F(x|x)a \\ p(x+1|a,x) &= 1 - F(x|x)a \end{aligned}$$

$F(x|x) \in (0,1)$ is probability that there is no change in state if $a = 1$

Firms' Optimization Problem

First firm to reach state N wins the race and receives prize Ω

Ties are broken by flip of a coin

Firms discount future costs and revenues at common rate $\beta < 1$

Firms' objective: maximize expected discounted payoffs

Equilibrium I

Restriction to pure Markov strategies

Firm i's strategy: $\sigma_i(\cdot) : X \times X \to A$

Expected discounted payoff: $V_i(\cdot) : X \times X \to \mathbb{R}$

Bellmann equation for $x_i, x_{-i} < N$,

$$V_i(x_i, x_{-i}) = \max_{a_i} \left\{ -C_i(a_i) + \beta \sum_{x'_i, x'_{-i}} p(x'_i | a_i, x_i) p(x'_{-i} | a_{-i}, x_{-i}) V_i(x'_i, x'_{-i}) \right\}$$

Equilibrium II

Boundary condition at terminal states

$$V_i(x_i, x_{-i}) = \begin{cases} \Omega, & \text{for } x_{-i} < x_i = N \\ \Omega/2, & \text{for } x_i = x_{-i} = N \\ 0, & \text{for } x_i < x_{-i} = N \end{cases}$$

Optimal strategies satisfy

$$\sigma_i(x_i, x_{-i}) = \arg\max_{a_i \in A} \left\{ -C_i(a_i) + \beta \sum_{x_i', x_{-i}'} p(x_i' | a_i, x_i) p(x_{-i}' | a_{-i}, x_{-i}) V_i(x_i', x_{-i}') \right\}$$

Our Equilibrium Equations

$$
\begin{aligned}
0 &= -V_i(x_i, x_{-i}) - c_i a_i^{\eta} + \beta \sum_{x'_i, x'_{-i}} p(x'_i | a_i, x_i) p(x'_{-i} | a_{-i}, x_{-i}) V_i(x'_i, x'_{-i}) \\
0 &= -\eta c_i a_i^{\eta-1} + \beta \sum_{x'_i, x'_{-i}} \frac{\partial}{\partial a_i} p(x'_i | a_i, x_i) p(x'_{-i} | a_{-i}, x_{-i}) V_i(x'_i, x'_{-i})
\end{aligned}
$$

Parameter specification: $\eta = 2$, $F(x_1, x_2) \equiv F$

Unknowns: $V_1(x_1, x_2)$, $V_2(x_1, x_2)$, $a_1(x_1, x_2)$, $a_2(x_1, x_2)$

Four equations per stage (x_i, x_{-i})

Upwind Gauss-Seidel: instead of solving all equations simultaneously

solve each stage game separately

Polynomial Equilibrium Equations

Equations for firm 1:

$$
\begin{aligned}
0 &= a_1^2(-c_1) + a_1 V_1 a_2(\beta F^2) + a_1 V_1(-\beta F) \\
&\quad + a_1 a_2 \left(\beta F^2 (V_{(11)} - V_{(01)} - V_{(10)})\right) + a_1(\beta F V_{(10)}) \\
&\quad + V_1 a_2(-\beta F) + V_1(\beta - 1) + a_2(\beta F V_{(01)}) \\
0 &= a_1(-2c_1) + V_1 a_2(\beta F^2) + V_1(-\beta F) \\
&\quad + a_2 \left(\beta F^2 (V_{(11)} - V_{(01)} - V_{(10)})\right) + (\beta F V_{(10)})
\end{aligned}
$$

Total degree $3 \times 2 \times 3 \times 2 = 36$

Linearity in V_1, V_2 allows reduction in number of equations and Bezout number

Solutions

$N = 3$ stages, $0, 1, 2,$ prize $\Omega = 10$

$F = \frac{1}{4},$ cost coefficients $c_1 = c_2 = 1,$ discount factor $\beta = 0.96$

Real and complex finite solutions

$(0,0)$, $(1,0)$, $(1,1)$	3 real, 4 complex
$(2,1)$	2 real, 4 complex
$(2,0)$	6 real, 0 complex
$(2,2)$	3 real, 4 complex

36 paths followed in less than 3 seconds

Only one economically meaningful solution: unique equilibrium

Other real solutions lead to negative transition probabilities

Unique Equilibrium

	a_1	V_1	a_2	V_2
$(2,2)$	1.373	2.697	1.373	2.697
$(2,1)$	0.939	6.725	0.317	0.205
$(2,0)$	0.567	7.653	0.035	0.004
$(1,1)$	0.904	1.911	0.904	1.911
$(1,0)$	0.755	4.776	0.275	0.192
$(0,0)$	0.673	1.419	0.673	1.419

