Lecture 2: Structural estimation of discrete decision problems (MPEC and NFXP)

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Road Map: Structural estimation and discrete decision problems

Lecture 2: Constrained versus unconstrained optimization approaches

- **PART I**: The Nested Fixed Point Algorithm (NFXP)
- **PART II**: Mathematical Programming with Equilibrium Constraints (MPEC)
- Leading example: Rust's Engine replacement model
- Matlab implementation

Lecture 3-5: CCP estimation based on the Hotz-Miller inversion (Miller)

- Conditional Independence and the Inversion Theorem
- Identification in Discrete Choice Models
- CCP Estimators

Lecture 6-7: Machine Learning of Dynamic Discrete Choice
Lecture 8: Matlab implementation of CCP, NPL, BBL and MSM

Structural Estimation in Microeconomics

Methods for solving Dynamic Discrete Choice Models

- Rust (1987): MLE using Nested-Fixed Point Algorithm (NFXP)
- Hotz and Miller (1993): CCP estimator - (two step estimator)
- Keane and Wolpin (1994): Simulation and interpolation
- Rust (1997): Randomization algorithm (breaks curse of dimensionality)
- Aguirregabiria and Mira (2002): Nested Pseudo Likelihood (NPL).
- Bajari, Benkard and Levin (2007): Two step-minimum distance (equilibrium inequalities).
- Arcidiacono Miller (2002): CCP with unobserved heterogeneity (EM Algorithm).
- Norets (2009): Bayesian Estimation (allows for serial correlation in ϵ)
- Su and Judd (2012): MLE using constrained optimization (MPEC)
- and MUCH more
- Any estimator method or solution algorithm of DDC models must confront *Harold Zurcher*

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PART I

Rust (ECTA, 1987):

OPTIMAL REPLACEMENT OF GMC BUS ENGINES: AN EMPIRICAL MODEL OF HAROLD ZURCHER

Overview of Rust (1987)

The economic question: For how long should one continue to operate and maintain a bus before it is optimal to replace or rebuild the engine?

Optimal replacement decision: Solution to a dynamic optimization problem that formalizes the trade-off between two conflicting objectives:

- *minimizing replacement costs*
- *minimizing operating and maintenance cost as well as unexpected engine failures*

(quality of engine declines over time, but replacing it is costly)

Empirical question: Did the decision maker (the superintendent of maintenance, Harold Zurcher) behave according to the optimal replacement rule implied by the dynamic discrete choice model?

Structural estimation: Using data on *monthly mileage and engine replacements* for a sample of GMC busses, Rust estimate the structural parameters in the engine replacement model using NFXP

Overview of Rust (1987)

Main contributions

1. Development and implementation of Nested Fixed Point Algorithm
2. Formulation of assumptions, that makes dynamic discrete choice models tractable.
3. Bottom-up approach: Micro-aggregated demand for machines
4. An illustrative application in a simple model of engine replacement.
5. The first researcher to obtain ML estimates of discrete choice dynamic programming models

Policy experiments:

- How does changes in replacement cost affect
  - the distribution of mileage
  - the demand for engines

General Behavioral Framework

The decision problem

- The decision maker chooses a sequence of actions to maximize expected discounted utility over a finite horizon

$$V_\theta(s_t) = \sup_{\Pi} E\left[\sum_{j=0}^{\infty} \beta^j U(s_{t+j}, d_{t+j}; \theta_1) \,|\, s_t, d_t\right]$$

where

- $\Pi = (f_t, f_{t+1}, ..,), d_t = f_t(s_t, \theta) \in C(x_t) = \{1, 2, .., J\}$
- $\beta \in (0, 1)$ is the discount factor
- $U(s_t, d_t; \theta_1)$ is a choice and state specific utitty function
- E summarizes expectations of future states given s_t and d_t

Rust's Assumptions

Assumption (CI)

State variables, $s_t = (x_t, \varepsilon_t)$ obeys a (conditional independent) controlled Markov process with probability density

$$p(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, d, \theta_2, \theta_3) = q(\varepsilon_{t+1} | x_{t+1}, \theta_3) p(x_{t+1} | x_t, d, \theta_2)$$

Assumption (AS (additive separability))

$$U(s_t, d) = u(x_t, d; \theta_1) + \varepsilon_t(d)$$

Assumption (XV)

The unobserved state variables, ε_t are assumed to be multivariate iid. extreme value distributed

Object of interest: $\theta = (\beta, \theta_1, \theta_2, \theta_3)$
The vector of structural parameters to be estimated.

