Optimal Taxation without State-Contingent Debt

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In an economy studied by Lucas and Stokey, tax rates inherit the serial correlation structure of government expenditures, belying Barro's earlier result that taxes should be a random walk for any stochastic process of government expenditures. To recover a version of Barro's random walk tax-smoothing outcome, we modify Lucas and Stokey's economy to permit only risk-free debt. Having only risk-free debt confronts the

A few days before he died on May 20, 1997, Rao Aiyagari refereed an earlier version of this paper and suggested the example that now appears in Sec. III. With the editors' encouragement, we thank Rao by including him as a coauthor. We thank V. V. Chari, Darrell Duffie, Kenneth Judd, Ramon Marimon, Martin Schneider, Stijn Van Nieuwerburgh, Stephanie Schmitt-Grohé, Nancy Stokey, François Velde, an anonymous referee, and, especially, Fernando Alvarez, Marco Bassetto, and Lars Hansen for useful comments. Marcet's research has been supported by DGICYT and CIRIT, and Sargent's by a grant from the National Science Foundation to the National Bureau of Economic Research. We thank the Naval Surface Warfare Center for making their library of mathematics subroutines in Fortran available to us and the Center for Scientific Computing (Espoo, Finland) and Seppo Honkapohja for computer resources.

[*Journal of Political Economy,* 2002, vol. 110, no. 6]

Ramsey planner with additional constraints on equilibrium allocations beyond one imposed by Lucas and Stokey's assumption of complete markets. The Ramsey outcome blends features of Barro's model with Lucas and Stokey's. In our model, the contemporaneous effects of exogenous government expenditures on the government deficit and taxes resemble those in Lucas and Stokey's model, but incomplete markets put a near–unit root component into government debt and taxes, an outcome like Barro's. However, we show that without ad hoc limits on the government's asset holdings, outcomes can diverge in important ways from Barro's. Our results use and extend recent advances in the consumption-smoothing literature.

> It appears to have been the common practice of antiquity, to make provision, during peace, for the necessities of war, and to hoard up treasures before-hand, as the instruments either of conquest or defence; without trusting to extraordinary impositions, much less to borrowing, in times of disorder and confusion. [David Hume, "Of Public Credit," 1777]

I. Introduction

Barro (1979) embraced an analogy with a permanent income model of consumption to conjecture that debt and taxes should follow random walks, regardless of the serial correlations of government expenditures.[1] Lucas and Stokey (1983) broke Barro's intuition when they formulated a Ramsey problem for a model with complete markets, no capital, exogenous Markov government expenditures, and state-contingent taxes and government debt. They discovered that optimal tax rates and government debt are not random walks and that the serial correlations of optimal tax rates are tied closely to those for government expenditures. Lucas and Stokey found that taxes should be smooth, not by being random walks, but in having a smaller variance than a balanced budget would imply.

However, the consumption model that inspired Barro assumes a consumer who faces incomplete markets and adjusts holdings of a risk-free asset to smooth consumption across time and states. By assuming complete markets, Lucas and Stokey disrupted Barro's analogy.[2]

[1] Hansen, Roberds, and Sargent (1991) describe the testable implications of various models including Barro's.

[2] We have heard V. V. Chari and Nancy Stokey conjecture that results closer to Barro's would emerge in a model that eliminates complete markets and permits only risk-free borrowing. An impediment to evaluating this conjecture has been that the optimal taxation problem with only risk-free borrowing is difficult because complicated additional constraints restrict competitive allocations (see Chari, Christiano, and Kehoe 1995, p. 366).

This paper recasts the optimal taxation problem in an incomplete markets setting. By permitting only risk-free government borrowing, we revitalize parts of Barro's consumption-smoothing analogy. Work after Barro, summarized and extended by Chamberlain and Wilson (2000), has taught us much about the consumption-smoothing model. We find that under some restrictions on preferences and the quantities of risk-free claims that the government can issue and own, the consumption-smoothing model allows us to reaffirm Barro's random walk characterization of optimal taxation. But dropping the restriction on government asset holdings or modifying preferences causes the results to diverge in important ways from Barro's.

Our interest in reinvigorating Barro's model is inspired partly by historical episodes that pit Barro's model against Lucas and Stokey's. For example, see the descriptions of French and British eighteenth-century public finance in Sargent and Velde (1995). Time-series graphs of Great Britain's debt resemble realizations of a martingale with drift and are much smoother than graphs of government expenditures, which show large temporary increases associated with wars. Barro's model implies behavior like those graphs whereas Lucas and Stokey's model does not.[3] Our adaptation of Lucas and Stokey's model to rule out state-contingent debt is capable of generating behavior like Britain's. Section VI illustrates this claim by displaying impulse responses to government expenditure innovations for both Lucas and Stokey's original model and our modification of it.

The remainder of this paper is organized as follows. Section II describes our basic model. It retains Lucas and Stokey's environment but modifies their bond market structure by having the government buy and sell only risk-free one-period debt. Confining the government to risk-free borrowing retains Lucas and Stokey's single implementability restriction on an equilibrium allocation and adds stochastic *sequences* of implementability restrictions. These additional restrictions emanate from the requirement that the government's debt be risk-free. We formulate a Lagrangian for the Ramsey problem and show how the additional constraints introduce two new state variables: the government debt level and a variable dependent on past Lagrange multipliers. The addition of these state variables to Lucas and Stokey's model makes taxes and government debt behave more as they do in Barro's model. First-order conditions associated with the saddle point of the Lagrangian form a system of expectational difference equations whose solution determines the Ramsey outcome under incomplete markets. These equa-

[3] Perhaps Lucas and Stokey's model does better at explaining France's behavior, with its recurrent defaults, which might be interpreted as occasionally low state-contingent payoffs.

tions are difficult to solve in general. Therefore, Section III analyzes a special case with utility linear in consumption but concave in leisure. This specification comes as close as possible to fulfilling Barro's intuition but requires additional restrictions on the government expenditure process and the government debt in order to align fully with Barro's conclusions. In particular, we show that if the government's *asset* level is not restricted, the Ramsey plan under incomplete markets will eventually set the tax rate to zero and finance all expenditures from a war chest.[4] However, if we arbitrarily put a binding upper limit on the government's asset level, the Ramsey plan's taxes and government debt will resemble the outcomes asserted by Barro.

Without the binding upper bound on government assets, the multiplier determining the tax rate converges in the example of Section III. Section IV introduces another example, one with an absorbing state for government expenditures, in which that multiplier also converges, but now to a nonzero value, implying a positive tax rate. Sections IV and V then study the generality of the result that the multiplier determining the tax rate converges. Together these sections show that the result is not true for general preferences and specifications of the government expenditure process. Section IV studies how far the martingale convergence approach used in the consumption-smoothing literature can take us. Section V takes a more direct approach to studying the limiting behavior of the multiplier in general versions of our model. Under a condition that the government expenditure process remains sufficiently random, we show that, in general, the multiplier will not converge to a nonzero value, meaning that the allocation cannot converge to that for a complete market Ramsey equilibrium. That result establishes the sense in which the previous examples are both special. Section VI briefly describes linear impulse response functions of numerically approximated equilibrium allocations. The computed examples have tax rates that combine a feature of Barro's policy (a unit root component) with aspects of Lucas and Stokey's Ramsey plan (strong dependence of taxes and deficits on current shocks).

Throughout this paper, we assume that the government binds itself to the Ramsey plan. Therefore, we say nothing about Lucas and Stokey's discussions of time consistency and the structure of government debt.

[4] See the first section of Hume (1777). The examples in Lucas and Stokey (1983), where the government faces a war at a known future date, also generate a behavior of debt consistent with our epigraph from Hume.

II. The Economy

Technology and preferences are those specified by Lucas and Stokey. Let c_t, x_t, and g_t denote consumption, leisure, and government purchases at time t. The technology is

$$c_t + x_t + g_t = 1. \tag{1}$$

Government purchases g_t follow a Markov process, with transition density $P(g'|g)$ and initial distribution π. We assume that (P, π) is such that $g \in [g_{\min}, g_{\max}]$. Except for some special examples, we also assume that P has a unique invariant distribution with full support $[g_{\min}, g_{\max}]$.

A representative household ranks consumption streams according to

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, x_t), \tag{2}$$

where $\beta \in (0, 1)$, and E_0 denotes the mathematical expectation conditioned on time 0 information.

The government raises all revenues through a time-varying flat rate tax τ_t on labor at time t. Households and the government make decisions whose time t components are functions of the history of government expenditures $g^t = (g_t, g_{t-1}, \ldots, g_0)$ and of initial government indebtedness b^g_{-1}.

Incomplete Markets with Debt Limits

Let $s_t \equiv \tau_t(1 - x_t) - g_t$ denote the time t net-of-interest government surplus. Households and the government borrow and lend only in the form of risk-free one-period debt. The government's budget and debt limit constraints are

$$b^g_{t-1} \leq s_t + p^b_t b^g_t, \quad t \geq 0, \tag{3}$$

and

$$\underline{M} \leq b^g_t \leq \overline{M}, \quad t \geq 0. \tag{4}$$

Here p^b_t is the price in units of time t consumption of a risk-free bond paying one unit of consumption in period $t+1$ for sure; b^g_t represents the number of units of time $t+1$ consumption that at time t the government promises to deliver. When (3) holds with strict inequality, we let the right side minus the left side be a nonnegative level of lump-sum transfers T_t to the household. The upper and lower debt limits $\overline{M}$ and $\underline{M}$ in (4) influence the optimal government plan. We discuss alternative possible settings for $\overline{M}$ and $\underline{M}$ below.