Learning Curve Game

Game with static and dynamic component

Firms play Cournot game in each period

Learning: output may lead to lower unit cost of production

Two goods and two firms, cost function $c^i(q_i, x_i) = x_i q_i$, $i = 1, 2$

State is unit cost $x_i \in X = \{\xi_1, \xi_2, \ldots, \xi_N\}$, absorbing state ξ_N

Transition probabilities depend on output (polynomial specification)

$$\begin{aligned} \Pr[x_{i,t+1} = \xi_{j+1} | x_{i,t} = \xi_j] &= F(x_{i,t}) q_i \\ \Pr[x_{i,t+1} = \xi_j | x_{i,t} = \xi_j] &= 1 - F(x_{i,t}) q_i \end{aligned}$$

Absorbing state ξ_N: $F(\xi_N) = 0$

Parametrization

Profit function of firm i, $\quad \Pi_i(q_1, q_2, x_i) = P_i(q_1, q_2)\, q_i - x_i\, q_i$

Price function $P_i(q_1, q_2) = \frac{\partial}{\partial q_i} u(q_1, q_2)$

$u(q_1, q_2) = 4\left(q_1^{1/2} + q_2^{1/2}\right)^{4/3} + M$

Parameter values: unit cost $x_i \in \{\frac{3}{2}, 1, \frac{1}{2}\}$, $F(x_i) = 0.001$ for $x_i > \frac{1}{2}$

Polynomial Equations

For eliminating rational exponents and clearing of denominators

$$0 = Q_1^2 - q_1, \quad 0 = Q_2^2 - q_2, \quad 0 = Q^3 - Q_1 - Q_2$$

Bellman equation for firm 1

$$\begin{aligned} 0 &= 4QQ_1 - (1-\beta)V_1 + (F_1\beta {W^1}_{1,0} - x_1)q_1 - \beta F_1 q_1 V_1 - \beta F_1 q_2 V_1 \\ &\quad +\beta F_1 F_2 q_1 q_2 V_1 + \beta F_1 {W^1}_{0,1} q_2 + (F_1 F_2 \beta {W^1}_{1,1} - \beta F_1 F_2 {W^1}_{0,1} - F_1 F_2 \beta {W^1}_{1,0}) q_1 q_2 \end{aligned}$$

First-order condition for firm 1

$$\begin{aligned} 0 &= 8Q_1 + 6Q_2 + (3\beta F_1 {W^1}_{1,0} - 3x_1)Q^2 Q_1 - 3\beta F_1 Q^2 Q_1 V_1 \\ &\quad +(3F_1 F_2 \beta {W^1}_{1,1} - 3F_1 F_2 \beta {W^1}_{0,1} - 3F_1 F_2 \beta {W^1}_{1,0}) Q^2 Q_1 q_2 + 3\beta F_1 F_2 Q^2 Q_1 q_2 V_1 \end{aligned}$$

Bezout number for $F_1 F_2 \neq 0$: $2 \cdot 2 \cdot 3 \cdot (3 \cdot 5)^2 = 2700$

Equilibrium

m_1	m_2	Bezout	7-Bezout	Real	q_1	q_2	V_1	V_2
$\frac{1}{2}$	$\frac{1}{2}$	432	14	5	203	203	1452	1452
$\frac{1}{2}$	1	576	31	7	190	113	1404	1056
$\frac{1}{2}$	$\frac{3}{2}$	576	31	7	172	33	1201	550
1	1	2700	177	11	103	103	1011	1011
1	$\frac{3}{2}$	2700	177	10	86	28	820	516
$\frac{3}{2}$	$\frac{3}{2}$	2700	177	11	21	21	372	372

Running time for 177 paths less than 95 sec

Summary

All-solution homotopy methods for polynomial systems

Real and complex solutions, solutions at infinity

Theoretical bounds on the number of solutions

Accounting for all finite and infinite solutions is possible

Find all solutions to equilibrium equations in economics

Computational approach to proving uniqueness

Difficulties

POLSYS_PLP currently cumbersome to use

Interface needed for solving many similar systems

Convergence problems due to manifolds at infinity

Other software packages have high set-up cost

Extensions

Complementarity conditions

Generic systems and Cheater's homotopy

Parallelization