# Approximation Methods

## Standard Least-squares Method

Given data $(x_i, y_i)$, and regressor functions $\phi_j(x)$, the least squares fit solves

$$\text{Min}_{\beta} \quad \sum_{i=1}^{n} \left( y_i - \sum_{j=1}^{m} \beta_j \, \phi_j(x_i) \right)^2$$

Problem: multicollinearity (a.k.a. ill-conditioning)

# L1 approximation

Given data $(x_i, y_i)$, and regressor functions $\phi_j(x)$, the $L_1$ fit solves

$$\text{Min}_{\beta} \quad \sum_{i=1}^{n} \left| y_i - \sum_{j=1}^{m} \beta_j \, \phi_j(x_i) \right|$$

## Computation

The optimization problem looks bad since the objective function is not differentiable, particularly if you end up with zero error at some data point. This is no problem since we can reformulate it as

$$\text{Min}_{\beta,\lambda,\mu} \qquad \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i$$

$$\text{s.t.} \qquad -\lambda_i \le y_i - \sum_{j=1}^{m} \beta_j \, \phi_j(x_i) \le \mu_i$$

$$\lambda_i, \; \mu_i \ge 0$$

## ■ Statistical application

This also called LAD.

Not used as much as least-squares because of nice asymptotic theory for least-squares (or so says the gossip).

Statistical issues are of no concern for me. I just want to fit curves to points.

### ■ Advantages

No problem with singular Hessians because there are no Hessians.

Regression functions can be collinear.

I don’t care about multiple solutions for parameters; I just want a good curve.

# Shape-preserving approximation

Standard methods do not preserve shape. Concave data can produce nonconcave curves. This is not acceptable!

## Shape-preservation as a semi-infinite optimization problem

Suppose that we know that the data came from a monotone function. Our $L_2$ problem becomes

$$\text{Min}_{\beta} \quad \sum_{i=1}^{n} \left( y_i - \sum_{j=1}^{m} \beta_j \, \phi_j(x_i) \right)^2$$

$$\text{s.t.} \qquad \frac{d}{d\,x} \left( \sum_{j=1}^{m} \beta_j \, \phi_j(x) \right) \geq 0, \ \forall \ x$$

which is a problem with a finite number of unknowns but an infinite number of constraints.

## Implementing shape-preservation

Pick some points $z_k$, $k = 1, \ldots, K$. For sufficiently large K, we can replace the semiinfinite problem with

$$\mathrm{Min}_{\beta} \quad \sum_{i=1}^{n} \left( y_i - \sum_{j=1}^{m} \beta_j \, \phi_j(x_i) \right)^2$$

$$\text{s.t.} \qquad \frac{d}{dz} \left( \sum_{j=1}^{m} \beta_j \, \phi_j(z_k) \right) \geq 0, \; k = 1, \ldots, K$$

## ■ L1 shape-preservation: the best way

The constrained optimization problem may still have ill-conditioning problems. The problem is a quadratic optimization problem, not the easiest kind of problem. The best approach for this is to use an L1 criterion:

$$\mathrm{Min}_{\beta} \quad \sum_{i=1}^{n} \left| y_i - \sum_{j=1}^{m} \beta_j \, \phi_j(x_i) \right|$$

$$\text{s.t.} \qquad \frac{d}{dz}\left( \sum_{j=1}^{m} \beta_j \, \phi_j(z_k) \right) \geq 0, \;\; k = 1, \;\ldots, \; K$$

This is now a linear programming problem, *the easiest kind of problem around!!!*