The Dynamic Programming Problem

- Under AS, the optimal decision solves the following DP problem

$$V_\theta(x_t, \varepsilon_t) = \max_{d \in C(x_t)} [u(x_t, d, \theta_1) + \varepsilon_t(d) + \beta E(V_\theta(x_{t+1}, \varepsilon_{t+1}) | x_t, \varepsilon_t, d)]$$

- Under (CI) and (XV) we can integrate out the unobserved state variables, such that the unknown function, EV_θ, no longer depends on ε_t.

$$EV_\theta(x, d) = \Gamma_\theta(EV_\theta)(x, d)$$

$$= \int_y \ln \left[\sum_{d' \in D(y)} \exp\left[u(y, d'; \theta_1) + \beta EV_\theta(y, d')\right] \right] p(dy | x, d, \theta_2)$$

  - (CI): significantly reduces the dimension of the state space
  - (XV): allows us to integrate out the unobserved state variables, ε_t,

- Γ_θ is a *contraction mapping* with unique fixed point EV_θ, i.e. $\|\Gamma(EV) - \Gamma(W)\| \leq \beta \|EV - W\|$

Zurcher's Bus Engine Replacement Problem

- Choice set: Binary choice set, $C(x_t) = \{0, 1\}$. Each bus comes in for repair once a month and Zurcher chooses between ordinary maintenance ($d_t = 0$) and overhaul/engine replacement ($d_t = 1$).
- State variables: Harold Zurcher observes:
  - x_t: mileage at time t since last engine overhaul
  - $\varepsilon_t = [\varepsilon_t(d_t = 0), \varepsilon_t(d_t = 1)]$: other state variable
- Utility function:

$$u(x_t, d, \theta_1) + \varepsilon_t(d_t) = \begin{cases} -RC - c(0, \theta_1) + \varepsilon_t(1) & \text{if } d_t = 1 \\ -c(x_t, \theta_1) + \varepsilon_t(0) & \text{if } d_t = 0 \end{cases} \qquad (1)$$

- State variables process x_t (mileage since last replacement)

$$p(x_{t+1}|x_t, d_t, \theta_2) = \begin{cases} g(x_{t+1} - 0, \theta_2) & \text{if } d_t = 1 \\ g(x_{t+1} - x_t, \theta_2) & \text{if } d_t = 0 \end{cases} \qquad (2)$$

- If engine is replaced, state of bus regenerates to $x_t = 0$.

Likelihood Function

Likelihood

- Under assumption (CI) the likelihood function ℓ^f has the particular simple form

$$\ell^f\left(x_1, \dots x_T, d_1, \dots d_r | x_0, d_0, \theta\right) = \prod_{t=1}^{T} P\left(d_t | x_t, \theta\right) p\left(x_t | x_{t-1}, d_{t-1}, \theta_2\right)$$

where $P\left(d_t | x_t, \theta\right)$ is the choice probability given the observable state variable, x_t.

How to compute the choice probability, $P\left(d_t | x_t, \theta\right)$

- Need to solve dynamic program

How to estimate the transition probability, $p\left(x_t | x_{t-1}, d_{t-1}, \theta_2\right)$

- Can be estimated without knowledge of θ_1 and EV (e.g. non-parametrically, NLS, MLE or...)

Conditional Choice Probabilities

- Under the extreme value assumption choice probabilities are multinomial logistic

$$P\left(d|x,\theta\right)=\frac{\exp\left\{u\left(x,d,\theta_1\right)+\beta EV_\theta\left(x,d\right)\right\}}{\sum_{j\in C(y)}\exp\left\{u\left(x,j,\theta_1\right)+\beta EV_\theta\left(x,j\right)\right\}}$$

- The expected value function is given by the unique fixed point to the contraction mapping Γ_θ, defined by

$$\begin{aligned} EV_\theta\left(x,d\right) &= \Gamma_\theta(EV_\theta)\left(x,d\right) \\ &= \int_y \ln\left[\sum_{d'\in D(y)}\exp\left[u(y,d';\theta_1)+\beta EV_\theta\left(y,d'\right)\right]\right] \\ &\quad p\left(dy|x,d,\theta_2\right) \end{aligned}$$

- Structural Estimation: Rust's *Nested Fixed Point Algorithm* (NFXP)

Structural Estimation: The Nested Fixed Point Algorithm

Since the contraction mapping Γ always has a unique fixed point, the constraint $EV = \Gamma_\theta(EV)$ implies that the fixed point EV_θ is an *implicit function* of θ.

