Optimal Taxation without State-Contingent Debt

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Abstract

To recover a version of Barro's (1979) 'random walk' tax smoothing outcome, we modify Lucas and Stokey's (1983) economy to permit only risk-free debt. This imparts near unit root like behavior to government debt, independently of the government expenditure process, a realistic outcome in the spirit of Barro's. We show how the risk-free-debt-only economy confronts the Ramsey planner with additional constraints on equilibrium allocations that take the form of a sequence of measurability conditions. We solve the Ramsey problem by formulating it in terms of a Lagrangian, and applying a Parameterized Expectations Algorithm to the associated first-order conditions. The first-order conditions and numerical impulse response functions partially affirm Barro's random walk outcome. Though the behaviors of tax rates, government surpluses, and government debts differ, allocations are very close for computed Ramsey policies across incomplete and complete markets economies.

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Introduction

This paper computes a Ramsey plan for fiscal policy in a representative household economy without capital, a time-varying flat rate tax on labor income, and only risk-free debt. Our model of optimal debt and taxes is 'in between' ones studied by Barro (1979) and Lucas and Stokey (1983). Barro embraced an analogy with a permanent income model of consumption to conclude that debt and taxes should follow random walks.[1] Lucas and Stokey studied a Ramsey problem for an equilibrium model with complete markets, no capital, exogenous Markov government expenditures, and state contingent taxes and government debt. Their Ramsey plan puts serial correlation of the tax rate close to that for the government expenditure process, contrary to the striking result of Barro that the serial correlation of taxes is independent of the government expenditure process.

A conjecture circulates that results closer to Barro's would emerge in a model that eliminates complete markets and permits only risk-free borrowing.[2] An impediment to evaluating this conjecture has been that the optimal taxation problem with only risk-free borrowing is difficult because complicated additional constraints restrict competitive allocations (see Chari, Christiano, and Kehoe (1995, p.366)).[3]

To focus on the above conjecture, we modify Lucas and Stokey's environment by letting the government issue or purchase only risk-free one period debt. We show how the restriction to risk-free borrowing imposes a *sequence* of measurability constraints *in addition*

[1] Hansen, Roberds, and Sargent (1991) describe the testable implications of various models like Barro's.

[2] We heard this conjecture from V. V. Chari and Nancy Stokey.

[3] Our interest in formalizing the conjecture about Barro's model originates partly from historical episodes which seem to pit Barro's model against Lucas and Stokey's. For example, see the descriptions of French and British 18th century public finance cited in Sargent and Velde (1995). Barro's model well accounts for many (but not all) broad features of Britain's deficit and debt policy, with its aversion to defaults; Lucas and Stokey's model does better at explaining France's behavior, with its recurrent defaults (occasionally low state-contingent payoffs?).

to a single implementability constraint required under complete markets. We formulate a Lagrangian for the Ramsey problem by which the measurability constraints introduce a Lagrange multiplier process and a state variable (the government debt level) that would disappear under complete markets. First order conditions associated with the saddle-point of this Lagrangian form a system of expectational difference equations that we solve numerically to illustrate features of the Ramsey outcome.

The incomplete debt market substantially alters the serial correlation of Ramsey taxes relative to Lucas and Stokey's model, separating it from that of government expenditures and moving the outcome toward that asserted by Barro. Impulse response functions of taxes with respect to innovations in government expenditures resemble a sum of the impulse response in the Lucas-Stokey model (i.e., a piece proportional to the impulse response of government expenditures) and the impulse response in Barro's model (a flat, unit-root-like impulse response).

Throughout this paper, we assume that the government has a commitment mechanism that binds it to the Ramsey plan. Therefore, we say nothing about Lucas and Stokey's discussions of time consistency and the structure of government debt.

Physical Setup

Technology and preferences are those specified by Lucas and Stokey. Let c_t, x_t, g_t denote consumption, leisure, and government purchases at time t. The technology is

$$c_t + x_t + g_t = 1. \tag{1}$$

Government purchases g_t follow a Markov process, with transition density $P(g'|g)$. A representative household ranks consumption streams according to

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, x_t), \tag{2}$$

where $\beta \in (0, 1)$, and E_0 denotes the mathematical expectation conditioned on time 0 information.

The government raises all revenues through a time-varying flat rate tax τ_t on labor at time t. Households and the government make decisions whose time t components are functions of the history of government expenditures $g^t = (g_t, g_{t-1}, \ldots, g_0)$, and of initial government indebtedness b^g_{-1}.

Incomplete Markets with Debt Limits

Let $\omega_t \equiv \tau_t(1-x_t)-g_t$ denote the time t net-of-interest government surplus. Households and the government borrow and lend only in the form of risk-free one-period debt. The government's budget and debt limit constraints are:

$$b^g_{t-1} = \omega_t + p^b_t b^g_t, \quad t \geq 0 \tag{3}$$

$$\underline{M} \leq b^g_t \leq \overline{M}, \quad t \geq 0. \tag{4}$$

Here p^b_t is the price of a risk-free bond paying one unit of consumption in period $t+1$, and b^g_t is the market value of the stock of government bonds issued at t, which pay off for sure at $t+1$. The debt limits in (4) play their routine role in incomplete markets settings.

The household's problem is to maximize (2) subject to the sequence of budget constraints

$$p^b_t b^g_t + c_t \leq (1-\tau_t)(1-x_t) + b^g_{t-1}, \; t \geq 0, \tag{5}$$

where b^g_t here denotes the household's holdings of government debt. The household also faces debt limits analogous to (4), which we assume are less stringent (in both directions) than those faced by the government. Therefore, in equilibrium, the household's problem always has an interior solution. The household's first-order conditions require that the price of risk-free debt satisfies

$$p^b_t = E_t \beta \frac{u_{c,t+1}}{u_{c,t}}, \forall t \geq 0, \tag{6}$$

and that taxes satisfy

$$\frac{u_{x,t}}{u_{c,t}(1-\tau_t)} = 1. \tag{7}$$

We use the following definitions.

DEFINITIONS I: A feasible *allocation* is a stochastic process $\{c_t, x_t, g_t\}$ satisfying (1). A *bond price process* is a stochastic process $\{p_t^b\}$ whose time t element is measurable with respect to $\{g^t, b_{-1}^g\}$. Given b_{-1}^g and a stochastic process $\{g_t\}$, a *government policy* is a stochastic process for $\{\tau_t, b_t^g\}$ whose time t element is measurable with respect to (g^t, b_{-1}^g).

DEFINITION II: Given b_{-1}^g and a stochastic process $\{g_t\}$, a *competitive equilibrium* is an allocation, a government policy, and a bond price process that solves the household's optimization problem and that satisfies the government's budget constraints (3) and (4).

DEFINITION III: The *Ramsey problem* is to maximize (2) over competitive equilibria. A *Ramsey outcome* is a competitive equilibrium that attains the maximum of (2).

We use a standard strategy of casting the Ramsey problem in terms of a constrained choice of allocation. We use (6) and (7) to eliminate asset prices and taxes from the government's budget and debt constraints, and thereby deduce sequences of restrictions on the government's allocation in *any* competitive equilibrium with incomplete markets. We shall establish the remarkable feature that incomplete markets competitive allocations must satisfy Lucas and Stokey's complete markets restriction on allocations, and others besides. From now on, we use (7) to represent the government surplus in terms of the allocation as $\omega_t \equiv \omega(c_t, g_t) = (1 - u_{x,t}/u_{c,t})(c_t + g_t) - g_t$.