The household's problem is to choose stochastic processes $\{c_t, x_t, b_t^g\}_{t=0}^{\infty}$ to maximize (2) subject to the sequence of budget constraints

$$p_t^b b_t^g + c_t \leq (1 - \tau_t)(1 - x_t) + b_{t-1}^g + T_t, \quad t \geq 0, \tag{5}$$

with prices and taxes $\{p_t^b, \tau_t, T_t\}$ taken as given; here b_t^g denotes the household's holdings of government debt. The t element of consumers' choices must be measurable with respect to (g^t, b_{-1}^g).

The household also faces debt limits analogous to (4), which we assume are less stringent (in both directions) than those faced by the government. Therefore, in equilibrium, the household's problem always has an interior solution. When u_i represents marginal utility with respect to variable i, the household's first-order conditions require that the price of risk-free debt satisfies

$$p_t^b = E_t \beta \frac{u_{c,t+1}}{u_{c,t}} \quad \forall t \geq 0 \tag{6}$$

and that taxes satisfy

$$\frac{u_{x,t}}{u_{c,t}(1 - \tau_t)} = 1. \tag{7}$$

Debt Limits

By analogy with Aiyagari's (1994) and Chamberlain and Wilson's (2000) analyses of a household savings problem, we shall study two kinds of debt limits, called "natural" and "ad hoc." Natural debt limits come from taking seriously the risk-free status of government debt and finding the maximum debt that could be repaid almost surely under an optimal tax policy. We call a debt or asset limit ad hoc if it is more stringent than a natural one. In our model, the natural asset and debt limits are in general difficult to compute. We compute and discuss them for an important special case in Section III.

Definitions

We use the following definitions.

Definition 1. Given b_{-1}^g and a stochastic process $\{g_t\}$, a feasible *allocation* is a stochastic process $\{c_t, x_t, g_t\}$ satisfying (1) whose time t elements are measurable with respect to (g^t, b_{-1}^g). A *bond price process* $\{p_t^b\}$ and a *government policy* $\{\tau_t, b_t^g\}$ are stochastic processes whose time t element is measurable with respect to (g^t, b_{-1}^g).

Definition 2. Given b_{-1}^g and a stochastic process $\{g_t\}$, a *competitive equilibrium* is an allocation, a government policy, and a bond price process that solve the household's optimization problem and that satisfy the government's budget constraints (3) and (4).

Because we have made enough assumptions to guarantee an interior solution of the consumer's problem, a competitive equilibrium is fully characterized by (1), (3), (4), (7), and (6).

DEFINITION 3. The *Ramsey problem* is to maximize (2) over competitive equilibria. A *Ramsey outcome* is a competitive equilibrium that attains the maximum of (2).

We use a standard strategy of casting the Ramsey problem in terms of a constrained choice of allocation. We use (6) and (7) to eliminate asset prices and taxes from the government's budget and debt constraints, and thereby deduce sequences of restrictions on the government's allocation in *any* competitive equilibrium with incomplete markets. Lucas and Stokey showed that under complete markets, competitive equilibrium imposes a single intertemporal constraint on allocations. We shall show that competitive equilibrium allocations in incomplete markets must satisfy the same restriction from Lucas and Stokey, as well as additional ones that impose that the government purchase or sell only risk-free debt.

From now on, we use (7) to represent the government surplus in terms of the allocation as $s_t \equiv s(c_t, g_t) \equiv [1 - (u_{x,t}/u_{c,t})](c_t + g_t) - g_t$. The following proposition characterizes the restrictions that the government's budget and behavior of households place on competitive equilibrium allocations.

PROPOSITION 1. Take the case $T_t = 0$, and assume that for any competitive equilibrium $\beta^t u_{c,t} \rightarrow 0$ almost surely.[5] Given b^g_{-1}, a feasible allocation $\{c_t, g_t, x_t\}$ is a competitive equilibrium if and only if the following constraints are satisfied:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} s_t = b^g_{-1}, \tag{8}$$

$$\underline{M} \leq E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} \leq \overline{M} \quad \forall t \geq 0,\ \forall g^t \in [g_{\min}, g_{\max}]^{t+1}, \tag{9}$$

and

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} \text{ is measurable with respect to } g^{t-1}$$
$$\forall t \geq 0,\ \forall g^t \in [g_{\min}, g_{\max}]^{t+1}. \tag{10}$$

[5] We assume zero lump-sum transfers for simplicity. It is trivial to introduce lump-sum transfers. The condition on marginal utilities can be guaranteed in a number of ways.

Proof. We relegate the proof to the Appendix.

In the complete markets setting of Lucas and Stokey, (8) is the *sole* "implementability" condition that government budget balance and competitive household behavior impose on the equilibrium allocation. The incomplete markets setup leaves this restriction intact but adds three *sequences* of constraints. Constraint (10) requires that the allocation be such that, at each date $t \geq 0$,

$$B_t \equiv E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j},$$

the present value of the surplus (evaluated at date t Arrow-Debreu prices), be known one period ahead.[6] Condition (9) requires that the debt limits be respected. Condition (8) is the time 0 version of constraint (10).

We approach the task of characterizing the Ramsey allocation by composing a Lagrangian for the Ramsey problem.[7] We use the convention that variables dated t are measurable with respect to the history of shocks up to t. We attach stochastic processes $\{\nu_{1t}, \nu_{2t}\}_{t=0}^{\infty}$ of Lagrange multipliers to the inequality constraints on the left and right of (9), respectively. We incorporate condition (10) by writing it as

$$b_{t-1}^g = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j},$$

multiplying it by u_{ct} and attaching a Lagrange multiplier $\beta^t \gamma_t$ to the resulting time t component. Then the Lagrangian for the Ramsey problem can be represented, after application of the law of iterated expectations and Abel's summation formula (see Apostol 1974, p. 194), as

$$L = E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t, 1 - c_t - g_t) - \psi_t u_{c,t} s_t + u_{c,t}(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b_{t-1}^g)], \quad (11)$$

where

$$\psi_t = \psi_{t-1} + \nu_{1t} - \nu_{2t} + \gamma_t \quad (12)$$

[6] There is a parallel constraint in the complete markets case in which B_t must be measurable with respect to g^t. But this constraint is trivially satisfied by the definition of $E_t(\cdot)$. Proposition 1 is reminiscent of Duffie and Shafer's (1985) characterization of incomplete markets equilibrium in terms of "effective equilibria" that, relative to complete markets allocations, require next-period allocations to lie in subspaces determined by the menu of assets. In particular, see the argument leading to proposition 1 in Duffie (1992, pp. 216–17).

[7] Chari et al. (1995, p. 366) call the Ramsey problem with incomplete markets a computationally difficult exercise because imposing the sequence of measurability constraints (10) seems daunting. For a class of special examples sharing features with the one in Sec. III, Hansen et al. (1991) focus on the empirical implications of the measurability constraints (10).

for $\psi_{-1} = 0$. Here $\gamma_0 \leq 0$, with equality only if the government's assets are large enough for the payouts on them to sustain the highest possible value of g at all periods with zero taxes. The multipliers $\psi_t \leq 0$ for $t \geq 0$; γ_t can be either positive or negative for $t > 0$. To see why $\gamma_0 < 0$, differentiate the Lagrangian with respect to b^g_{-1}, and notice that $u_{c,0}\gamma_0$ can be regarded as the effect on the welfare of the representative household of an increase in the present value of government purchases. The nonpositive random multiplier ψ_t measures the effect on the representative household's welfare of an increase in the present value of government expenditures from time t onward. The multiplier γ_t measures the marginal impact of news about the present value of government expenditures on the maximum utility attained by the planner.[8]

The Ramsey problem under complete markets amounts to a special case in which $\gamma_{t+1} = \nu_{1t} = \nu_{2t} \equiv 0$ for all $t \geq 0$, and γ_0 is the (scalar) multiplier on the time 0 present value government budget constraint: these specifications imply that $\psi_t = \psi_0 = \gamma_0$ for complete markets. Relative to the complete markets case, the incomplete markets case augments the Lagrangian with the appearances of b^g_{t-1}, γ_t, ν_{1t}, ν_{2t}, for all $t \geq 1$, and $\underline{M}$ and $\overline{M}$ in the Lagrangian, and the effects of γ_t, ν_{1t}, ν_{2t} on ψ_t in (12).[9]

We want to investigate whether the additional constraints on the Ramsey allocation move us toward Barro's tax-smoothing outcome. For $t \geq 1$, the first-order condition with respect to c_t can be expressed as

$$u_{c,t} - u_{x,t} - \psi_t \kappa_t + (u_{cc,t} - u_{cx,t})(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b^g_{t-1}) = 0, \qquad (13)$$

where[10]

$$\kappa_t = (u_{cc,t} - u_{cx,t})s_t + u_{ct}s_{c,t}. \qquad (14)$$

[8] The present value is evaluated at Arrow-Debreu prices for markets that are reopened at time t after g_t is observed.

[9] As is often the case in optimal taxation problems, it is not easy to establish that the feasible set of the Ramsey problem is convex, so it is not easy to guarantee that the saddle point of L is the solution to the optimum. But since the first-order conditions of the Lagrangian are necessary and our solutions rely on only the first-order conditions of the Lagrangian, it is enough to check (as we do) that only one solution to the first-order conditions of the Lagrangian can be found.

Because future control variables appear in the measurability constraints, the optimal choice at time t is not a time-invariant function of the natural state variables (b^g_{t-1}, g_t) as in standard dynamic programming. Nevertheless, the Lagrangian in (11) and the constraint (12) suggest that a recursive formulation can be recovered if ψ_{t-1} is included in the state variables. Indeed, this fits the "recursive contracts" approach described in Marcet and Marimon (1998); they show, under some assumptions, that the optimal choice at time t is a time-invariant function of state variables $(\psi_{t-1}, b^g_{t-1}, g_t)$. Appendix B of Marcet, Sargent, and Seppälä (1995) describes in detail how to map the current problem into the recursive contracts framework.