Hence, NFXP solves the *unconstrained* optimization problem

$$\max_\theta L(\theta, EV_\theta)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function $L(\theta, EV_\theta)$ is maximized w.r.t. θ
- Quasi-Newton algorithm: Usually BHHH, BFGS or a combination.
- Each evaluation of $L(\theta, EV_\theta)$ requires solution of EV_θ

Inner loop (fixed point algorithm):

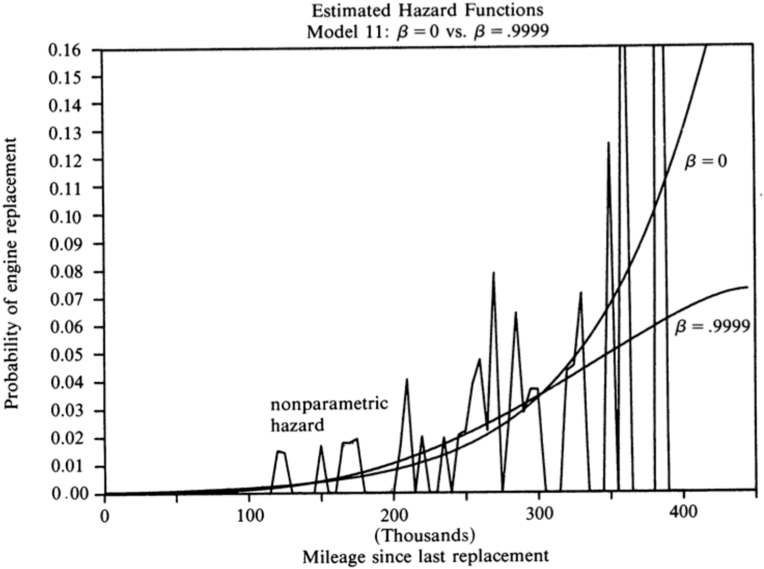
The implicit function EV_θ defined by $EV_\theta = \Gamma(EV_\theta)$ is solved by:

- Successive Approximations (SA)
- Newton-Kantorovich (NK) Iterations

Data

- Harold Zurcher's Maintenance records of 162 busses
- Monthly observations of mileage on each bus (odometer reading)
- Data on maintenance operations
  1. Routine, periodic maintenance (e.g. brake adjustments)
  2. Replacement or repair at time of failure
  3. Major engine overhaul and/or replacement
- Rust focus on 3)

Estimated Hazard Functions



Specification Search

TABLE VIII

SUMMARY OF SPECIFICATION SEARCH[a]

	Bus Group		
Cost Function	1, 2, 3	4	1, 2, 3, 4
Cubic $c(x, \theta_1) = \theta_{11}x + \theta_{12}x^2 + \theta_{13}x^3$	Model 1 −131.063 −131.177	Model 9 −162.885 −162.988	Model 17 −296.515 −296.411
quadratic $c(x, \theta_1) = \theta_{11}x + \theta_{12}x^2$	Model 2 −131.326 −131.534	Model 10 −163.402 −163.771	Model 18 −297.939 −299.328
linear $c(x, \theta_1) = \theta_{11}x$	Model 3 −132.389 −134.747	Model 11 −163.584 −165.458	Model 19 −300.250 −306.641
square root $c(x, \theta_1) = \theta_{11}\sqrt{x}$	Model 4 −132.104 −133.472	Model 12 −163.395 −164.143	Model 20 −299.314 −302.703
power $c(x, \theta_1) = \theta_{11}x^{\theta_{12}}$	Model 5[b] N.C. N.C.	Model 13[b] N.C. N.C.	Model 21[b] N.C. N.C.
hyperbolic $c(x, \theta_1) = \theta_{11}/(91 - x)$	Model 6 −133.408 −138.894	Model 14 −165.423 −174.023	Model 22 −305.605 −325.700
mixed $c(x, \theta_1) = \theta_{11}/(91 - x) + \theta_{12}\sqrt{x}$	Model 7 −131.418 −131.612	Model 15 −163.375 −164.048	Model 23 −298.866 −301.064
nonparametric $c(x, \theta_1)$ any function	Model 8 −110.832 −110.832	Model 16 −138.556 −138.556	Model 24 −261.641 −261.641