The following proposition characterizes the restrictions that the government's budget and competitive behavior of households place on competitive equilibrium allocations:[4]

EQUILIBRIUM ALLOCATIONS PROPOSITION: Given b_{-1}^g, a stochastic process for $\{c_t, g_t, x_t\}$ satisfies (3), (4), (7), (6) if and only if the following constraints are satisfied:

[4] This proposition extends remarks of Chari, Christiano, and Kehoe (1995, p. 366), by reformulating the measurability constraint from a 'difference equation' to an 'isoperimetric' form.

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \omega_t = b_{-1}^g, \tag{8}$$

$$\underline{M} \le E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} \le \overline{M} \text{ for all } t \tag{9}$$

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} \text{ is measurable with respect to information known at time } t-1. \tag{10}$$

Proof: First we show that the constraints (3), (4), and (6) imply (9) and (10). From (3) and the household's first-order conditions with respect to bonds we have

$$\omega_t + \beta \ E_t \left(\frac{u_{c,t+1}}{u_{c,t}} \ b_t^g \right) = b_{t-1}^g.$$

Using forward substitution on b_t^g and also the law of iterated expectations, we have

$$E_t \sum_{j=0}^{T-1} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} + \beta^T \ E_t \left(\frac{u_{c,t+T}}{u_{c,t}} \ b_{t+T-1}^g \right) = b_{t-1}^g,$$

for all T, which implies

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j} = b_{t-1}^g,$$

Since according to Definition I, b_{t-1}^g is known at $t-1$ and (4) is satisfied, the last equation implies that (9) and (10) are satisfied.

To prove the reverse implication, we have

$$B_t \equiv \omega_t + E_t \sum_{j=1}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j}$$

$$= \omega_t + \beta E_t \sum_{j=1}^{\infty} \beta^{j-1} \frac{u_{c,t+1}}{u_{c,t}} \ \frac{u_{c,t+j}}{u_{c,t+1}} \omega_{t+j}.$$

Applying the law of iterated expectations, we can condition the term inside E_t on information at $t+1$ to get

$$\omega_t + \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} \ E_{t+1} \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+1+j}}{u_{c,t+1}} \omega_{t+j+1} \right]$$

$$= \omega_t + \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} \ B_{t+1} \right] = \omega_t + \beta E_t \left(\frac{u_{c,t+1}}{u_{c,t}} \right) B_{t+1},$$

using (10) in the last line. With formula (6) for bond prices we have:

$$B_t = \omega_t + p_t^b \ B_{t+1},$$

which guarantees that (3) and (4) are satisfied precisely for $b_{t-1}^g = B_t$. ∎

In the complete markets setting of Lucas and Stokey, (8) is the *sole* 'implementability' condition that government budget balance and competitive household behavior impose on the equilibrium allocation. The incomplete markets setup leaves this restriction intact, but adds two *sequences* of constraints. Constraint (10) requires the allocation to be such that at each date $t \geq 0$, B_t, the present value of the surplus (evaluated at date t Arrow-Debreu prices), be known one period ahead.[5], [6] Condition (9) requires that the debt constraints be respected. Condition (8) is but the time 0 version of constraint (10).

The Ramsey problem with incomplete markets has been called a 'computationally difficult exercise' (Chari, Christiano, and Kehoe (1995, p. 366)) because imposing the sequence of measurability constraints (10) seems daunting. We approach this task by composing a Lagrangian for the Ramsey problem, and attaching a Lagrange multiplier to each measurability constraint. We use the convention that variables dated t are measurable with respect to the history of shocks up to t. We incorporate condition (10) by writing it as $b_{t-1}^g = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} \omega_{t+j}$, multiplying it by u_{ct}, and attaching a Lagrange multiplier

[5] There is a parallel 'constraint' in the complete markets case, since B_t needs to be measurable with respect to information at t in that case. But this constraint is trivially satisfied by the definition of $E_t(\cdot)$.

[6] This proposition is reminiscent of Duffie and Shafer's (1985) characterization of incomplete markets equilibrium in terms of 'effective equilibria' that, relative to complete markets allocations, require next-period allocations to lie in subspaces determined by the menu of assets. In particular, see the argument leading to Proposition 1 in Duffie (1992, p. 216-217).

$\beta^t \gamma_t$ to the resulting time t component. Then the Lagrangian for the Ramsey problem can be represented, after applying the law of iterated expectations and summation by parts, as

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, 1 - c_t - g_t) - \psi_t u_{c,t} \omega_t + u_{c,t}(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b^g_{t-1}) \right\} \tag{11}$$

where

$$\psi_t = \psi_{t-1} + \nu_{1t} - \nu_{2t} + \gamma_t, \tag{12}$$

where γ_t can be positive or negative and $\psi_{-1} = 0$. The Ramsey problem under complete markets is the special case in which $\gamma_{t+1} = \nu_{1t} = \nu_{2t} \equiv 0 \ \forall t \geq 0$, and γ_0 is the (scalar) multiplier on the time 0 present value government budget constraint: these specifications imply that $\psi_t = \psi_0 = \gamma_0$. Relative to the complete markets case, the incomplete markets case augments the Lagrangian with the appearances of $b^g_{t-1}, \gamma_t, \ \forall t \geq 1$, and $\underline{M}, \overline{M}$ in the Lagrangian, and the effects of $\gamma_t, \nu_{1t}, \nu_{2t}$ on ψ_t in (12).

It is well known that the Ramsey problem is not recursive: because future control variables appear in the measurability constraints, the Bellman equation does not hold and the optimal choice at time t is *not* a time invariant function of the natural state variables, namely, (b^g_{t-1}, g_t). Nevertheless, the Lagrangian in (11) and the constraint (12) suggest that a recursive formulation can be recovered if ψ_{t-1} is included in the state variables. Indeed, the 'recursive contracts' approach described in Appendix B can be used to show formally that the optimal choice at time t is a time invariant function of state variables $(\psi_{t-1}, b^g_{t-1}, g_t)$. This observation motivates the simulations below.

We want to investigate how far these steps move us in the direction of Barro's tax smoothing outcome. For $t \geq 1$, the first-order condition with respect to c_t can be expressed as

$$u_{c,t} - u_{x,t} - \psi_t \kappa_t + (u_{cc,t} - u_{cx,t})(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b^g_{t-1}) = 0, \tag{13}$$

where[7]

$$\kappa_t = (u_{cc,t} - u_{cx,t})\omega_t + u_{ct}\omega_{c,t}. \tag{14}$$

[7] In the definition of κ_t, it is understood that total differentiation of the function $u = u(c, 1 - c - g)$ with respect to c is occurring. Evidently, $\kappa_t = (u_{ct} - u_{xt}) + c_t(u_{cc,t} - 2u_{cx,t} + u_{xx,t}) + g_t(u_{xx,t} - u_{cx,t})$.

It is useful to study this condition under both complete and incomplete markets.

Complete Markets

With complete markets, $\nu_{1t} = \nu_{2t} = \gamma_{t+1} = 0 \ \forall t \geq 0$ causes (13) to collapse to

$$u_{c,t} - u_{x,t} - \gamma_0 \kappa_t = 0, \tag{15}$$

which is a version of Lucas and Stokey's condition (2.9) for $t \geq 1$. From its definition (14), κ_t depends only on the level of government purchases at t. Therefore, given the multiplier γ_0, (15) determines the allocation and associated tax rate τ_t as a function only of the 'natural' time t state variable g_t. The *sole* intertemporal link is through the requirement that γ_0 must take a value to assure the time 0 present value government budget constraint. Equation (15) implies that, to a linear approximation, τ_t and all other endogenous variables mirror the serial correlation properties of the g_t process. The 'tax-smoothing' that occurs in this complete markets model is 'across states' and is reflected in the variability of tax rates and revenues relative to government purchases, but *not* in any propagation mechanism imparting more pronounced serial correlation to tax rates than to government purchases. Evidently, in the complete markets model, the government debt defined as B_t above also inherits its serial correlation properties entirely from g_t.

