Solving Equations

Suppose

$f : R^n \to R^n$

Consider

$f(x) = 0$

Typical algorithms

Newtons' method:

- fast, but
- can diverge
- may try to evaluate f(x) where it does not exist

Homotopy method:

- reliable, but
- slow

Alternative: Reformulate as an optimization problem

$$\text{Min}_x \; 1$$

$$\text{s.t.} \qquad f(x) = 0$$

Advantages

- **Variety of solvers**

  Can use KNITRO, CONOPT, Filter, SNOPT, NPSOL, etc

- **Can impose constraints**
- **Impose domain conditions**

Suppose f(x) is not defined for x<= 0. Then solve

$$\text{Min}_x \quad 1$$

$$\text{s.t.} \quad f(x) = 0$$
$$x \geq \epsilon$$

for some small $\epsilon > 0$. Can't use x=>0 because then solver may consider some $x_i = 0$.

Use auxiliary information about solution

Suppose you know that the solution to f(x)=0 satisfies a < x < b (a and b are vectors). We can use that information in

$$\text{Min}_x \quad 1$$

$$\text{s.t.} \quad f(x) = 0$$
$$a \leq x \leq b$$

More generally, if we know that $a \leq g(x) \leq b$, for some a, b, and g, then solve

$$\text{Min}_x \quad 1$$

$$\text{s.t.} \quad f(x) = 0$$

$$a \leq g(x) \leq b$$

- **Stabilize algorithm with L_2 penalty**

  Suppose that the Jacobian of f is nearly singular near the solution. Then, the following quadratic penalty formulation stabilizes the algorithm:

  $\text{Min}_x \quad \|x\|_2^2$

  s.t. f(x) =0

  where P is some penalty parameter, preferably small.

- **Stabilize algorithm with L_1 penalty**

  An L_1 penalty might also help

  $\text{Min}_x \quad \|x\|_1$

  s.t. f(x) =0

  where P is some penalty parameter, preferably small. (More later about how to do this.)

- **Stabilize via relaxation**

We can instead try to find something that nearly solves the equations:

Min$_x$ $\|\lambda\|_1$

s.t. $-\lambda_i \leq f(x) \leq \lambda_i$

$\lambda_i \geq 0$

This will give you something instead of just saying “Can’t find a solution”

- **Find multiple solutions**

If there are many solutions, one could resolve the following problem several times for different π parameters:

Min$_x$ π . x

s.t. f(x) = 0

This will go after solutions on the boundary of the set of solutions.

The following will go after the solution closest to some chosen x_0:

Min$_x$ $\|x_0 - x\|_2{}^2$

s.t. f(x) = 0