[10] In the definition of κ_t, it is understood that total differentiation of the function $u = u(c, 1 - c - g)$ with respect to c is occurring. Evidently,

$$\kappa_t = (u_{ct} - u_{xt}) + c_t(u_{cc,t} - 2u_{cx,t} + u_{xx,t}) + g_t(u_{xx,t} - u_{cx,t}).$$

It is useful to study this condition under both complete and incomplete markets.

Complete Markets

Complete markets amount to $\nu_{1t} = \nu_{2t} = \gamma_{t+1} = 0$ for all $t \geq 0$, which causes (13) to collapse to

$$u_{c,t} - u_{x,t} - \gamma_0 \kappa_t = 0, \tag{15}$$

which is a version of Lucas and Stokey's condition (2.9) for $t \geq 1$. From its definition (14), κ_t depends on the level of government purchases only at t. Therefore, given the multiplier γ_0, (15) determines the allocation and associated tax rate τ_t as a time-invariant function of g_t only. Past g's do not affect today's allocation. The *sole* intertemporal link occurs through the requirement that γ_0 must take a value to satisfy the time 0 present value government budget constraint. Equation (15) implies that, to a linear approximation, τ_t and all other endogenous variables mirror the serial correlation properties of the g_t process.[11] The "tax smoothing" that occurs in this complete markets model occurs "across states" and is reflected in the diminished variability of tax rates and revenues relative to the taxes needed to balance the budget in all periods, but *not* in any propagation mechanism imparting more pronounced serial correlation to tax rates than to government purchases. Evidently, in the complete markets model, the government debt B_t also inherits its serial correlation properties entirely from g_t. For example, if g_t is first-order Markov, then B_t is a function only of g_t (see Lucas and Stokey 1983).

Incomplete Markets

In the incomplete markets case, equation (12) suggests that ψ_t changes (permanently) each period because γ_t is nonzero in all periods. Being of either sign, γ_t causes ψ_t to increase or to decrease permanently. The multiplier ψ_t is a risk-adjusted martingale, imparting a unit root component to the solution of (13). Taking the derivative of the Lagrangian with respect to b_t^g, we get

$$E_t[u_{c,t+1}\gamma_{t+1}] = 0. \tag{16}$$

This implies that γ_t can be positive or negative and that ψ_t can rise or

[11] If utility is quadratic as in some examples of Lucas and Stokey, τ_t is a linear function of g_t, and all variables inherit their serial correlation directly from g_t.

fall in a stochastic steady state. Assuming that the debt constraints do not bind at t, $\nu_{1,t+1}$, $\nu_{2,t+1} = 0$, and using (12) gives

$$\psi_t = (E_t[u_{c,t+1}])^{-1}E_t[u_{c,t+1}\psi_{t+1}]. \tag{17}$$

With the definition of conditional covariance, equation (17) can be further decomposed as

$$\psi_t = E_t[\psi_{t+1}] + (E_t[u_{c,t+1}])^{-1}\,\mathrm{Cov}_t\,(u_{c,t+1},\ \psi_{t+1}).$$

Equation (13) shows that this approximate martingale result is not precisely Barro's, first, because ψ is not a pure martingale and, second, because (13) makes τ_t depend also on $\gamma_t b^g_{t-1}$, and so distorts the pure martingale outcome. In Section IV, we pursue how much information can be extracted from (17).

Example 1: Serially Uncorrelated Government Purchases

The case in which government expenditures are independently and identically distributed (i.i.d.) provides a good laboratory for bringing out the implications of prohibiting state-contingent debt. With complete markets, the one-period state-contingent debt falling due at t satisfies

$$m_{t-1}(g_t) = B_t = s_t + E_t\left[\sum_{j=1}^{\infty}\beta^j\frac{u_{c,t+j}}{u_{c,t}}s_{t+j}\right],$$

where $m_{t-1}(g)$ means the quantity of claims purchased at $t-1$ contingent on $g_t = g$. With a serially independent g_t process, and since both consumption and s are time-invariant functions of g_t, the expectation conditional on g_t equals an unconditional expectation, constant through time, implying

$$u_{c,t}m_{t-1}(g_t) = u_{c,t}s_t + \beta Eu_cB, \tag{18}$$

where $Eu_cB = Eu_cs/(1-\beta)$. Equation (18) states that, measured in marginal utility units, the gross payoff on government debt equals a constant plus the time t surplus, which is serially uncorrelated. In marginal utility units, the time t value of the state-contingent debt with which the government leaves every period is a constant, namely, βEu_cB. The one-period rate of return on this debt is high in states when the surplus s_t is pushed up because g_t is low, and it is low in states when high government expenditures drive the surplus down. There is no propagation mechanism from government purchases to the value of debt with which the government leaves each period, which is constant.[12]

[12] For serially correlated government spending, it can be shown that the portfolio m is time-invariant. This follows, e.g., from Marcet and Scott (2001, proposition 1).

With incomplete markets, the situation is very different. Government debt evolves according to

$$B_{t+1} = r_t(B_t - s_t), \tag{19}$$

where $r_t \equiv (p_t^b)^{-1}$ and B_{t+1} is denominated in units of time $t+1$ consumption goods. Since the gross real interest rate r is a random variable exceeding one, this equation describes a propagation mechanism by which even a serially independent government surplus process s_t would influence future levels of debt and taxes. In fact, if the government tried to implement the complete markets solution, which generates an i.i.d. surplus, equation (19) is explosive in debt, and with probability one, debt will go to plus or minus infinity; therefore, the complete markets solution is not implementable, so that even with i.i.d. government expenditures, the absence of complete markets causes the surplus process itself to be serially correlated, as described above.

Reason for Examples

So far, we have shown that the optimal tax is determined jointly by g_t, b_{t-1}^g, and a state variable that resembles a martingale, namely ψ_t. Dependence on g_t induces effects like those found by Lucas and Stokey. Dependence on ψ_t impels a martingale component, like that found by Barro. It is impossible to determine which effect dominates at this level of generality. To learn more, we now restrict the curvature of the one-period utility function to create a workable special example.

III. An Example Affirming Barro

In the Ramsey problem, the government simultaneously chooses taxes and manipulates intertemporal prices. Manipulating prices substantially complicates the problem, especially with incomplete markets. We can simplify by adopting a specification of preferences that eliminates the government's ability to manipulate prices. This brings the model into the form of a consumption-smoothing model (e.g., Aiyagari 1994; Chamberlain and Wilson 2000) and allows us to adapt results for that model to the Ramsey problem. We shall establish a martingale result for tax rates under an arbitrary restriction on the level of risk-free assets that the government can *acquire*.

Example 2: Constant Marginal Utility of Consumption

We assume that $u(c, x) = c + H(x)$, where H is an increasing, strictly concave, three times continuously differentiable function. We assume $H'(0) = \infty$ and $H'(1) < 1$ to guarantee that the first-best has an interior solution for leisure, and $H'''(x)(1-x) > 2H''(x)$ for all $x \in (0, 1)$ to guarantee existence of a unique maximum level of revenue.[13]

Making preferences linear in consumption ties down intertemporal prices. Then (6) and (7) become

$$p_t^b = \beta \tag{20}$$

and

$$H'(x_t) = 1 - \tau_t. \tag{21}$$

Equation (20) makes the price system independent of the allocation.

Government revenue is $R(x) = [1 - H'(x)](1 - x)$ with derivatives

$$R'(x) = -H''(x)(1-x) - [1 - H'(x)] \tag{22}$$

and

$$R''(x) = -H'''(x)(1-x) + 2H''(x). \tag{23}$$

Our assumptions on H guarantee that $R'' < 0$. Hence R is strictly concave. Letting x_1 be the first-best choice for leisure satisfying $H'(x_1) = 1$, we know that $x_1 < 1$. Since $R'(x_1) > 0$ and $R(x_1) = R(1) = 0$, strict concavity of R implies that there is a unique $x_2 \in (x_1, 1)$ that maximizes the revenue and satisfies $R'(x_2) = 0$. The government wants to confine x_t to the interval $[x_1, x_2]$. Concavity of R implies that R' is monotone and, therefore, that R is monotone increasing on $[x_1, x_2]$.

Natural Debt Limits

Aiyagari and others define an agent's "natural debt limit" to be the maximum level of indebtedness for which the debt can be repaid almost surely, given the agent's income process. Here the natural debt limit for the government is evidently

$$\overline{M} = \frac{1}{1-\beta}[R(x_2) - g_{\max}].$$

To discover a natural *asset* limit, we write the government budget constraint with zero revenues and transfers at the maximum government expenditure level as

$$b_{t-1}^g = -g_{\max} + p^b b_t^g,$$

where $p^b = \beta$. Evidently the natural asset limit for the government is $\underline{M} = -g_{\max}/(1-\beta)$. The government has no use for more assets because it can finance all expenditures from interest on its assets even in the highest government expenditure state.

[13] The latter assumption is satisfied, e.g., if $H''' > 0$.

Imposing $c_t \geq 0$ gives a natural borrowing limit for the consumer,

$$\bar{b}^c \leq \frac{H'(x_2)(1 - x_2)}{1 - \beta},$$

where the numerator is the lowest after-tax income of the household.

We assume that parameters are such that $\bar{b}^c > -\underline{M}$.

Ramsey Problem and an Associated Permanent Income Model

Under this specification, the Ramsey problem acquires the form of the consumption-smoothing problem. Because the revenue function is monotone on $[x_1, x_2]$, we can invert it to get the function $x = x(R)$ for $R \in \mathcal{R} \equiv [0, R(x_2)]$. This means that utility can be expressed in terms of revenue and, since the term $1 - g_t$ is exogenous, it can be dropped from the objective of the government to let us express the government's one-period return function as $W(R) = -x(R) + H(x(R))$. Notice that $W(R)$ equals minus the deadweight loss from raising revenues R and thus matches Barro's one-period return function.