Incomplete Markets

In the incomplete markets case, ψ_t changes (permanently) each period because γ_t is non-zero in all periods; being of either sign, γ_t causes ψ_t either to increase or to decrease. When a debt limit doesn't bind, we can show that the multiplier ψ_t is a risk-adjusted martingale, imparting a unit-root component to the solution of (13). Taking the derivative of the Lagrangian with respect to b_t^g we get

$$E_t[u_{c,t+1}\gamma_{t+1}] = 0.$$

This implies that γ_t can be positive or negative, and that ψ_t can rise or fall in the steady state. Assuming that the debt constraints don't bind and using (12) gives

$$\psi_t = (E_t[u_{c,t+1}])^{-1} E_t[u_{c,t+1}\psi_{t+1}].$$

Using the definition of conditional covariance, the above equation can be further decomposed as

$$\psi_t = E_t[\psi_{t+1}] + (E_t[u_{c,t+1}])^{-1}\text{cov}_t[u_{c,t+1}, \psi_{t+1}].$$

Notice the similarity of this equation with the expression for the generalized version of the pure expectations theory of the term structure presented in Sargent (1987, ch. 3). Like the forward price in the generalized version of the pure expectations theory, ψ_t is a martingale adjusted for the risk premium $(E_t[u_{c,t+1}])^{-1}\text{cov}_t[u_{c,t+1}, \psi_{t+1}]$. Equation (13) shows that this 'approximate' martingale result is not precisely Barro's: (13) makes $u_{c,t}\tau_t$ depend also on $\gamma_t b^g_{t-1}$, and so distorts the pure martingale outcome.

Serial Correlation of taxes and market completeness

There is an irony here, exhibited by considering two cases: the complete markets case of Lucas and Stokey and the case of a balanced budget (i.e., $b^g_t = 0$ for all t). These are polar cases because, in the first instance, the possibilities for intertemporal and inter-state smoothing of taxes are as large as possible, while in the second instance there is no room for intertemporal smoothing by borrowing or lending. In both cases, the tax rate is a function of g_t only. Though it lies between these polar cases, our incomplete markets model imparts an additional martingale-like component to the dynamics.

Reason for computations

So far, we have shown that the optimal tax is determined jointly by g_t, b^g_{t-1} (when the third derivative of u with respect to c doesn't vanish), and by a state variable that has a martingale-like behavior (namely ψ_t). Dependence on g_t reproduces the effects discussed by Lucas and Stokey; dependence on ψ_t impels a martingale component, like Barro's. It is impossible to determine which effect dominates at this level of generality; for this purpose, we resort to numerical simulations in the next section.

Parameterized Expectations Algorithm

We shall describe computed solutions of both complete and incomplete markets economies with a serially independent government purchase process. We have also computed solutions for serially correlated government purchase processes, but focus on serially independent government purchases because of the clear and sharply different implications this case carries for the serial correlation of tax rates under the different market assumptions.

Parameterizations

In the computations, we rescaled the feasibility constraint so that $c_t + x_t + g_t = 100$, and set government purchases to have mean 30. The stochastic process for g_t is as follows

$$g_{t+1} = (1-\rho)\overline{g} + \rho g_t + \frac{\epsilon_{t+1}}{\alpha},$$

where ϵ_t is an independently and identically distributed sequence distributed $\mathcal{N}(0,1)$, and α is a scale factor. Our utility function is

$$u(c,x) = \frac{c^{1-\sigma_1}-1}{1-\sigma_1} + \eta\left(\frac{x^{1-\sigma_2}-1}{1-\sigma_2}\right). \tag{16}$$

Let $z_t \equiv [\psi_{t-1}, b^g_{t-1}, g_t]'$ be the *state* of the economy, and $y_t \equiv [c_t, x_t, \tau_t, p^b_t, \nu_{1t}, \nu_{2t}]'$ be a vector of endogenous variables depending on the state. For the incomplete markets economy, the Ramsey plan evidently has a non-linear state-space representation

$$\begin{aligned} \begin{bmatrix} \psi_t \\ b^g_t \end{bmatrix} &= h(z_t) \\ y_t &= k(z_t), \end{aligned} \tag{17}$$

where $h(z_t), k(\cdot)$ are the policy and transition functions that we want to compute and describe. To approximate a solution, we apply the parameterized expectations algorithm (PEA) of Marcet (1988). We use a complete markets case as a benchmark against which to measure the incomplete markets version.

Complete markets economy

We start with a given b_{-1}^g and a realization for $\{g_t\}_{t=0}^T$. For a fixed γ_0, we can solve (15) and (1) for a realization of an allocation. We can then generate a large number N realizations of $\sum_{t=0}^T \beta^t \frac{u_{c,t}}{u_{c,0}}\omega_t$, average across the N realizations to estimate $E_0 \sum_{t=0}^T \beta^t \frac{u_{c,t}}{u_{c,0}}\omega_t$, then check whether (8) is satisfied for this γ_0. If not, we adjust γ_0 and iterate until (8) is satisfied for our fixed b_{-1}^g.

Incomplete markets economy

For our incomplete markets formulation, we can generate a long pseudo-random realization for $(c_t, \psi_t, \gamma_t, b_t^g, \nu_{1t}, \nu_{2t})$ that satisfy the first-order conditions associated with the Lagrangian for the Ramsey problems. For convenience, we summarize, with some slight rewriting of some of the equations, in the following system:

$$\omega_t + b_t^g \beta \frac{E_t(u_{c,t+1})}{u_{c,t}} = b_{t-1}^g \tag{18}$$

$$u_{c,t} - u_{xt} - \psi_t \kappa_t + (u_{cc,t} - u_{cx,t})(\nu_{1t}\overline{M} - \nu_{2t}\underline{M} + \gamma_t b_{t-1}^g) = 0 \tag{19}$$

$$E_t\Big(u_{c,t+1}(\psi_{t+1} - \nu_{1t+1} + \nu_{2t+1})\Big) = E_t(u_{c,t+1})\psi_t \tag{20}$$

$$\psi_t = \psi_{t-1} + \gamma_t + \nu_{1t} - \nu_{2t} \tag{21}$$

$$(b_t^g - \overline{M})\nu_{1t} = (\underline{M} - b_t^g)\nu_{2t} = 0 \tag{22}$$

$$\nu_{1t}, \nu_{2t} \geq 0 \tag{23}$$

$$(b_t^g - \overline{M}), (\underline{M} - b_t^g) \leq 0 \tag{24}$$

The structure of these equations suggests that it is natural to parameterize the two functions $E_t(u_{c,t+1})$ and $E_t[u_{c,t+1}(\psi_{t+1} - \nu_{1t+1} + \nu_{2t+1})]$. With parameterizations in hand, it is straightforward to generate simulations that satisfy versions of these equations, given particular parameter settings for these two functions. For time periods t when the constraints on debt don't bind, we can find ψ_t from (20), then set $\nu_{1t} = \nu_{2t} = 0$, find γ_t from (21), c_t from (19), and b_t^g from (18). If b_t^g threatens to violate the upper bound, we can set $b_t^g = \overline{M}$, find consumption from (18), γ_t, ν_{1t} from (19) and (21), set $\nu_{2t} = 0$. The treatment is symmetric at the other bound.

In implementing our calculations, we use several transformations of variables, which bear interpretations in terms of an alternative formulation of the Ramsey problem. Details are described in Appendix A.