Structural Estimates, n=90

TABLE IX

STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$

FIXED POINT DIMENSION = 90

(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates/ Log-Likelihood	Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic ($df = 4$)	Marginal Significance Level
$\beta = .9999$	RC	11.7270 (2.602)	10.0750 (1.582)	9.7558 (1.227)	85.46	1.2E−17
	θ_{11}	4.8259 (1.792)	2.2930 (0.639)	2.6275 (0.618)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3489 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2708.366	−3304.155	−6055.250		
$\beta = 0$	RC	8.2985 (1.0417)	7.6358 (0.7197)	7.3055 (0.5067)	89.73	1.5E−18
	θ_{11}	109.9031 (26.163)	71.5133 (13.778)	70.2769 (10.750)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3488 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2710.746	−3306.028	−6061.641		
Myopia test:	LR Statistic ($df = 1$)	4.760	3.746	12.782		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0292	0.0529	0.0035		

Structural Estimates, n=175

TABLE X

STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$

FIXED POINT DIMENSION = 175

(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates Log-Likelihood	Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic (df = 6)	Marginal Significance Level
$\beta = .9999$	RC	11.7257 (2.597)	10.896 (1.581)	9.7687 (1.226)	237.53	1.89E−48
	θ_{11}	2.4569 (.9122)	1.1732 (0.327)	1.3428 (0.315)		
	θ_{30}	.0937 (.0047)	.1191 (.0050)	.1071 (.0034)		
	θ_{31}	.4475 (.0080)	.5762 (.0075)	.5152 (.0055)		
	θ_{32}	.4459 (.0080)	.2868 (.0069)	.3621 (.0053)		
	θ_{33}	.0127 (.0018)	.0158 (.0019)	.0143 (.0013)		
	LL	−3993.991	−4495.135	−8607.889		
$\beta = 0$	RC	8.2969 (1.0477)	7.6423 (.7204)	7.3113 (0.5073)	241.78	2.34E−49
	θ_{11}	56.1656 (13.4205)	36.6692 (7.0675)	36.0175 (5.5145)		
	θ_{30}	.0937 (.0047)	.1191 (.0050)	.1070 (.0034)		
	θ_{31}	.4475 (.0080)	.5762 (.0075)	.5152 (.0055)		
	θ_{32}	.4459 (.0080)	.2868 (.0069)	.3622 (.0053)		
	θ_{33}	.0127 (.0018)	.0158 (.0019)	.0143 (.0143)		
	LL	−3996.353	−4496.997	−8614.238		
Myopia tests:	LR Statistic ($df = 1$)	4.724	3.724	12.698		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0297	0.0536	.00037		

Estimating parameters, bustypes 1,2,3,4 (model 19)

Output from `run_busdata.m`

```
Structural Estimation using busdata from Rust(1987)
Beta            =    0.99990
n               =  175.00000
Sample size     = 8156.00000


    Param.                        Estimates          s.e.         t-stat
----------------------------------------------------------------------
    RC                               9.7498        1.2249         7.9596
    c                                1.3385        0.3143         4.2589
    p                  (1)           0.1070        0.0034        31.2107
    p                  (2)           0.5152        0.0055        93.0556
    p                  (3)           0.3622        0.0053        68.0405
    p                  (4)           0.0143        0.0013        10.8946
    p                  (5)           0.0009        0.0003         2.6469
----------------------------------------------------------------------
log-likelihood     = -8607.89002
runtime (seconds) =     0.36119
```

Identification - scale of cost function

Identification problem?

- We only identify RC/σ and $c(x, \theta_1)/\sigma = 0.001 * \theta_1/\sigma * x$, (where σ is parameter that index the scale of the cost function).
- σ is unidentified form mileage and replacement data

How to deal with identification problem related to scale of utility?