With these assumptions, $W(R)$ is a twice continuously differentiable, strictly concave function on $\mathcal{R}$. To see this, note that

$$W'(R) = [H'(x(R)) - 1]x'(R),$$
$$W''(R) = [H'(x(R)) - 1]x''(R) + H''(x(R))[x'(R)]^2.$$

The fact that $R'' < 0$ implies that $x'' > 0$, and since H is concave, the formula above for W'' implies that W is concave. Furthermore, $W(R)$ has a strict maximum at $R = 0$, associated with the first-best tax rate of $x_1 = 0$.

Then the Ramsey problem can be expressed as

$$\max_{\{R_t, b_t^g\}} E_0 \sum_{t=0}^{\infty} \beta^t W(R_t) \tag{24}$$

subject to

$$b_t^g \geq \beta^{-1}(g_t + b_{t-1}^g - R_t) \tag{25}$$

and

$$b_t^g \in [\underline{M}, \overline{M}]. \tag{26}$$

We restrict revenues to be in $\mathcal{R}$ and the sequence of revenues to be in the infinite Cartesian product $\mathcal{R}^\infty$.

We can map our model into the consumption problem by letting R

play the role of consumption, $W(R)$ the one-period utility function of the consumer, g_t exogenous labor income, and b_t^g the household's debt.[14]

As Chamberlain and Wilson (2000) describe, the solution of the consumption problem depends on the utility function, the relation of the interest rate to the discount factor, and whether there persists sufficient randomness in the income process. Problem (24)–(26) corresponds to a special consumption problem with a finite bliss level of consumption and the gross interest rate times the discount factor identically equal to unity. For such a problem, if income remains sufficiently stochastic, then under the natural debt limits, consumption converges to bliss consumption and assets converge to a level sufficient to support that consumption.

As we shall see in the next subsection, there is a related result for the general case of the Ramsey plan under incomplete markets: tax revenues converge to zero and government assets converge to a level always sufficient to support government purchases from interest earnings alone, with lump-sum transfers being used to dispose of earnings on government-owned assets. To sustain randomly fluctuating tax rates in the limit requires arresting such convergence. Putting a binding upper limit on assets prevents convergence, as we shall show by applying results from the previous section to the special utility function of this section.

Incomplete Markets, "Natural Asset Limit"

For example 2, the definition of κ_t in (14) implies

$$\kappa_t = -R'(x_t) \leq 0 \quad \text{for } x_t \in [x_1,\ x_2]. \tag{27}$$

The variables $(\tau_t,\ x_t,\ \psi_t)$ are then determined by (12), which we repeat for convenience, and the following specialization of (13):

$$\psi_t = \psi_{t-1} + \nu_{1t} - \nu_{2t} + \gamma_t \tag{28}$$

and

$$\tau_t = 1 - H'(x_t) = -\psi_t R'(x_t). \tag{29}$$

Under the natural asset limit and the ability to make positive lump-sum transfers, $\nu_{2t} \equiv 0$. Then (12), $u_{ct} = 1$, and (16) imply that

$$E_{t-1}\psi_t \geq \psi_{t-1}. \tag{30}$$

Inequality (30) asserts that the nonpositive stochastic process ψ_t is a

[14] See Aiyagari (1994), Chamberlain and Wilson (2000), and their references for treatments of this problem. Hansen et al. (1991) pursue the analogy between the consumption- and tax-smoothing problems.

submartingale.[15] It is bounded above by zero. Therefore, the submartingale convergence theorem (see Loève 1978) asserts that ψ_t converges almost surely to a nonpositive random variable. There are two possibilities.

1. If the Markov process for g has a unique nontrivial invariant distribution, then our lemma 3 in Section V shows that ψ_t converges almost surely to zero. In that case, (29) implies that τ_t converges to the first-best tax rate $\tau_t = 0$, and leisure converges to the first-best x_1. The level of government assets converges to the level $g_{\max}/(1-\beta)$ sufficient to finance $g_{\max}$ from interest earnings. Transfers are eventually zero when $g_t = g_{\max}$ but positive when $g_t < g_{\max}$.

2. If the Markov process for g has an absorbing state, then ψ_t can converge to a strictly negative value; ψ converges when g_t enters the absorbing state. From then on, taxes and all other variables in the model are constant.

Barro's Result under an Ad Hoc Asset Limit

Thus, under the natural asset limit, this example nearly sustains Barro's martingale characterization for the tax rate, since ψ_t is a martingale and taxes are a function of ψ_t. But the government accumulates assets, and in the limit, the allocation is first-best and taxes are zero. We now show that imposing an ad hoc asset limit makes outcomes align with Barro's even in the limit as t grows, at least away from corners. When $\underline{M} > -g_{\max}/(1-\beta)$, the lower limit on debt occasionally binds. This puts a nonnegative multiplier ν_{2t} in (12) and invalidates the martingale implication (30). This markedly alters the limiting behavior of the model in the case in which the Markov process for g_t has a unique invariant distribution. In particular, rather than converging almost surely, ψ_t can continue to fluctuate randomly if randomness in g persists sufficiently. Off corners (i.e., if $\nu_{2,t+1} = \nu_{1,t+1} = 0$ almost surely given information at t), ψ_t fluctuates as a martingale. But on the corners, it will not. If we impose time-invariant ad hoc debt limits $\underline{M}$ and $\overline{M}$, the distribution of government debt will have a nontrivial distribution with randomness that does not disappear even in the limit. Also, ψ will have the following type of "inward-pointing" behavior at the boundaries. If government assets are at the lower bound and $g_{t+1} = g_{\max}$, then taxes are set at $1 - H'(x_2)$ and government assets stay at the lower bound. If $g_{t+1} < g_{\max}$, then taxes will be lower and government assets will drift up. If government assets are at the upper bound and $g_{t+1} = g_{\min}$, then just enough taxes are collected to keep assets at the upper bound; if $g_{t+1} > g_{\min}$, then assets will drift downward.

[15] Inequality (30) differs from (17) because here we allow the asset limit to bind.

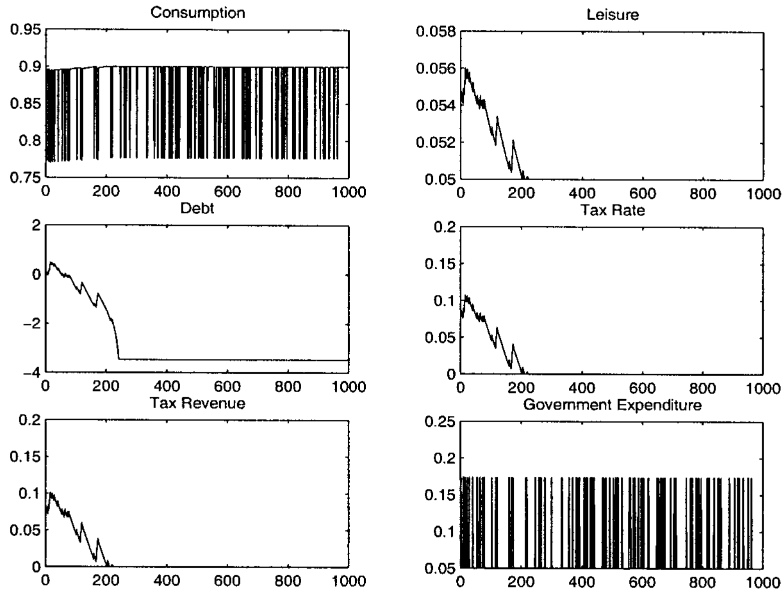

FIG. 1.—Outcomes for an incomplete markets economy in example 1 with natural debt limits on government assets.

In the case in which g_t is i.i.d., by using an argument similar to those in Huggett (1993) and Aiyagari (1994), one can show that an ergodic distribution of assets exists. Figures 1 and 2 illustrate the difference between natural and ad hoc asset limits. They show simulations of two economies in each of which government expenditures follow a two-state Markov process and the consumers have quasi-linear preferences. The two economies are identical except for their debt limits. In both economies, $H(x) = 0.05 \log(x)$, and g_t can take only values 0.1736 (war) or 0.05 (peace) with the transition matrix

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}.$$

In the economy displayed in figure 1, the government faces natural asset and debt limits, $(\underline{M}, \overline{M}) = (-3.472, 8.584)$, whereas in the figure 2 economy it faces more stringent ad hoc limits, $(\underline{M}, \overline{M}) = (-1, 1)$. The different asset limits lead to dramatically different results in the outcomes. While the first economy displays convergence to the first-best, the second economy exhibits Barro-like random walk behavior of taxes and debt within boundaries.

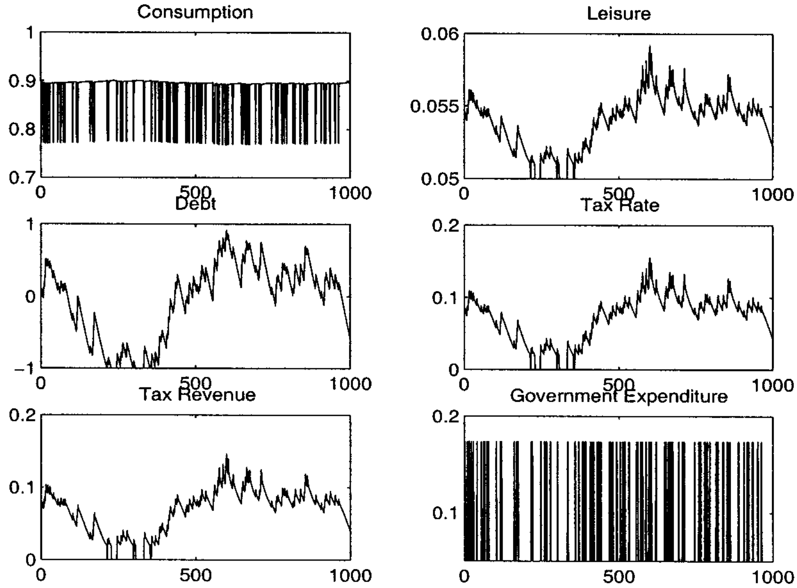

FIG. 2.—Outcomes for the incomplete markets economy in example 1 with an ad hoc limit on assets.