Description

We shall describe our results in two ways. First, we summarize the linear structure of the dynamics by computing moving average representations for elements of y_t and z_t. We can construct this representation by imitating Wold and applying regression to the orthogonal basis for g_t formed by the history of ϵ_t's. For a long simulation, we regress y_t against $\{\epsilon_{t-s}\}_{s=1}^{L}$ for large L, thereby estimating the impulse response function for y_t. Second, we shall display cross-sections of the Ramsey policy functions $h(\cdot), k(\cdot)$ in (17).

Serially uncorrelated government purchases

The case in which government expenditures are serially independent provides a good laboratory for bringing out the implications of allowing or prohibiting state-contingent debt. With complete markets, the one-period state contingent debt falling due at t satisfies $B_t(g_t) = \omega_t + \beta E_t[\frac{u_{c,t+1}}{u_{c,t}} B_{t+1}(g_{t+1})]$. With a serially independent g_t process, the expectation conditional on g_t equals an unconditional expectation, implying

$$B_t(g_t) = \omega_t + \frac{\beta E u_c B}{u_{c,t}}, \tag{25}$$

where $E u_c B = \frac{E u_c \omega}{1-\beta}$. Equation (25) states that the gross payoff on government debt equals a constant plus the time t surplus, which is serially uncorrelated. The time t value of the state contingent debt with which the government leaves period t is a constant, in marginal utility units, namely, $\frac{\beta E u_c B}{u_{c,t}}$, which is evidently uncorrelated with the level of government expenditures. There is no propagation mechanism from government expenditures to debt.

With incomplete markets, the situation is very different. Government debt evolves according to

$$B_{t+1} = R_t[B_t - \omega_t], \tag{26}$$

where $R_t \equiv (p_t^b)^{-1}$; recall that B_{t+1} is denominated in units of time $t+1$ consumption goods. Since the gross real interest rate is a random variable exceeding one, this equation describes a propagation mechanism by which even a serially independent government surplus process ω_t would impart close to unit root behavior to the debt level. Of course, even with serially independent government expenditures, the absence of complete markets causes the surplus process itself to be serially correlated, as described above.

Numerical Results

We use the parameter values $(\beta, \sigma_1, \sigma_2, \eta) = (.95, .5, 2, 1)$ and $(\overline{g}, \rho, \alpha, b^g_{-1}) = (30, 0, .4, 0)$. We set the debt limits at $(\underline{M}, \overline{M}) = (-1000, 1000)$.[8]

Complete markets

Figure 1 displays impulse response functions with the above parameter settings. The figures confirm that every variable of interest inherits the serial correlation pattern of government purchases. We can estimate the variance of each variable by squaring the coefficient at zero lag, then multiplying by the innovation variance of g_t. Notice that the tax rate τ_t has very low variance, as indicated by its low zero-lag coefficient of about 7×10^{-4}. These impulse response functions tell us how extensively the government relies on the proceeds of the 'insurance' it has purchased from the private sector. In particular, the net-of-interest deficit is about 93 percent of the innovation to government purchases. The deficit is covered by state-contingent payments from the private sector.

[8] Because we used the parameterization described in the appendix, we actually applied the debt limits to $p_t^b b_t^g$ rather than to b_t^g.

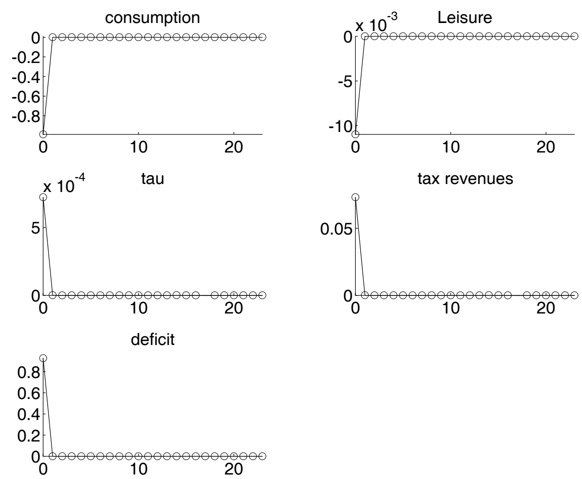

Figure 1. Impulse response functions for complete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse response functions for consumption, leisure, tax rate, tax revenues, and the government deficit.

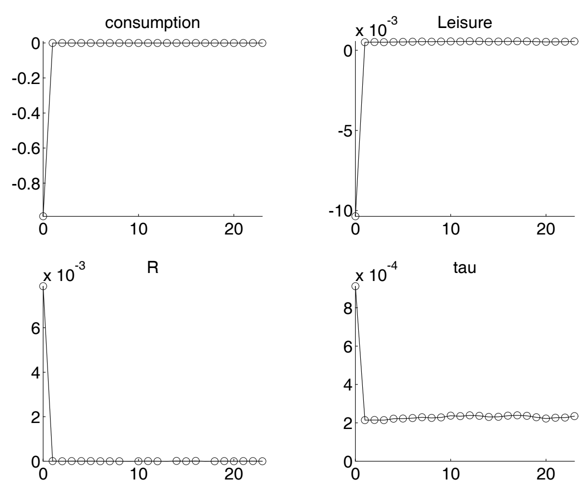

Figure 2a. Impulse response functions for incomplete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse responses of consumption, leisure, the gross real interest rate, and the tax rate.

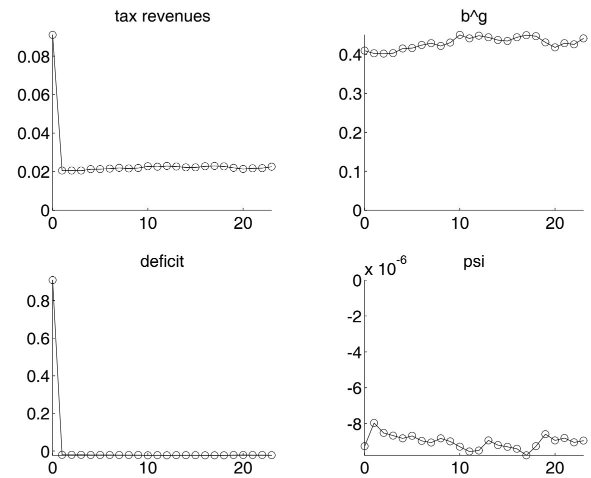

Figure 2b. Impulse response function for incomplete markets economy, serially independent government purchases. From left to right, top to bottom, are impulse responses of tax revenues, the debt level b^g, the deficit, and the multiplier ψ_t.

Incomplete Markets

Figures 2 through 9 display aspects of the results for the incomplete markets economy. The impulse response function for b_t^g in figure 2 shows what a good approximation it is to assert, as Barro did, that an innovation in government expenditures induces a permanent increase in debt. This contrasts sharply with the pattern under complete markets with serially independent g_t, for which an innovation in government expenditures has *no* effect on the present value of debt passed into future periods. Figure 2 shows that ψ_t is close to a martingale. The impulse response functions for the tax rate τ_t and tax revenues deviate from the 'random walk' predicted by Barro only in their first-period responses. (Barro's random walk prediction states that these would be perfectly flat, reflecting purely a unit root.) These impulse response functions resemble a weighted sum of the random walk response predicted by Barro and the white noise response predicted by Lucas and

Stokey.[9]

Notice that the lag zero impulse coefficient for the tax rate is about 1/4 higher than for the complete markets case, so that the one-step ahead prediction error variance is correspondingly higher. Because of the near-unit root behavior of the tax rate under incomplete markets, the j-step ahead prediction error variance grows steadily with j, at least for a long while. The unconditional variance of tax rates under incomplete markets is therefore much higher than under complete markets.