- Using replacement cost data and structural estimates we can obtain a scale estimate
- Scale the estimates with observed average replacement costs

Average Engine Replacement Costs

TABLE III

AVERAGE ENGINE REPLACEMENT COSTS[a]

	Bus Group		
Operation	1, 2, 3	4	1, 2, 3, 4
Labor time[b] to drop engine	$ 150	$ 150	$ 150
Labor time[b] to overhaul engine	3373	2870	3032
Parts required to overhaul engine	5826	4343	4730
Labor time[b] to re-install engine	150	150	150
Total cost of replacement	$9499	$7513	$8062

- Replacement costs are *higher* for group 1,2,3 although engine replacements for this group occur 57.000 miles and 15 month *earlier*
- Presumably operating and maintenance costs for these busses increase much faster

Identification - scale of cost function

- Using replacement cost data (prev. slide) and structural estimates from Table IX (next slide) we can obtain a scale estimate

$$
\begin{aligned}
\sigma_{bus\ 1,2,3} &= \frac{RC}{RC/\sigma} \\
&= \$9499/11.7257 \\
\sigma_{bus\ 4} &= \$7513/10.0750
\end{aligned}
$$

- We can the obtain a dollar estimate of $c(x, \theta_1)$ (i.e monthly maintenance costs per accumulated 5000 miles)

$$
\begin{aligned}
c(x, \theta_1)_{bus\ 1,2,3} &= \sigma * 0.001\theta_{11}/\sigma * x \\
&= \$9499/11.725 * 0.001 * 4.82 * x = \$3.9 * x \\
c(x, \theta_1)_{bus\ 4} &= \$7513/10.0750 * 0.001 * 2.2930 * x = \$1.7 * x
\end{aligned}
$$

- Hence, a bus with mileage of 300.000 (i.e. $x = 300.000/5.000$) is $(3.9 - 1.7) * 300000/5000 = \$132$ more expensive to operate per month

Structural Estimates

TABLE IX
STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$
FIXED POINT DIMENSION = 90
(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
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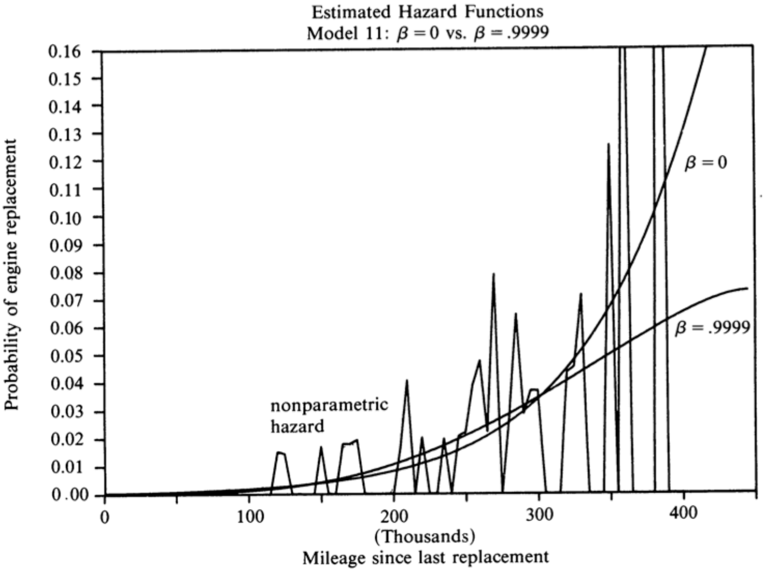
Why a dynamic model?

Suppose the "true" β is > 0, but we estimate the model with $\beta = 0$

- Our estimate of the replacement cost function will be biased.
- Parameters RC and θ_1 would be biased too
  (RC is upward biased and θ_1 is downward biased.)
- Predictions using the estimated model will be biased for two reasons:
  1. parameter estimates are biased
  2. the static model is not correct.
- Though the biases introduced by (1) and (2) might partly compensate each other, it will be a very unlikely coincidence that they compensate each other to make the bias negligible.
- Effect on equilibrium demand and hazard functions are very different!