Complete Markets: Constant Tax Rates

For comparison, it is useful to describe what the allocation and taxes would be under complete markets in example 2. In the complete markets case, restrictions (25) and (26) are replaced by the following version of Lucas and Stokey's single implementability constraint:

$$b^g_{-1} = E_0 \sum_{t=0}^{\infty} \beta^t (R_t - g_t). \tag{31}$$

The policy that maximizes (24) subject to (31) sets revenues and tax rates equal to constants and transfers to zero. This can be shown directly, but it is instructive to show it simply by applying the results earlier in this section. Then equations (27) and (15) imply

$$\tau_t = 1 - H'(x_t) = -\gamma_0 R'(x_t). \tag{32}$$

Recall that $R'(x) \geq 0$ for $x \in [x_1, x_2)$ and that $\gamma_0 \leq 0$. The restrictions on $R(x)$ on $[x_1, x_2]$ derived above imply that there is a unique $x_t = x^{\mathrm{CM}}$ that solves (32). Thus, under complete markets, the tax rate and leisure are constant over time and across states.

Although the incomplete markets economy under the natural asset limits eventually obtains the first-best allocation, with taxes and hence the distortions that they bring converging to zero, at time 0 the consumers are better off in the complete markets economy with its persistent distortions. The explanation, of course, is that it takes a long time for the incomplete markets economy to reach the first-best. In the example presented in figure 1, it takes about 200 periods before the economy converges to the first-best.

Example 2 ties down $u_{c,t}$ by assuming linear utility. The next two sections study whether taxes can be expected to converge under more general utility specifications.

IV. Nonconvergence of ψ_t

Example 2 showed how a martingale property under the natural debt and asset limits guaranteed that ψ converges almost surely. Furthermore, in that case, the limit would often be zero.

In this section we explore whether it is possible to obtain a general result about convergence by exploiting the martingale property of ψ_t. We study the interaction of the convergence of ψ_t and $u_{c,t}$ under more general preferences. We shall show that if we can determine the asymptotic behavior of the predictability of $u_{c,t}$, then we can also show convergence of ψ_t. We proceed to ask whether ψ_t can converge when $u_{c,t}$ does not. We show that, in general, if $u_{c,t}$ does not converge, as happens in most models, then we can say very little about convergence of ψ_t.

We have already argued that if the debt limits can bind, then ψ_t should not be expected to converge. Throughout this section we assume that the natural debt and asset limits are imposed, so that the asset and debt limits never bind.[16] Then (17) holds, and it is convenient to rewrite it as

$$\psi_t = E_t\left[\frac{u_{c,t+1}}{E_t[u_{c,t+1}]}\psi_{t+1}\right]. \tag{33}$$

We also assume throughout this section that $u_c(c, x) > 0$ for all feasible c, x.

With terminology common in finance, (33) and the fact that $E_t[u_{c,t+1}/E_t[u_{c,t+1}]] = 1$ make ψ a "risk-adjusted martingale." Risk-adjusted martingales converge under suitable conditions. One strategy to prove convergence involves finding an equivalent measure that satisfies a particular boundary condition (see, e.g., Duffie 1996, chap. 4).

[16] Some standard regularity conditions need to be imposed in order to guarantee existence of natural debts limits, in particular, to guarantee that the interest rate is bounded away from zero.

We follow a related approach of Chamberlain and Wilson (2000) and give an example in which the required boundary condition is satisfied. We shall also show that, unfortunately, the standard boundary conditions are violated in the general case.

Martingale Convergence

We begin with what seems like an encouraging result. Let

$$\theta_t \equiv \prod_{\tau=1}^{t} \frac{u_{c,\tau}}{E_{\tau-1}[u_{c,\tau}]}.$$

LEMMA 1. $\{\theta_t\psi_t\}$ is a martingale. Therefore, it converges almost surely to a random variable $\overline{\theta\psi}$ that is finite with probability one.

Proof. By assumption, the debt limits are never binding, and (33) holds for all periods with probability one. Multiplying both sides of (33) by θ_t, we have

$$\theta_t\psi_t = E_t[\theta_{t+1}\psi_{t+1}] \tag{34}$$

almost surely. Since $\theta_t \geq 0$, $\theta_t\psi_t \leq 0$, and this product converges almost surely to a finite variable by theorem A of Loève (1978, p. 59). Q.E.D.

Lemma 1 implies convergence of ψ_t only if we can say something about the asymptotic behavior of θ_t. In particular, if θ_t converges to a nonzero limit, then lemma 1 allows us to conclude that ψ_t converges.[17] This can be guaranteed in an interesting special case.

Example 3: Absorbing States Imply That ψ_t Converges

Assume that $\{g_t\}$ has absorbing states in the sense that $g_t = g_{t-1}$ almost surely for t large enough, so that fluctuations cease and $u_{c,t}(\omega) = E_{t-1}[u_{c,t}](\omega)$. Since Lucas and Stokey also consider examples with absorbing states, it is instructive to compare in what sense the incomplete markets equilibrium replicates the complete markets one.

The arguments of Lucas and Stokey show that given an initial level of debt b_{-1}^g, the Lagrange multiplier is constant through time. Let us make this dependence explicit and denote by $\gamma_0^{CM}(b_{-1}^g)$ the multiplier that obtains given a level of initial debt under complete markets.

Under incomplete markets, since $0 < u_{c,t} < \infty$, it is clear that θ_t converges to a positive number almost surely. Then lemma 1 implies that $\psi_t \rightarrow \psi_\infty$ almost surely and the limiting random variable ψ_∞ plays the role of Lucas and Stokey's single multiplier for that tail allocation. Once g has reached an absorbing state, the incomplete markets allocation co-

[17] This is the same proof strategy of Chamberlain and Wilson (2000). Our lemma 1 is analogous to their theorem 1. However, in their model, θ_t is exogenous.

incides with the complete markets allocation that would have occurred under the same shocks, but for a different initial debt. More precisely, for each realization ω, the incomplete markets allocations coincide with those under complete markets, under the assumption that initial debt under complete markets had been equal to a value $\bar{b}(\omega)$ satisfying $\gamma_0^{\mathrm{CM}}(\bar{b}(\omega)) = \psi_\infty(\omega)$.

The value of ψ_∞ depends on the realization of the government expenditure path. If the absorbing state is reached after many bad shocks (high g), the government will have accumulated high debt, and convergence occurs to a complete market economy with high initial debt. One can state sufficient conditions to guarantee that the absorbing state is reached with positive probability before the first-best is attained, so that $P(\psi_\infty < 0) > 0$. This will be the case, for example, if the initial level of debt is sufficiently high and if there is a positive probability of reaching the absorbing state in one period. But even with an absorbing state, a Markov process (P, π) can put a positive probability on an arbitrarily long sequence of random government expenditures that gives the government the time and incentive to accumulate enough assets to reach the first-best.

Therefore, in example 3, taxes always converge. It is easy to construct examples in which there is a positive probability of converging to a Ramsey (Lucas and Stokey) equilibrium with nonzero taxes. But if θ_t converges to zero, lemma 1 becomes silent about convergence of ψ_t and the Ramsey allocation under risk-free government debt.[18] So our next task is to say something about the asymptotic behavior of θ_t.

LEMMA 2. (*a*) $\{\theta_t\}$ is a nonnegative martingale. Therefore, $\theta_t \to \bar{\theta}$ almost surely for a random variable $\bar{\theta}$ that is finite with probability one. (*b*) Fix a realization ω. If $\theta_t(\omega) \to \bar{\theta}(\omega) > 0$, then $u_{c,t}(\omega)/\{E_{t-1}[u_{c,t}](\omega)\} \to 1$ as $t \to \infty$.

Proof. To prove part *a*,

$$E_t[\theta_{t+1}] = \theta_t E_t\left[\frac{u_{c,t+1}}{E_t[u_{c,t+1}]}\right] = \theta_t.$$

To prove part *b*, notice that if $\theta_t(\omega) \to \bar{\theta}(\omega) > 0$, then

$$\log \theta_t(\omega) = \sum_{\tau=1}^{t} \{\log u_{c,\tau}(\omega) - \log E_{\tau-1}[u_{c,\tau}(\omega)]\} \to \log \bar{\theta}(\omega) > -\infty$$

as $t \to \infty$. Convergence of this sum implies $\log u_{c,t}(\omega) - \log E_{t-1}[u_{c,t}(\omega)] \to 0$ and $u_{c,t}(\omega)/\{E_{t-1}[u_{c,t}](\omega)\} \to 1$ as $t \to \infty$. Q.E.D.

There are three interesting possibilities for the asymptotic behavior

[18] Note that Chamberlain and Wilson do not have many results for the case in which θ_t converges to zero, a possibility that they exclude by making the appropriate assumptions on their (exogenous) interest rate.

of the allocations under incomplete markets: (i) convergence to the first-best (as in example 2), (ii) convergence to a Lucas and Stokey equilibrium (as in example 3), and (iii) convergence to a stationary distribution (different from the distributions of cases i and ii). Part *a* of lemma 2 might appear to be a hopeful, positive result that will help us in discerning which of these cases occurs, since convergence of θ_t together with lemma 1 may allow us to conclude something about convergence of ψ_t. But corollary 1 shows that, in general, θ_t converges to zero under all the cases above, in which case lemma 1 is silent about convergence of the allocations.

COROLLARY 1. (*a*) If the allocation converges to a stationary distribution with $u_{c,t} \neq E_{t-1}[u_{c,t}]$ with positive probability, then $\theta_t \to 0$ almost surely. (*b*) If, for any multiplier $\gamma_0 > 0$, the complete markets Ramsey equilibria converge to a distribution such that $u_{c,t}^{\mathrm{CM}} \neq E_{t-1}[u_{c,t}^{\mathrm{CM}}]$ with positive probability, then $\theta_t \to 0$ almost surely.