Figure 3 shows the histogram of b_t^g for a simulation of length 10,000. Notice how much of the time the debt level is *negative*. Evidently, the government typically uses an accumulated stock of claims on the public to 'self-insure'. Notice that neither bound on the debt was attained in our sample.[10]

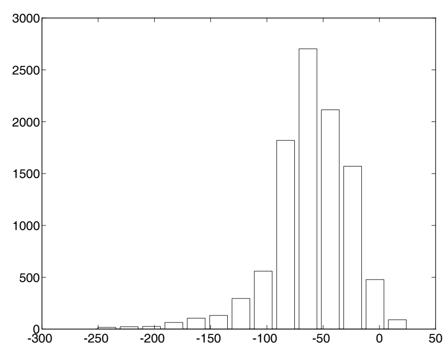

Figure 3. Histogram of debt levels, serially independent government purchases.

[9] The impulse response functions for tax rates and for tax revenues reveal that these variables are well approximated as univariate processes whose first differences are first order moving averages.

[10] The analysis of Magill and Quinzii (1994) prompts us to conjecture that if the debt limit band $(\underline{M}, \overline{M})$ is widened without limit, the debt limits in the Ramsey equilibrium will never bind. To attain a solution in which the debt bounds did not bind in the sample, we computed a sequence of equilibria with wider and wider bounds. While suggestive, Magill and Quinzii's analysis does not apply without modification to our Ramsey problem.

Welfare comparison

Despite differences of behaviors for taxes, surpluses, and debts, the impulse response functions are for consumption and leisure, respectively, in the complete and incomplete market economies (Figures 1 and 2) are very close. The proximity of the impulse response functions for (c_t, x_t) implies proximity of the Ramsey allocations in the two economies. This is confirmed by welfare calculations. We calculated the expected utility of the household to be 298.80 in the complete markets economy and 298.79 in the incomplete markets economy.[11] This comparison indicates the capacity of tax-smoothing over time to substitute for tax-smoothing across states.

Decision Rules

Figures 4 through 8 display various cross-sections of the decision rules for the incomplete markets economy. Figure 4 displays decisions and various endogenous variables as functions of b^g, holding g_t fixed at its mean of 30, and setting ψ at its mean value conditional on b_g.[12] Figure 5 through 8 display decisions and various endogenous variables as functions of g_t for various levels of b^g.

The decision rules are mostly linear to a good approximation, except for the curvature displayed by ψ and R.

---

[11] For pairs of economies with $\rho = .75$, we calculated expected utilities 299.04 and 298.97.

[12] It happens that ψ and b^g are very highly correlated, so that g_t, b_t^g are 'nearly' the entire state.

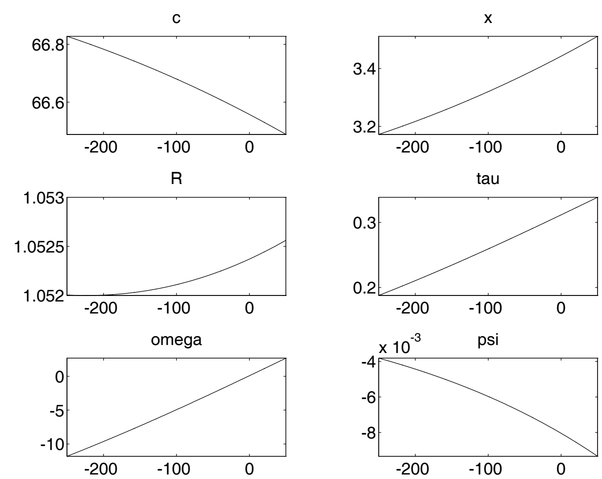

Figure 4. Various variables as functions of b^g, with $g = 30$. From left to right, top to bottom, the variables portrayed are, respectively, consumption, leisure, the gross real interest rate, the tax rate, the government surplus, and the multiplier ψ.

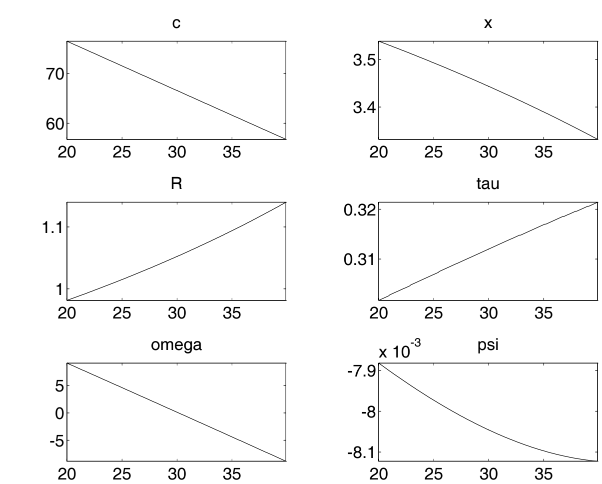

Figure 5. Various variables as functions of g, with $b^g = 0$. From left to right, top to bottom, the variables portrayed are, respectively, consumption, leisure, the gross real interest rate, the tax rate, the government surplus, and the multiplier ψ.

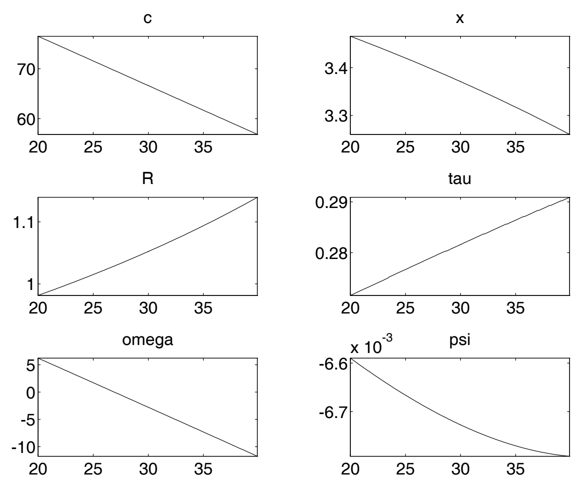

Figure 6. Various variables as functions of g, with $b^g = -60$. From left to right, top to bottom, the variables portrayed are, respectively, consumption, leisure, the gross real interest rate, the tax rate, the government surplus, and the multiplier ψ.

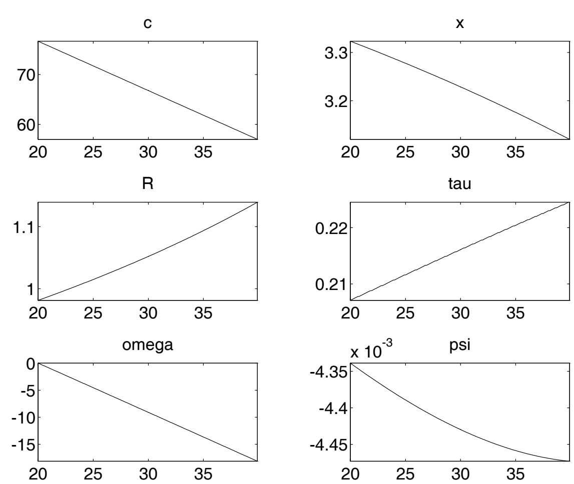

Figure 7. Various variables as functions of g, with $b^g = -100$. From left to right, top to bottom, the variables portrayed are, respectively, consumption, leisure, the gross real interest rate, the tax rate, the government surplus, and the multiplier ψ.