Estimated Hazard Functions



Equilibrium bus mileage and demand for enigines

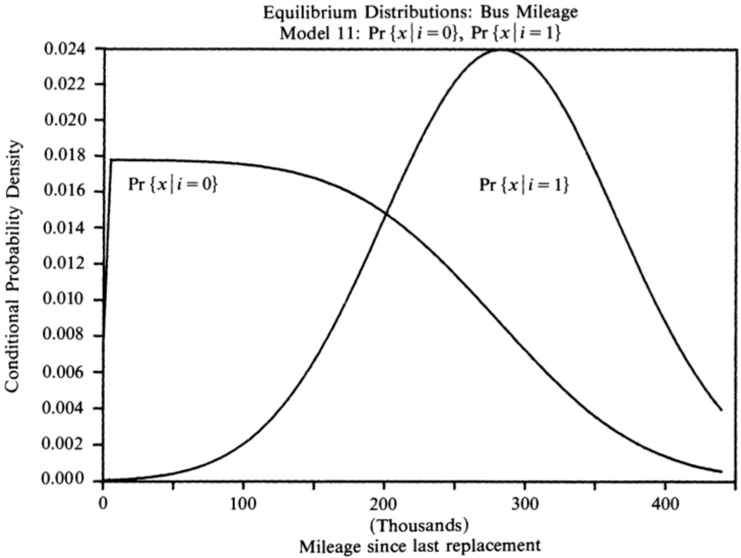
- Let π be the long run stationary (or equilibrium) distribution of the controlled process $\{i_t, x_t\}$
- π is then given by the unique solution to the functional equation

$$\pi(x, i) = \int_y \int_j P(i|x, \theta)p(x|y, j, \theta_3)\pi(dy, dj)$$

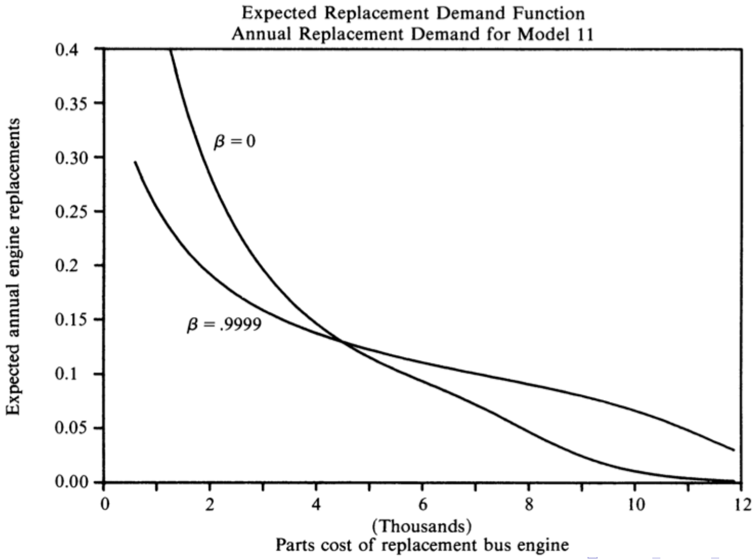
- Carly the equilibrium distribution of π is an implicit function of the structural parameters θ, which we emphasize by the notation π_θ
- Given π_θ, we can also obtain the following simple formula for annual equilibrium demand for engines as a function of RC

$$d(RC) = 12M \int_0^\infty \pi(dx, 1)$$

Equilibrium Bus mileage, bus group 4



Demand Function, bus group 4



Why not a reduced form for demand?

Reduced form

- Regress engine replacements on replacement costs

Problem: Lack of variation in replacement costs

- Data would be clustered around the intersection of the demand curves for $\beta = 0$ and $\beta = 0.9999$
(both models predict that RC is around the actual RC of $4343)
- Demand also depends on how operating costs varies with mileage
- Need exogenous variation in RC
.... that doesn't vary with operating costs
- Even if we had exogenous variation, this does not help us to understand the underlying economic incentives

Structural Approach

Attractive features

- structural parameters have a transparent interpretation
- evaluation of (new) policy proposals by counterfactual simulations.
- economic theories can be tested directly against each other.
- economic assumptions are more transparent and explicit (compared to statistical assumptions)

Less attractive features

- We impose more structure and make more assumptions
- Truly "structural" (policy invariant) parameters may not exist
- The curse of dimensionality
- The identification problem
- The problem of multiplicity and indeterminacy of equilibria
- Intellectually demanding and a huge amount of work

PART II

Constrained and Unconstrained Optimization Approaches to Structural Estimation (MPEC vs. NFXP)

MPEC is used in multiple contexts

Single-Agent Dynamic Discrete Choice Models

- Rust (1987): Bus-Engine Replacement Problem
- Nested-Fixed Point Problem (NFXP)
- Su and Judd (2012): Constrained Optimization Approach