Proof. Part *a*: In this case, $u_{c,t}/E_{t-1}[u_{c,t}]$ does not converge to one almost surely. Then the contrapositive of part *b* of lemma 2 implies that the probability that θ_t has a positive limit is equal to zero.

Part *b*: Consider a realization for which $\bar{\theta}(\omega) > 0$. Then lemma 1 implies that $\psi_t(\omega)$ converges, γ_t converges to zero, and the first-order conditions for optimality indicate that the Ramsey allocation converges to a complete markets equilibrium. Hence marginal utility converges to some complete market Ramsey equilibrium, under the assumption stated in part *b* $u_{c,t}/E_{t-1}[u_{c,t}]$ cannot converge to one, and the statement is implied by the contrapositive of part *b* of lemma 2. Q.E.D.

Notice that the conditions of part *b* of corollary 1 are satisfied if u has some curvature and g has persistent randomness. In example 2, u has insufficient curvature, and in example 3, g has insufficient randomness, so that is why convergence of ψ could occur in those cases.

One can interpret this corollary as saying that in the general case we are unable to use lemma 1 to determine the asymptotic behavior of the allocations. This is a negative conclusion, because it means that the martingale approach cannot be used in some important cases. For example, we could be interested in exploring the possibility that (c_t, b_t^g, g_t) converges to a stationary nondegenerate distribution. At this point we cannot say whether this is the case. But if this were the case, then part *a* of the corollary would imply that lemma 1 is silent, so the martingale approach could not be used. In Section V, we shall show that if part *b* applies, convergence to complete markets allocation is not a possibility.[19]

[19] There is a literature in finance stating conditions to guarantee that risk-adjusted martingales converge. But the case $\theta_t \to 0$ corresponds to the case in which the boundary conditions for existence of the equivalent measure used in that approach fail to hold, so that approach is also unavailable to study the limiting properties of the model. See Duffie

V. Another Nonconvergence Result

In Section IV, we discovered that the martingale approach is often inconclusive about the asymptotic behavior of the equilibrium. However, in example 3 the incomplete markets Ramsey allocation and tax policy converge to their complete markets counterparts. In this section, we explore whether the convergence in example 3 can be extended to more general government expenditure processes. It cannot. By working directly with the government budget constraints, under general conditions on the government expenditure process, we rule out convergence to the Ramsey equilibrium under complete markets (to be called the Lucas-Stokey equilibrium). Thus we strengthen the results of the last section by ruling out another type of convergence.

The budget constraint of the government without lump-sum transfers and for any debt limits can be rewritten as

$$b_t^g - b_{t-1}^g = \left(\frac{1}{p_t^b} - 1\right)\left[\frac{g_t - \tau_t(1 - x_t)}{1 - p_t^b} + b_{t-1}^g\right]. \tag{35}$$

Here $g_t - \tau_t(1 - x_t)$ is the net-of-interest or "primary deficit." Let $D(f, g_t) \equiv [g_t - \tau_t^f(1 - x_t^f)]/(1 - p_t^{b,f})$, where the f superscript denotes the Lucas-Stokey equilibrium with a multiplier $\gamma_0 = f$.

Definition 4. Given f, we say that $D(f, g_t)$ is *sufficiently random* if there exists an $\epsilon > 0$ such that, for t large enough and any constant K, either

$$P(D(f, g_j) > K + \epsilon \text{ for all } j = t, \ldots, t + k \mid g_{t-1}, \ldots, g_0) > 0 \tag{36}$$

or

$$P(D(f, g_j) < K - \epsilon \text{ for all } j = t, \ldots, t + k \mid g_{t-1}, \ldots, g_0) > 0 \tag{37}$$

for all $k > 0$ for almost all realizations.[20]

Clearly, $D(f, g_t)$ is *in*sufficiently random if g_t converges almost surely, as in example 3. But if g_t is stationary with positive variance, most utility functions imply that D is sufficiently random for all f.[21]

Notice that convergence of the incomplete markets allocation to the Lucas-Stokey equilibrium requires that ψ_t converges to a nonzero value and that the multipliers ν of the debt limits become zero. The following lemma shows that if there is sufficient randomness in D, the incomplete markets allocation cannot converge to a Lucas-Stokey allocation.

Lemma 3. Assume that the interest rate is bounded away from zero

(1996) for a precise description of the conditions that the equivalent measure approach requires.

[20] Notice that ϵ can depend on f, the t "large enough" can depend on ϵ, f, but they have to be uniform on K and k.

[21] For stationary g, insufficient randomness could occur only if the complete markets solution implied a constant D.

with probability one. Also, assume that the first-order conditions for optimality in the Ramsey problem (13) define a continuous function mapping $(\psi_t, \gamma_t, b^g_{t-1})$ to the endogenous variables (τ_t, x_t, p^b_t). Then

$$P(\omega : \psi_t(\omega) \to \varsigma(\omega) < 0 \text{ as } t \to \infty \text{ and}$$

$$D(\varsigma(\omega), g_t) \text{ sufficiently random}) = 0.$$

Furthermore, for a particular realization in which $\psi_t(\omega) \to \varsigma(\omega)$, we have $b^g_t(\omega) \to D(\varsigma(\omega), g_t)$.

Proof. Consider a realization ω such that $\psi_t(\omega) \to \varsigma(\omega) < 0$. In this case, $(\psi_t - \psi_{t-1})(\omega) \to 0$ and (13) implies that (τ_t, x_t, p^b_t) converge to the Lucas-Stokey equilibrium with Lagrange multiplier $\varsigma(\omega)$, and

$$\left| \frac{g_t - \tau_t(1 - x_t)}{1 - p^b_t}(\omega) - D(\varsigma(\omega), g_t) \right| \to 0.$$

Now if $D(\varsigma(\omega), g_t)$ is sufficiently random, there is an $\epsilon > 0$ (possibly dependent on $\varsigma(\omega)$) as in the definition of sufficient randomness. Since the endogenous variables converge to the Lucas-Stokey equilibrium with Lagrange multiplier $\varsigma(\omega)$, there is a t such that, for all $\bar{t} \geq t$, we have

$$\left| \frac{g_{\bar{t}} - \tau_{\bar{t}}(1 - x_{\bar{t}})}{1 - p^b_{\bar{t}}}(\omega) - D(\varsigma(\omega), g_{\bar{t}}(\omega)) \right| < \frac{\epsilon}{2}.$$

Now if $D(\varsigma(\omega), g_t)$ is sufficiently random, either (36) or (37) is satisfied. Let us say that for $K = -b^g_{t-1}$ it is (36) that occurs. Using equation (35), we have that with positive probability

$$\begin{aligned} b^g_{\bar{t}} - b^g_{\bar{t}-1} &> \left(\frac{1}{p^b_{\bar{t}}} - 1\right)\left[D(\varsigma(\omega), g_{\bar{t}}(\omega)) - \frac{\epsilon}{2} + b^g_{\bar{t}-1}\right] \\ &> \left(\frac{1}{p^b_{\bar{t}}} - 1\right)\left(\epsilon - b^g_{t-1} - \frac{\epsilon}{2} + b^g_{\bar{t}-1}\right) \end{aligned}$$

for all $\bar{t} \geq t$, where the first inequality follows from convergence to the Lucas-Stokey equilibrium and the second inequality from equation (36) for $K = -b^g_{t-1}$. This equation for $\bar{t} = t$ implies that $b^g_t - b^g_{t-1} > 0$ so that, by induction, $-b^g_{t-1} + b^g_{\bar{t}-1} > 0$ and

$$b^g_{\bar{t}} - b^g_{\bar{t}-1} > \left(\frac{1}{p^b_{\bar{t}}} - 1\right)\frac{\epsilon}{2}$$

for all $\bar{t} \geq t$. Since $1/p^b_{\bar{t}}$ is larger than, and bounded away from, one, this equation implies that the debt grows without bound and that the upper bound of debt would be violated with positive conditional probability. Similarly, if we had (37) holding for $K = -b^g_{t-1}$, the lower bound on debt would be violated. Therefore, with sufficient randomness of D, it is impossible for the allocation to converge to a Lucas-Stokey allocation. Q.E.D.

Summary

In general, with sufficient randomness we can rule out the example 3 outcome that the Ramsey allocation with only risk-free debt converges to a Ramsey allocation with state-contingent debt. But at least two interesting possibilities remain: ψ_t may have a nondegenerate distribution in the limit or it may converge to the first-best, as in example 2 under the natural asset limit.

To illustrate features of the model that we cannot tell analytically, next we describe simulations.

VI. Numerical Examples

Sections IV and V tell why it is generally difficult to characterize the Ramsey allocation for the incomplete markets economy for more general preferences than those for example 2. It is reasonable to emerge from Sections III, IV, and V with the prejudice that in the general case the allocation would exhibit behavior somehow between those of examples 2 and 3. The results in this section support that prejudice by presenting approximate Ramsey plans for both complete and incomplete markets economies with a serially independent government purchase process.

