Perturbation Methods for Barro Debt and Tax Model

Setup

States: 𝔇 (debt)

𝔇plus: next period's debt

Gov spending: $\mathbb{G} + \delta \epsilon z$

Innovation: z: mean zero, unit variance, iid rv

ϵ: standard deviation of spending shocks

Choice: tax

Preferences: ℂ[tax]: cost function of raising tax dollars in revenue

β: discount factor

Tax policy : Tx[k, ϵ]

Parameter: δ: perturbation parameter which begins at zero and ends at one.

NOTE: The taxation policy function is expressed as Tx[𝔇, ϵ]. Remember the following distinctions: 𝔇 is the state variable, ϵ is a parameter for the spending shocks

Tastes, Technology, and the Bellman Equation

In[1071]:=
```
xxqq = 0; Remove["Global`*"]
```

In[1072]:=
```
debt = 𝔇; gee = 𝔾;
```

We define the Euler equation by building it up from key functions and expressions

Law of motion for debt

In[1073]:=
```
𝔇plus = R (𝔇 + δ d) + 𝔾 + δ ϵ z - tax;
```

Social cost of tax revenues is ℂ[tax]

Express Bellman equation in the form bell=0, and where V is the current value function and Vn is next period's value function.

In[1074]:=
```
bell = -V[𝔇 + δ d, δ ϵ] - ℂ[tax] + β V[𝔇plus, δ ϵ]
```

Out[1074]= $-V[\mathfrak{D} + d\,\delta, \delta\,\epsilon] + \beta\,V[-tax + \mathbb{G} + R\,(\mathfrak{D} + d\,\delta) + z\,\delta\,\epsilon, \delta\,\epsilon] - \mathbb{C}[tax]$

Assume the following to get the Barro result of a constant tax with constant expenditures

In[1075]:=
```
β = 1 / R;
```

The dynamic programming system has two equations -- foc and env -- for the two unknown functions - V and ℂ.

Compute first-order condition

In[1076]:=
```
foc = D[bell, tax]
```

Out[1076]= $-\mathbb{C}'[tax] - \frac{V^{(1,0)}[-tax + \mathbb{G} + R\,(\mathfrak{D} + d\,\delta) + z\,\delta\,\epsilon, \delta\,\epsilon]}{R}$

Compute envelope equation

In[1077]:=
```
env = D[bell, 𝔇]
```

Out[1077]= $-V^{(1,0)}[\mathfrak{D} + d\,\delta, \delta\,\epsilon] + V^{(1,0)}[-tax + \mathbb{G} + R\,(\mathfrak{D} + d\,\delta) + z\,\delta\,\epsilon, \delta\,\epsilon]$

In[1078]:=
```
dpsys = {foc, env};
```

When δ=0, we have the purely deterministic problem

In[1079]:=
```
dpsys0 = dpsys /. δ -> 0;
% // TableForm
```

Out[1080]//TableForm=

$$-\mathbb{C}'[\text{tax}] - \frac{V^{(1,0)}[-\text{tax}+R\,\mathbb{D}+\mathbb{G},0]}{R}$$

$$-V^{(1,0)}[\mathbb{D},0] + V^{(1,0)}[-\text{tax}+R\,\mathbb{D}+\mathbb{G},0]$$

We now set tax equal to its policy function Tx:

In[1081]:=
```
foceq = foc /. tax → Tx[𝔻 + δ d, δ ∈ ]
enveq = env /. tax → Tx[𝔻 + δ d, δ ∈ ]
belleq = bell //. tax → Tx[𝔻 + δ d, δ ∈ ]
```

In[∘]:= Define dpsys to be the functional equation system that defines V and Tx :

In[1084]:=
```
dpsys = {foceq, enveq};
dpsys // TableForm
```

Out[1085]//TableForm=

$$-\mathbb{C}'[\text{Tx}[\mathbb{D}+d\,\delta,\delta\in]] - \frac{V^{(1,0)}[\mathbb{G}+R\,(\mathbb{D}+d\,\delta)+z\,\delta\in-\text{Tx}[\mathbb{D}+d\,\delta,\delta\in],\delta\in]}{R}$$

$$-V^{(1,0)}[\mathbb{D}+d\,\delta,\delta\in] + V^{(1,0)}[\mathbb{G}+R\,(\mathbb{D}+d\,\delta)+z\,\delta\in-\text{Tx}[\mathbb{D}+d\,\delta,\delta\in],\delta\in]$$

We know that the tax policy is to set taxes equal to 𝔾 plus the interest on debt when there is no uncertainty. The following substitutions define the $\delta = 0$ case,

In[1092]:=
```
ss = {θ → 0, δ → 0, Tx[𝔇, 0] -> (R - 1) 𝔇 + 𝔾};
```

In[1093]:=
```
eqs = dpsys //. ss // Simplify
```

Out[1093]= $\left\{-\frac{R\,\mathbb{C}'[(-1+R)\,\mathbb{D}+\mathbb{G}]+V^{(1,0)}[\mathbb{D},0]}{R},\ 0\right\}$

eqs = 0 at the steady state, which allows us to solve for $V^{(1,0)}[\mathbb{D}, 0]$

In[1094]:=
```
sol = Solve[eqs == 0, V^(1,0)[𝔇, 0]]
```

Out[1094]= $\{\{V^{(1,0)}[\mathbb{D}, 0] \to -R\,\mathbb{C}'[(-1+R)\,\mathbb{D}+\mathbb{G}]\}\}$

Let rDG = (R - 1) $\mathbb{D}$ + $\mathbb{G}$ which is interest cost plus expenditure in state $\mathbb{D}$ in the deterministic case