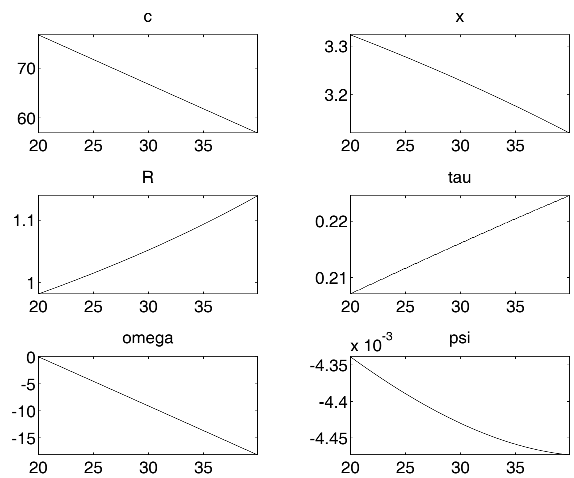

Figure 8. Various variables as functions of g, with $b^g = -200$. From left to right, top to bottom, the variables portrayed are, respectively, consumption, leisure, the gross real interest rate, the tax rate, the government surplus, and the multiplier ψ.

Conclusions

Lucas and Stokey (1983, p. 77) drew three lessons: (1.) Budget balance in a present value or average sense must be respected;[13] (2.) No case can be made for budget balance on a continual basis; (3.) State-contingent debt is an important feature of an optimal policy.[14] Our results support 1, amplify 2, but qualify 3. Our incomplete markets Ramsey allocation is very close to the complete markets Ramsey allocation, testimony to our Ramsey policy's use of 'self-insurance' through risk-free borrowing and lending with households.

[13] According to Keynes, 'What the government spends, the public pays for.'

[14] Lucas and Stokey write: "... even those most skeptical about the efficacy of actual government policy may be led to wonder why governments forego gains in everyone's welfare by issuing only debt that purports to be a *certain* claim on future goods." Our calculations do not diminish the relevance of this statement as a comment about the role of state-contingent debt in making possible a *debt structure* that renders their Ramsey tax policy time-consistent.

In affirming Barro's characterization of tax-smoothing as imparting near-unit root components to tax rates and government debt, our incomplete markets model enlivens a view of 18th century British fiscal outcomes as Ramsey outcomes. The time series of debt service and government expenditure for 18th century Britain resemble a simulation of Barro's model or ours, not a complete markets model.[15]

Appendix A: Computations

Reformulation of Incomplete Markets

Although the measurability conditions (10) are useful in characterizing equilibria and provide an interesting interpretation of the multipliers, the Ramsey problem can also be cast in a recursive contracts framework with the original budget constraints (3). This appendix describes such an alternative formulation and applies the method of parameterized expectations.

For convenience, we slightly modify the boundedness constraints to become

$$\underline{M} < p_t^b b_t^g < \overline{M}. \tag{27}$$

We impose the implementability conditions (6) and (3) on the allocation. Associated with the Ramsey problem is the Lagrangian:

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \{ \ u(c_t, 1 - c_t - g_t) + \lambda_t \ [(1 - \frac{u_{x,t}}{u_{c,t}}) \ (c_t + g_t) + p_t^b b_t^g - g_t - b_{t-1}^g] \\ + \mu_t [u_{c,t} \ p_t^b - \beta \ u_{c,t+1}] + \theta_{1t} \ [\overline{M} - p_t^b b_t^g] + \theta_{2t} [p_t^g b_t^g - \underline{M}] \}. \end{aligned} \tag{28}$$

for $\theta_{1t} \geq 0, \theta_{2t} \geq 0$. Here λ_t, μ_t, θ_{1t}, θ_{2t} are Lagrange multipliers corresponding to (3), (6), and (27). We have used an iterated expectations argument to incorporate (6) in (28). To obtain a recursive expression, we use summation by parts to rewrite the Lagrangian as

$$\begin{aligned} L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t, 1 - c_t - g_t) + \lambda_t \ [(1 - \frac{u_{x,t}}{u_{c,t}}) \ (c_t + g_t) + p_t^b b_t^g - g_t - b_{t-1}^g] \\ u_{c,t}(\mu_t p_t^b - \mu_{t-1} \ ) + \theta_{1t} \ [\overline{M} - p_t^b b_t^g] \ + \theta_{2t} [p_t^b b_t^b - \underline{M}] \}, \end{aligned} \tag{29}$$

with restrictions $\theta_{1t} \geq 0, \theta_{2t} \geq 0$, the transition function for b_t^g, (3), and with initial conditions

$$b_{-1}^g = 0, \qquad \mu_{-1} = 0.$$

Arguments of Marcet and Marimon (1995) can be applied to show that $(b_{t-1}^g, \mu_{t-1}, g_t)$ form the *state variables*, and that the solution of the Ramsey problem can be represented as a time-invariant function of these state variables.

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[15] See Figure 2 of Sargent and Velde (1995).

The Euler inequalities

For convenience, we introduce the transformations $R_t^{-1} \equiv p_t^b, R_{t-1}b_{t-1} \equiv b_{t-1}^g$. Let $\delta_t \equiv \partial[(1 - \frac{u_{x,t}}{u_{c,t}})(c_t + g_t)]/\partial c_t)$. The first order conditions satisfied by the saddle point of L are given by[16]

$$u_{c,t} - u_{x,t} + u_{cc,t}\ (\mu_t\ R_t^{-1} - \mu_{t-1}\ ) + \lambda_t \delta_t = 0 \tag{30}$$

$$-\theta_{1t} + \theta_{2t} + \lambda_t = \beta R_t E_t\ (\lambda_{t+1}) \tag{31}$$

$$-u_{c,t}\ \mu_t R_t^{-2} =\ b_t\ \beta E_t\ (\lambda_{t+1}) \tag{32}$$

$$(b_t - \overline{M})\ \theta_{1t} = 0, \qquad , \theta_{1t} \geq 0 \tag{33}$$

$$(\underline{M} - b_t)\theta_{2t} = 0, \qquad \theta_{2t} \geq 0 \tag{34}$$

$$b_t = -(1 - \frac{u_{x,t}}{u_{c,t}})\ (c_t + g_t)\ +\ g_t\ +\ R_{t-1}b_{t-1}; \tag{35}$$

$$u_{c,t}R_t^{-1} = \beta\ E_t\ u_{c,t+1}. \tag{36}$$

and (27). The above equations are the first-order conditions with respect to consumption, debt, and interest rate, and the Kuhn-Tucker condition.

Observe that the solution is restricted by seven equations, namely, (30), (31), (32), (33), (34), (35), and (36). This is a system of difference equations with two expectations in it (namely, $E_t\ (\lambda_{t+1} R_t)$ and $E_t\ (u_{c,t+1})$) that can be used to solve for the stochastic process of seven variables: $\{c_t, R_t, b_t, \lambda_t, \theta_{1t}, \theta_{2t}, \mu_t\}_{t=0}^{\infty}$.

Equivalence of formulations

The formulation in the text is equivalent with the present one, which can be verified by adopting the following correspondences:

Form 1	Form 2	
κ_t	$u_{c,t}\delta_t + (u_{cc,t} - u_{cxt})\omega_t$	(37)
ψ_t	$-\ \dfrac{\lambda_t}{u_{c,t}}$	(38)
ψ_t	$\dfrac{\mu_t R_t^{-1}}{b_t}$	(39)
b_{t-1}^g	$R_{t-1}b_{t-1}$	(40)
	$R_{t-1}b_{t-1} = \omega_t + b_t$	(41)

[16] Along with Lucas and Stokey (1983), Chari, Christiano, and Kehoe (1994), and others, we hope that these first-order conditions are sufficient for an allocation problem for which the concavity of the objective function is at question.