From the point of view of someone used to solving dynamic programming problems by discretizing the state space and iterating on the Bellman equation, obtaining numerical solutions of this model seems daunting. First of all, the solution is time-inconsistent, so that the policy function (as a function of the history of the states g_t) changes every period. Second, there are several endogenous continuous state variables, so that discretization is very costly computationally, and linear approximations are likely to be inexact. We approach the first issue by using the framework of recursive contracts to characterize the (time-inconsistent) optimal solution by a recursive dynamic Lagrangian problem with few state variables. As we argued in Section II, a sufficient set of state variables is $(g_t, b^g_{t-1}, \psi_{t-1})$. Then we can solve the first-order conditions by numerically approximating the law of motion with some continuous flexible functional form.[22]

[22] See Marcet et al. (1995) for a description of these and other computational details. (Their paper can be found at ftp://zia.stanford.edu/pub/sargent/webdocs/research/albert8.ps.) To approximate a solution, we apply the parameterized expectations algorithm of Marcet (1988). This approach is convenient since it avoids discretization of the state

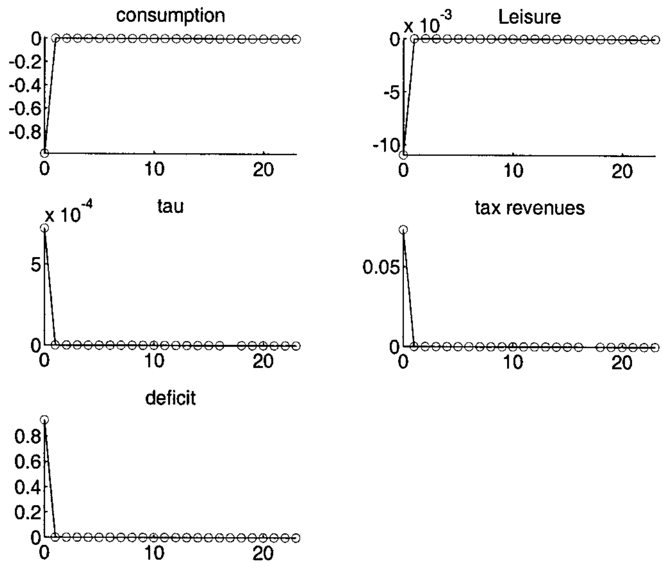

FIG. 3.—Impulse response functions for the complete markets economy, serially independent government purchases in the numerical example of Sec. VI. From left to right, top to bottom, are impulse response functions for consumption, leisure, tax rate, tax revenues, and the government deficit.

Parameters

We rescaled the feasibility constraint so that $c_t + x_t + g_t = 100$ and set government purchases to have mean 30. The stochastic process for g_t is

$$g_{t+1} = \bar{g} + \frac{\epsilon_{t+1}}{\alpha},$$

where ϵ_t is an i.i.d. sequence distributed $\mathcal{N}(0, 1)$, and α is a scale factor. Our utility function is

$$u(c, x) = \frac{c^{1-\sigma_1} - 1}{1 - \sigma_1} + \eta\left(\frac{x^{1-\sigma_2} - 1}{1 - \sigma_2}\right). \quad (38)$$

We set $(\beta, \sigma_1, \sigma_2, \eta) = (.95, .5, 2, 1)$, $(\bar{g}, \alpha, b_{-1}^{g}) = (30, .4, 0)$, and $(\underline{M}, \overline{M}) = (-1{,}000, 1{,}000)$.

For the complete markets Ramsey plan, figure 3 displays linear impulse response functions to the innovation in government expenditures.

variables, and in our problem we have at least two endogenous continuous state variables. A number of other approaches to solve this kind of first-order condition are also available in the literature.

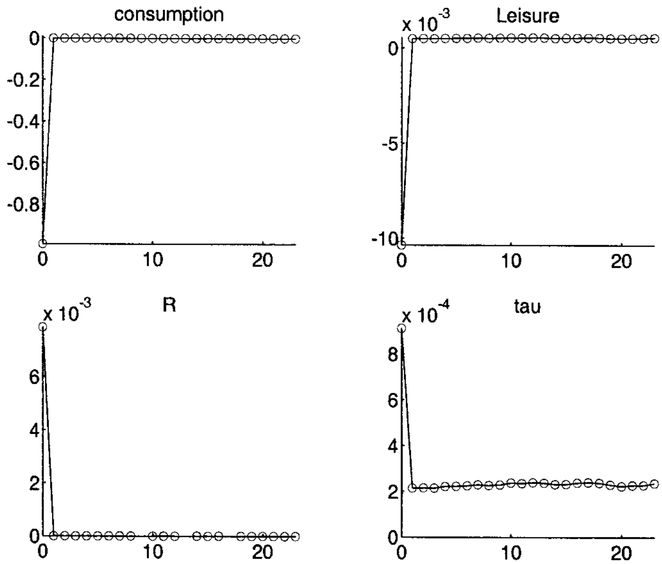

FIG. 4.—Impulse response functions for the incomplete markets economy, serially independent government purchases in the numerical example of Sec. VI. From left to right, top to bottom, are impulse responses of consumption, leisure, the gross real interest rate, and the tax rate.

The impulse responses confirm that every variable of interest inherits the serial correlation pattern of government purchases. We can estimate the variance of each variable by squaring the coefficient at zero lag and then multiplying by the innovation variance of g_t. Notice that the tax rate τ_t has very low variance, as indicated by its low zero-lag coefficient of about 7×10^{-4}. These impulse response functions tell us how extensively the government relies on the proceeds of the "insurance" it has purchased from the private sector. In particular, the net-of-interest deficit is about 93 percent of the innovation to government purchases. The deficit is covered by state-contingent payments from the private sector.

Figures 4 and 5 display linear impulse responses for the incomplete markets economy. The impulse response function for b_t^g shows what a good approximation it is to assert, as Barro did, that an innovation in government expenditures induces a permanent increase in debt. This contrasts sharply with the pattern under complete markets with serially independent g_t, for which an innovation in government expenditures has *no* effect on the present value of debt passed into future periods. Figure 5 shows that ψ_t is well approximated by a martingale. The impulse

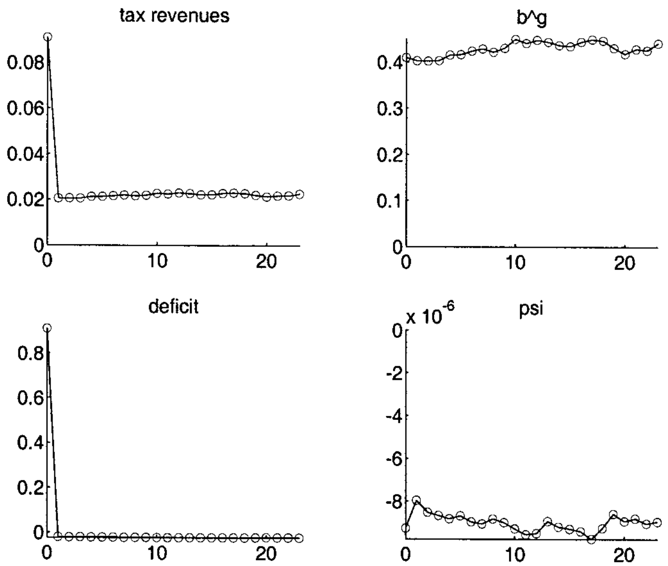

FIG. 5.—Impulse response function for the incomplete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse responses of tax revenues, the debt level b^g, the deficit, and the multiplier ψ_t.

response functions for the tax rate τ_t and tax revenues deviate from the "random walk" predicted by Barro mainly in their first-period responses. (A random walk would have a perfectly flat impulse response function.) These impulse response functions resemble a weighted sum of the random walk response predicted by Barro and the white-noise response predicted by Lucas and Stokey.[23]

Notice that the lag zero impulse coefficient for the tax rate is about one-fourth higher than for the complete markets case, so that the one-step-ahead prediction error variance is correspondingly higher. Because of the near–unit root behavior of the tax rate under incomplete markets, the j-step-ahead prediction error variance grows steadily with j, at least for a long while. The unconditional variance of tax rates under incomplete markets is therefore much higher than under complete markets.

Another way to see the difference between complete and incomplete markets is to compare autoregressions for tax rates. Table 1 presents

[23] The impulse response functions for tax rates and for tax revenues reveal that these variables are well approximated as univariate processes whose first differences are first-order moving averages.

TABLE 1
Autoregressions of the Tax Rate

	Complete Markets	Incomplete Markets
$E[\tau]$	.3108	.2776
std(τ)	.0018	.0191
a	.3125	.0031
b	$-.0054$	.9888
R^2	2.9128×10^{-5}	.9944

the first two unconditional moments for tax rates and the results from a least-squares regression

$$\tau_t = a + b\tau_{t-1} + \epsilon_t$$

for both economies, where ϵ_t is a least-squares residual that is orthogonal to τ_{t-1}. The enormous differences in b and R^2 are a testimony to the presence of a unit root component under incomplete markets.

Under complete markets the tax rate inherits the serial correlation properties of the exogenous shocks, and under incomplete markets tax rates have serial correlation coefficients near unity. Notice also that while taxes are, on average, lower under incomplete markets, they are also much more volatile.

Welfare Comparison

Despite differences of behavior for taxes, surpluses, and debts, the impulse response functions for consumption and leisure, respectively, in the complete and incomplete market economies are very close. The proximity of the impulse response functions for (c_t, x_t) implies proximity of the Ramsey allocations in the two economies. This is confirmed by some welfare calculations. We calculated the expected utility of the household to be 298.80 in the complete markets economy and 298.79 in the incomplete markets economy. In order to make the consumer indifferent between complete markets and incomplete markets, his consumption in the incomplete markets economy would have to be increased by only 0.0092 percent in all periods.[24] This comparison indicates the capacity of tax smoothing over time to substitute for tax smoothing across states.

[24] For similar pairs of economies with first-order autoregressive government expenditures with first-order autoregressive coefficient $\rho = .75$ and the same values of the other parameters, we calculated that indifference would be achieved by increasing consumption by 0.0409 percent.