In[738]:= **taxlaw = Collect[%, ϵ]**

Out[738]= $\text{Tx}[\mathbb{D}, 0] + \frac{r\,\epsilon^2\,\mathbb{C}^{(3)}[\text{rDG}]}{2\,\mathbb{C}''[\text{rDG}]} + \frac{r^2\,\epsilon^3\,\lambda\,\mathbb{C}^{(4)}[\text{rDG}]}{6\,\mathbb{C}''[\text{rDG}]}$

In[743]:= **debtlaw = Collect[%, ϵ]**

Out[743]= $\mathbb{D} + z\,\epsilon - \frac{r\,\epsilon^2\,\sigma^2\,\mathbb{C}^{(3)}[\text{rDG}]}{2\,\mathbb{C}''[\text{rDG}]} - \frac{r^2\,\epsilon^3\,\lambda\,\mathbb{C}^{(4)}[\text{rDG}]}{6\,\mathbb{C}''[\text{rDG}]}$

Presumptions

$\mathbb{C}''[\text{rDG}] < 0$

$\sigma^2 = 1$

$\mathbb{C}^{(3)}[\text{rDG}]$ and $\mathbb{C}^{(4)}[\text{rDG}]$ likely < 0 for many values

Examples

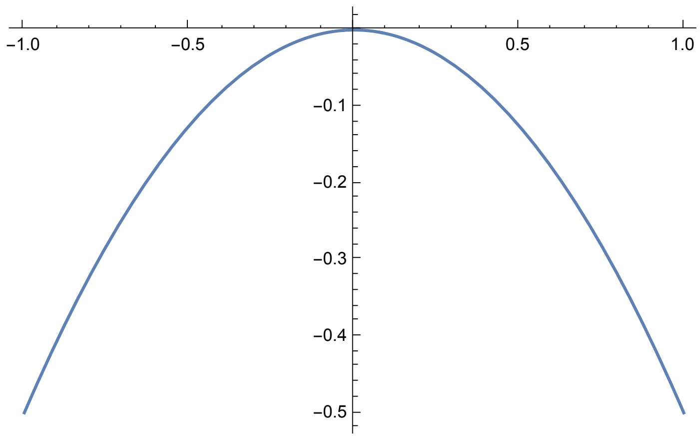
quadratic

In[1101]:=
```
v[x_] = -x^2 / 2
```

Out[1101]= $-\frac{x^2}{2}$

In[1102]:=
```
Plot[v[x], {x, -1, 1}]
```

Out[1102]=



In[746]:=
```
taxlaw /. ℂ -> v
```

Out[746]= Tx[𝔇, 0]

In[747]:=
```
debtlaw /. ℂ -> v
```

Out[747]= $\mathbb{D} + z\,\epsilon$

In[748]:=
```
% - 𝔇
```

Out[748]= $z\,\epsilon$

Quadratic case

i) implies Barro’s claim

ii) can NEVER be the true cost of taxation because it implies there is no limit to possible tax revenue!!!

example : $-x^2 / (1 - x)$

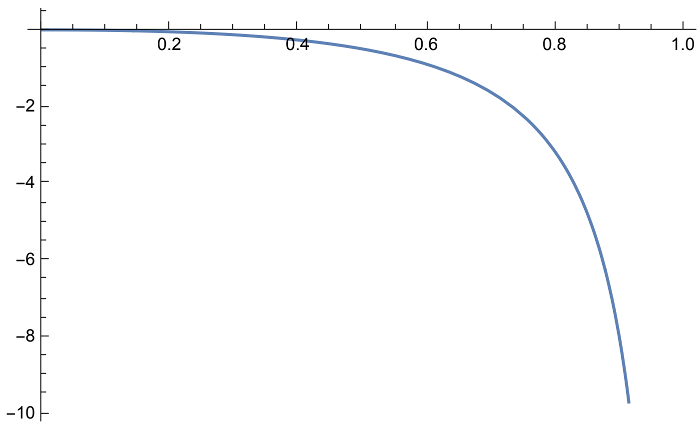
This is more sensible because the max revenue is x = 1.

In[1099]:=
```
v[x_] = -x^2 / (1 - x)
```

Out[1099]= $-\frac{x^2}{1-x}$

In[1100]:=
```
Plot[v[x], {x, 0, 1}]
```

Out[1100]=




```
debtlaw /. ℂ -> v // Simplify;

% /. z -> 0 /. ϵ -> 1
```

Out[794]= $\mathbb{D} + \frac{r\left(-4 r \lambda + 3 (-1 + rDG) \sigma^2\right)}{2 (-1 + rDG)^2}$

rDG = must be less than 1. Therefore, drift in debt must be negative unless λ is strongly negative.

example exponential

In[795]:=
```
v[x_] = -Exp[A x] + 1 + x
```

Out[795]= $1 - e^{A x} + x$

In[796]:=
```
taxlaw /. ℂ -> v /. ϵ -> 1
```

Out[796]= $\frac{A r}{2} + \frac{1}{6} A^2 r^2 \lambda + \text{Tx}[\mathcal{D}, 0]$

In[797]:=
```
debtlaw /. ℂ -> v /. ϵ -> 1
```

Out[797]= $z + \mathcal{D} - \frac{1}{6} A^2 r^2 \lambda - \frac{1}{2} A r \sigma^2$

Again, negative drift unless λ is strongly negative

Conclusions

Barro claim - “tax smoothing”

Debt and taxes should follow random walks.

Judd result using perturbation

Long-run taxes should be small, with government asset income covering much of government expenditures.