To verify the match, the key step is to show how to deduce (13) from (30). To reduce the notation, we present the calculations only for the case in which the debt limits do not bind. Use (39) in (30) to get

$$u_{c,t} - u_{x,t} + (u_{cc,t} - u_{cx,t})(\psi_t b_t - \psi_{t-1} b_{t-1} R_{t-1}) + \lambda_t \delta_t = 0.$$

Next use (41) and (38) to get

$$u_{c,t} - u_{x,t} + (u_{cc,t} - u_{cx,t})(\psi_t R_{t-1} b_{t-1} - \psi_t \omega_t - \psi_{t-1} b_{t-1} R_{t-1}) - \psi_t u_{c,t} \delta_t = 0.$$

Use (40) to get

$$u_{c,t} - u_{x,t} + (u_{cc,t} - u_{cx,t})(b^g_{t-1}(\psi_t - \psi_{t-1}) - \psi_t \omega_t) - \psi_t u_{c,t} \delta_t = 0.$$

Rearranging gives

$$u_{c,t} - u_{x,t} - \psi_t \Big((u_{cc,t} - u_{cx,t}) \omega_t + u_{c,t} \delta_t \Big) + (u_{cc,t} - u_{cx,t}) b^g_{t-1} \gamma_t = 0.$$

Using (37) gives (13).

Numerical Solution of the Ramsey Problem

The PEA algorithm replaces agents' conditional expectations about functions of future variables with an approximating function in state variables and coefficients on these variables. The approximating function is used to generate T realizations of variables of interest. Next, a nonlinear least-squares routine is used to reestimate the coefficients of the approximating function. Using these new coefficients, a new set of economic time series is created. Iterations continue until the regression coefficients converge. For more information about the details of PEA, see den Haan and Marcet (1990) and Marcet and Marshall (1994).

We chose to parameterize conditional expectations $E_t[u_{c,t+1}]$ and $E_t[R_t \lambda_{t+1}]$ appearing in (36) and (31). The state variables are $\mathbf{x}_t = [Rb_{t-1}, \mu_{t-1}, g_t]'$. We use the family of approximating functions mapping $\Re^3_+$ into $\Re_+$

$$\psi(\beta_i, \mathbf{x}_t) = \exp(P_n(\mathbf{x}_t)),$$

where P_n denotes a polynomial of degree n and the parameters β_i are the coefficients in the polynomial. To assure that the variable in the nonlinear least squares problem are of similar orders of magnitude, we applied function $\varphi : (\underline{k}, \overline{k}) \mapsto (-1, 1)$ to each state variable separately, where $\underline{k}$ and $\overline{k}$ are prespecified lower and upper bound for the argument of φ, k. That is,

$$\varphi(k) = 2\frac{k - \underline{k}}{\overline{k} - \underline{k}} - 1,$$

and our ψ is

$$\psi(\beta_i, \mathbf{x}_t) = \exp(P_n(\varphi(\mathbf{x}_t))).$$

After parameterizing the two conditional expectations as finite-dimensional functions of $(Rb_{t-1}, \mu_{t-1}, g_t)$, the next step of PEA is to obtain a long simulation, a process that we manage by considering two cases:

a. *Unconstrained periods.* Where $\theta_{1t} = \theta_{2t} = 0$ (in which case debt is not constrained), the variables can be obtained as follows: λ_t is given by (31); now we have four equations left ((32), (30), (36) and (35)) to solve for the remaining four variables (μ, c, R, b). Notice that

$$\mu_t R_t^{-1} = -\ b_t\ E_t\ (R_t \lambda_{t+1})/u_{c,t}. \tag{42}$$

Substitute $\mu_t R_t^{-1}$ into (30) and substitute b_t from (35) to obtain one non-linear equation for c which only contains state variables. This gives one non-linear equation to be solved. Next, μ, b and R can be easily found. Once this is done, we check if $\underline{M} \leq b_t \leq M$; if that is the case, go to the next period; otherwise, we have to solve this period as

b. *Constrained periods.* We find b immediately by setting $b_t = \overline{M}$ or $b_t = \underline{M}$. Consumption is given by the transition equation for debt, the interest rate by the (36), μ is obtained from (42), λ is obtained from (30) and θ_1 or θ_2 from (31).

In applying PEA, the usual iterations are performed to find the policy function at the steady state, and the short run simulations to find the transition for the multipliers (as Marcet and Marimon (1992)).

To elaborate, we want to find the parameter β_f with the following property: if agents use β_f in order to form the expectations of the Euler equation, then $\psi(\beta_f, \mathbf{x}_t)$ is the best predictor among functions $\psi(\cdot, \mathbf{x}_t)$. The mechanics for finding β_f are the following. We start with an initial $\beta_i = \beta_0$.

Step 1. Fix β_i. Substitute the conditional expectations in (36) and (31) by ψ to obtain:

$$u_{c,t} R_t^{-1} = \beta \psi(\beta_i^1, \mathbf{x}_t)$$
$$-\theta_{1t} + \theta_{2t} + \lambda_t = \beta \psi(\beta_i^2, \mathbf{x}_t),$$

where $[\beta_i^1, \beta_i^2]' = \beta_i$.

Step 2. Obtain a long series of endogenous variables that solves the Ramsey problem for this particular β_i. Call this series $\{\mathbf{z}_t(\beta_i)\}$.

Step 3. For this series calculate the expressions inside the conditional expectations in (3) and (8) and perform a nonlinear regression of these expressions on $\psi(\cdot, \mathbf{x}_t)$; let $S(\beta_i)$ be the result of this regression.

Step 4. Using a relaxation parameter $\alpha \in (0, 1]$ update β_i by

$$\beta_{i+1} = (1-\alpha)\beta_i + \alpha S(\beta_i).$$

Step 5. Iterate the procedure until $\|\beta_i - \beta_{i-1}\|_2 < \epsilon$, where $\|\cdot\|_2$ denotes the ℓ_2 norm and small $\epsilon > 0$. β_i is used to approximate β_f. By continuity, $\|\{\mathbf{z}_t(\beta_i) - \mathbf{z}_t(\beta_{i-1})\}\|_2 < \delta$, for small $\delta > 0$.

The procedure for obtaining the short simulations is the same except that step 2 is modified as follows:

Step 2b. Obtain a large number N of (independent) realizations of length T, that solve the Ramsey problem; in each series the initial conditions are fixed to be $\mathbf{x}_0 = [Rb_{-1}, \mu_{-1}, g_0]'$.

To increase accuracy, we can increase N. Here, T is selected to be long enough for the economy to get in the range of the steady state distribution.

The algorithm is implemented in Fortran-77 in Unix-environment.[17] The NLLS-solver we used is a modified form of the Levenberg-Marquardt algorithm documented in Morris (1990). In long simulations we used $T = 10000$, and in short simulations $N = 99$ and $T = 100$. The degree of polynomial was chosen by using the accuracy tests introduced in den Haan and Marcet (1994). It turned out that Rb_{t-1} and μ_{t-1} were highly correlated, and several higher degree elements were redundant. Hence, excluding them or μ_{t-1} does not reduce the predictive power of the parameterized expectations.

Appendix B: Recursive Saddle Point Formulation

According to Definition III, the Ramsey problem under incomplete markets is to maximize the utility of the household subject to measurability conditions expressed as

$$E_t \sum_{j=0}^{\infty} \beta^j \; u_{c,t+j} \omega_{t+j} = u_{c,t} \, b^g_{t-1} \,, \; \forall t \geq 0 \tag{43}$$

with b^g_{-1} given. The presence of future choice variables in constraints (43) makes the problem non-recursive in terms of the natural state variables and renders the solution time-inconsistent. Further, the solution is *not* of the form $c_t = f(g_t, b^g_{t-1})$; rather, the policy functions are time-dependent and may have the whole past history of g's as arguments. Finding a solution for taxes in terms of past g's is, therefore, demanding.