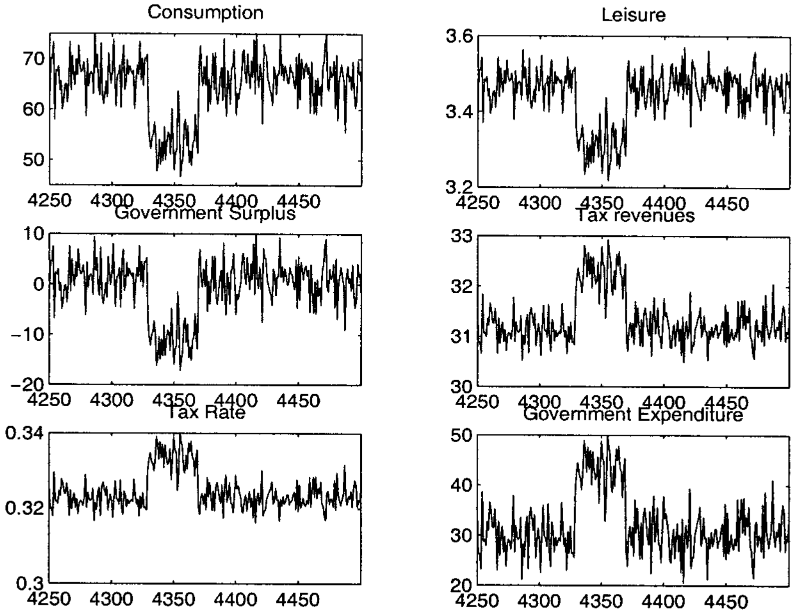

FIG. 6.—Simulation of peace and war economy with complete markets

War Finance under Complete and Incomplete Markets

We computed another example with regime-switching government expenditure shocks. Now the conditional mean of the government expenditure process follows a two-state Markov process. In particular, the stochastic process for g_t is

$$g_t = \bar{g}_t + \frac{\epsilon_t}{\alpha},$$

where ϵ_t continues to be i.i.d. $\mathcal{N}(0, 1)$, and $\bar{g}_t$ can have two different realizations, 30 and 42.5, corresponding to a peace state and a war state, respectively. We assume that both $\bar{g}_t$ and g_t are observed. The probability of remaining in peace next period given that the current state is peace is set to .99, and the probability of remaining in war next period given that the current state is war is set to .9. In other words, a large war happens with low probability (10 percent), but when it happens it lasts for some time (10 years). All other parameter values were set as above, except for $\alpha = .25$. We used the same algorithm as earlier in the section, except that now agents distinguish between peacetime and wartime.

Figures 6 and 7 illustrate the difference between complete and incomplete markets in war finance. They show simulations of two econ-

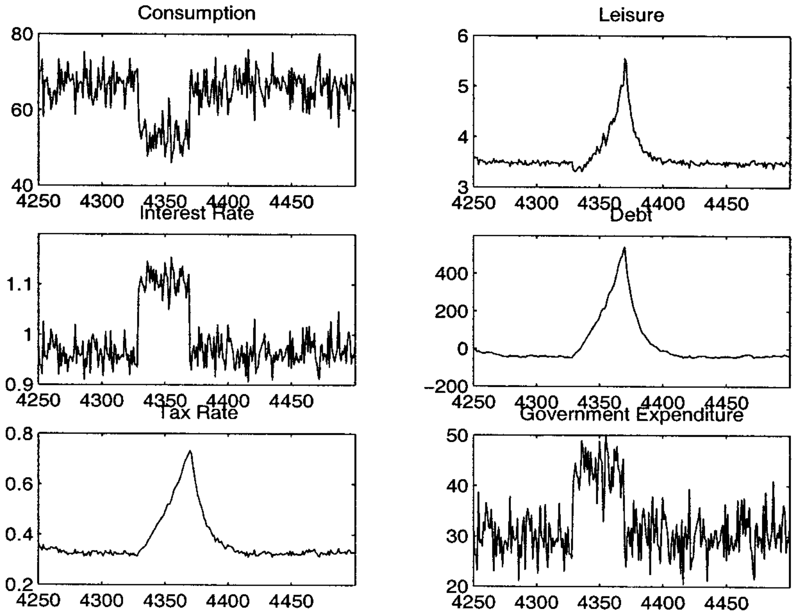

FIG. 7.—Simulation of peace and war economy with incomplete markets

omies with identical parameter values and government expenditure processes but different market structures. Under complete markets (fig. 6), when the economy goes to war, taxes are increased immediately as government expenditures rise. Similarly, when expenditures fall at the war's end, taxes decrease immediately. Notice that the actual tax increase is relatively small.

Under incomplete markets (fig. 7), during peacetime the government runs surpluses and lends to the consumers. War is financed by both considerable increases in taxes and borrowing from the public. Once the war ends, taxes are cut and the government debt is paid down at the same relatively fast pace.

The much higher persistence and variance of the government expenditure process make the welfare loss associated with incomplete markets higher than in the previous example. The expected utility of the household is 297.26 in the complete markets economy and 295.7 in the incomplete markets economy. To make the consumer indifferent between complete markets and incomplete markets, his consumption in the incomplete markets economy would have to be increased by 0.96 percent of his current consumption.

VII. Concluding Remarks

Lucas and Stokey (1983, p. 77) drew three lessons: (1) Budget balance in a present-value sense must be respected.[25] (2) No case can be made for budget balance on a continual basis. (3) State-contingent debt is an important feature of an optimal policy under complete markets.[26] Our results support lesson 1, amplify lesson 2, but may qualify lesson 3, depending on the persistence and variance of government purchases. For our first computed example, which has serially uncorrelated government expenditures, the welfare achieved by the incomplete markets Ramsey allocation is close to the complete markets Ramsey allocation, testimony to the efficacy of the incomplete market Ramsey policy's use of "self-insurance." The government uses debt as a buffer stock, just as savings allow smooth consumption in the "savings problem." For a general equilibrium version of a model whose residents all face versions of the savings problem, Krusell and Smith (1998) display incomplete markets allocations close to ones under complete markets.[27]

The analogy to the literature on the savings problem helps us to understand why our two computed examples differ in how close their Ramsey allocations are under complete and incomplete market structures. For a given random expenditure process, the proximity of the complete and incomplete markets Ramsey allocations will depend sensitively on (*a*) the persistence of the government expenditure process and the volatility of innovations to it, (*b*) the curvature of the household's utility function, and (*c*) the debt and asset limits set for the government.[28] More persistent government expenditure processes are more difficult for a government to self-insure, as our calculations for

[25] According to Keynes (1924, pp. 68–69), "What a government spends the public pays for."

[26] Lucas and Stokey write that "even those most skeptical about the efficacy of actual government policy may be led to wonder why governments forego gains in everyone's welfare by issuing only debt that purports to be a *certain* claim on future goods" (p. 77). Our computations do not diminish the relevance of this statement as a comment about the role of state-contingent debt in making possible a *debt structure* that renders their Ramsey tax policy time-consistent.

[27] Angeletos (2000) and Buera and Nicolini (2001) show how, if randomness has only finitely many possible outcomes and enough longer-term risk-free bonds are available, the Ramsey planner can implement Lucas and Stokey's allocation. The planner puts state-contingent fluctuations into the term structure of interest rates and exchanges longer- for shorter-term debt in order to duplicate the state-contingent payoffs on government debt required by Lucas and Stokey. Buera and Nicolini show that very large transactions can be required. Schmitt-Grohé and Uribe (2001) solve a Ramsey problem for an economy with sticky prices and a government that issues only one-period risk-free nominal debt. They compute very small welfare reductions from their market frictions (sticky prices and incomplete state-contingent debt).

[28] Thus, for their settings of other parameters, Krusell and Smith's allocations under complete and incomplete markets would be brought even closer together if they replaced the no-borrowing constraint they impose with the natural debt limits.

the war and peace economy illustrate, increasing the relevance of Lucas and Stokey's lesson 3 for highly persistent processes.

In affirming Barro's characterization of tax smoothing as imparting near–unit root components to tax rates and government debt, our incomplete markets model enlivens a view of eighteenth-century British fiscal outcomes as Ramsey outcomes. The time series of debt service and government expenditure for eighteenth-century Britain resemble a simulation of Barro's model or ours, not a complete markets model (see Sargent and Velde 1995, fig. 2).

Appendix

Proof of Proposition 1

First we show that the constraints (3), (4), and (6) imply (9) and (10). From (3) and the household's first-order conditions with respect to bonds, we have

$$s_t + \beta E_t\left[\frac{u_{c,t+1}}{u_{c,t}} b_t^g\right] = b_{t-1}^g.$$

Using forward substitution on b_t^g and also the law of iterated expectations, we have

$$E_t \sum_{j=0}^{T-1} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} + \beta^T E_t\left[\frac{u_{c,t+T}}{u_{c,t}} b_{t+T-1}^g\right] = b_{t-1}^g$$

for all T, which implies

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} = b_{t-1}^g.$$

Since, according to definition 1, b_{t-1}^g is known at $t-1$ and (4) is satisfied, the last equation implies that (9) and (10) are satisfied.

To prove the reverse implication, take any feasible allocation that satisfies (8), (9), and (10); we have

$$B_t \equiv s_t + E_t \sum_{j=1}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} \tag{A1}$$

$$= s_t + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} \frac{u_{c,t+1}}{u_{c,t}} \frac{u_{c,t+j}}{u_{c,t+1}} s_{t+j}. \tag{A2}$$

Applying the law of iterated expectations, we can condition the term inside E_t on information at $t+1$ to get

$$s_t + \beta E_t\left[\frac{u_{c,t+1}}{u_{c,t}} E_{t+1} \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+1+j}}{u_{c,t+1}} s_{t+j+1}\right] = s_t + \beta E_t\left[\frac{u_{c,t+1}}{u_{c,t}} B_{t+1}\right]$$

$$= s_t + \beta E_t\left[\frac{u_{c,t+1}}{u_{c,t}}\right] B_{t+1},$$

using (10) in the last equality. With formula (6) for bond prices, we have

$$B_t = s_t + p_t^b B_{t+1},$$

which guarantees that (3) and (4) are satisfied precisely for $b_{t-1}^g = B_t$. Q.E.D.

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