In this appendix, we re-formulate the problem with an eye to recovering a recursive structure and facilitating computation. We use the apparatus of Marcet and Marimon (1996) (M&M)[18], and keep our notation close to theirs.

Consider the following problem

Program 1

$$\sup_{\{X_t\}} \; E_0 \sum_{t=0}^{\infty} \beta^t \; U(X_t, X_{t+1}, S_t)$$

subject to

$$\mathcal{T}(X_t, \, X_{t+1}, \, S_t) \geq 0, \;\; t \geq 0; \; X_0 = \overline{X} \tag{44}$$

[17] The program is available on request from the authors.

[18] There are several approaches in the literature that can be used to find a recursive structure in models where the Bellman equation is not satisfied. Nevertheless, they are not applicable without modifications to our problem: the approach of Kydland and Prescott (1980) does not incorporate uncertainty; the recursive formulation of Abreu, Pierce and Stachetti (1990) only provides necessary conditions for an optimum, but is not designed for solving Ramsey problems. The M&M approach is closely related to that used for the linear-quadratic case by Hansen, Epple, and Roberds (1985).

$$E_t \sum_{j=0}^{\infty} \beta^j \ \mathcal{V}(S_{t+j}, X_{t+j+1}) \geq \Phi(X_t, X_{t+1}, S_t), \quad \forall t \geq 0 \tag{45}$$

$$X_{t+1} \text{ measurable with respect to } \quad (S_0 \ldots, S_{t-1}, S_t). \tag{46}$$

Here, the initial values S_0, $\overline{X}$, the constant β, and the mappings $u, \mathcal{V}, \Phi$ and $\mathcal{T}$ are given, and $\{S_t\}$ is a stochastic Markov process.

The approach uses three steps:

Step 1:

Show that solving *Program 1* is equivalent with solving

Program 2

$$\inf_{\{\Gamma_t, \mu_t\}} \sup_{\{X_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \ \left(U(X_t, X_{t+1}, S_t) + \mu_{t+1} \ \mathcal{V}(S_t, X_t, X_{t+1}) \ - \ \Gamma_{t+1} \Phi(X_t, X_{t+1}, S_t)\right)$$

subject to

$$\mu_{t+1} \ = \mu_t + \Gamma_{t+1} \quad \text{for all } \ t, \ \mu_0 \ = 0 \tag{47}$$

and constraints (44) and (46).

This step follows from algebra. The form of Program 2 offers hope for finding a recursive approach, because future variables appear neither in the constraints nor return functions. Nevertheless, we cannot formulate a Bellman equation immediately for Program 2, because it is a saddle point, not a maximization, problem. We require a theory for recursive saddle point problems.

Thus, define a

Recursive Saddle Point Problem (RSPP):

$$\inf_{\{\mu_t\}} \sup_{\{X_t\}} E_0 \sum_{t=0}^{\infty} \beta^t h(X_t, \ X_{t+1}, \ \mu_t, \ \mu_{t+1}, \ S_t)$$

subject to

$$\mathcal{T}(X_t, \ X_{t+1}, \ S_t) \ \geq 0 \tag{48}$$

$$\mathcal{Q}(\mu_t, \ \mu_{t+1}, \ S_t) \geq 0 \tag{49}$$

$$X_0 = \overline{X}, \ \ \mu_0 = \overline{\mu}, \ \ S_0 = \overline{S} \tag{50}$$

$$(X_{t+1}, \mu_{t+1}) \text{ measurable with respect to } \ (S_0 \ldots, S_{t-1}, S_t), \tag{51}$$

and let W be the value function for this problem; i.e., $W(\overline{X}, \overline{\mu}, \overline{S})$ is the value attained by the objective function at the saddle point for the given initial conditions.

Step 2:

Show that the value function W of a RSPP satisfies the following analog of a Bellman equation:

Saddle Point Functional Equation (SPFE)

$$W(X, \mu, S) \ = \ \inf_{\mu'} \sup_{X'} \left\{h(X, \ X', \ \mu, \ \mu', \ S) \ + \ \beta E\left[W(X', \mu', S') | \ S\right]\right\}$$

$$\text{s.t.} \quad \mathcal{T}(X,\, X',\, S) \geq 0$$
$$\mathcal{Q}(\mu,\, \mu',\, S) \geq 0$$

Letting $f(X, \mu, S)$ be the optimal choice in the right side of the SPFE, it can be shown that the optimal solution to program 2 satisfies $X_t = f(X_{t-1}, \mu_{t-1}, S_{t-1})$ for all t and with $(X_{-1}, \mu_{-1}, S_{-1}) = (\overline{X}, 0, \overline{S})$.

Step 3:
Evidently, *Program 2* is a special case of a RSPP, because we can take

$$h(X_t,\, X_{t+1},\, \mu_t,\, \mu_{t+1},\, S_t) = U(X_t, X_{t+1}, S_t) + \mu_{t+1}\ \mathcal{V}(S_t, X_t, X_{t+1}) \ -\ \Gamma_{t+1}\ \Phi(X_t, X_{t+1})$$

$$\mathcal{Q}(\mu, \Gamma,\, \mu', \Gamma'\, S) \equiv \begin{cases} \mu' - \mu - \Gamma' \\ -\mu' + \mu + \Gamma' \end{cases}$$

and

$$\overline{\mu} = 0.$$

Recall that Step 1 implies that the solutions to Program 1 and 2 are equivalent; therefore, the solution to Program 1 satisfies $X_t = f(X_{t-1}, \mu_{t-1}, S_{t-1})$.

The Ramsey problem in the current paper is a special case of program 1 if we take

$$\begin{aligned} X_t \equiv (c_t, b_t^g), S_{t-1} &\equiv g_t, \mu_t \equiv \psi_t \\ U(X_t, X_{t+1}) &\equiv u(c_{t+1}, 1 - c_{t+1} - g_{t+1}), \\ \mathcal{V}(S_t, X_{t+j+1}) &\equiv u_{c,t+1}\omega_{t+1} \\ \Phi(X_t, X_{t+1}, S_t) &\equiv u_{c,t+1}\ b_t^g \\ T(X_t, X_{t+1}, S_t) &\equiv \left\{ \begin{array}{c} b_t^g - \overline{M} \\ \\ -b_t^g + \underline{M} \end{array} \right\} \end{aligned}$$

The corresponding RSPP (or *Program 2*) is displayed in equation (11) in the main text. Therefore, the corresponding SPFE for the Ramsey problem is

$$W(b^g, \psi, g) \ = \inf_{\mu', \gamma', \nu_1', \nu_2'} \sup_{X'} \left\{ u(c', 1 - c' - g) + \psi' u_c' \omega \ - \right.$$

$$\left. u_{c'}\ (b^g \gamma + \nu_1 \overline{M} - \nu_2 \underline{M}) + \ \beta E\left[W(b^{g\prime}, \psi', g') | \, g \right] \right\}$$

$$\begin{aligned} \text{s.t.} \qquad \underline{M} \leq b^{g\prime} &\leq \overline{M} \\ \nu_1', \nu_2' &\geq 0 \\ \psi' &= \gamma' + \nu_1' + \nu_2' + \psi \end{aligned}$$

Using the framework in M&M, we conclude that the solution of the Ramsey problem satisfies:

$$c_t = h(g_t, \mu_{t-1}, b_{t-1}^g) \qquad \text{for all } t,$$

and $(\mu_{-1}, b_{-1}^g) = (0, \overline{b}^g)$, where h is the decision function for the above SPFE.

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