Random-Coefficients Logit Demand Models

- BLP (1995): Random-Coefficients Demand Estimation
- Nested-Fixed Point Problem (NFXP)
- Dube, Fox and Su (2012): Constrained Optimization Approach

Estimating Discrete-Choice Games of Incomplete Information

- Aguirregabiria and Mira (2007): NPL (Recursive 2-Step)
- Bajari, Benkard and Levin (2007): 2-Step
- Pakes, Ostrovsky and Berry (2007): 2-Step
- Pesendorfer and Schmidt-Dengler (2008): 2-Step
- Pesendorfer and Schmidt-Dengler (2010): comments on AM (2007)
- Kasahara and Shimotsu (2012): Modified NPL
- Su (2013), Egesdal, Lai and Su (2014): Constrained Optimization

Zurcher's Bus Engine Replacement Problem

- Choice set: Each bus comes in for repair once a month and Zurcher chooses between ordinary maintenance ($d_t = 0$) and overhaul/engine replacement ($d_t = 1$)
- State variables: Harold Zurcher observes:
  - x_t: mileage at time t since last engine overhaul
  - $\varepsilon_t = [\varepsilon_t(d_t = 0), \varepsilon_t(d_t = 1)]$: other state variable
- Utility function:

$$u(x_t, d, \theta_1) + \varepsilon_t(d_t) = \begin{cases} -RC - c(0, \theta_1) + \varepsilon_t(1) & \text{if } d_t = 1 \\ -c(x_t, \theta_1) + \varepsilon_t(0) & \text{if } d_t = 0 \end{cases} \tag{3}$$

- State variables process x_t (mileage since last replacement)

$$p(x_{t+1} | x_t, d_t, \theta_2) = \begin{cases} g(x_{t+1} - 0, \theta_2) & \text{if } d_t = 1 \\ g(x_{t+1} - x_t, \theta_2) & \text{if } d_t = 0 \end{cases} \tag{4}$$

- If engine is replaced, state of bus regenerates to $x_t = 0$.

Structural Estimation

Data: $(d_{i,t}, x_{i,t})$, $t = 1, ..., T_i$ and $i = 1, ..., n$

Likelihood function

$$\ell_i^f(\theta) = \sum_{t=2}^{T_i} log(P(d_{i,t}|x_{i,t}, \theta)) + \sum_{t=2}^{T_i} log\left(p(x_{i,t}|x_{i,t-1}, d_{i,t-1}, \theta_2)\right)$$

where

$$P(d|x, \theta) = \frac{\exp\{u(x, d, \theta_1) + \beta EV_\theta(x, d)\}}{\sum_{d' \in \{0,1\}} \{u(x, d', \theta_1) + \beta EV_\theta(x, d')\}}$$

and

$$\begin{aligned} EV_\theta(x, d) &= \Gamma_\theta(EV_\theta)(x, d) \\ &= \int_y \ln \left[\sum_{d' \in \{0,1\}} \exp[u(y, d'; \theta_1) + \beta EV_\theta(y, d')] \right] p(dy|x, d, \theta_2) \end{aligned}$$

The Nested Fixed Point Algorithm

NFXP solves the *unconstrained* optimization problem

$$\max_{\theta} L(\theta, EV_\theta)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function $L(\theta, EV_\theta)$ is maximized w.r.t. θ
- Quasi-Newton algorithm: Usually BHHH, BFGS or a combination.
- Each evaluation of $L(\theta, EV_\theta)$ requires solution of EV_θ

Inner loop (fixed point algorithm):
The implicit function EV_θ defined by $EV_\theta = \Gamma(EV_\theta)$ is solved by:

- Successive Approximations (SA)
- Newton-Kantorovich (NK) Iterations

Mathematical Programming with Equilibrium Constraints

MPEC solves the *constrained* optimization problem

$$\max_{\theta, EV} L(\theta, EV) \text{ subject to } EV = \Gamma_\theta(EV)$$

using general-purpose constrained optimization solvers such as KNITRO

Su and Judd (Ecta 2012) considers two such implementations:

MPEC/AMPL:

- AMPL formulates problems and pass it to KNITRO.
- Automatic differentiation (Jacobian and Hessian)
- Sparsity patterns for Jacobian and Hessian

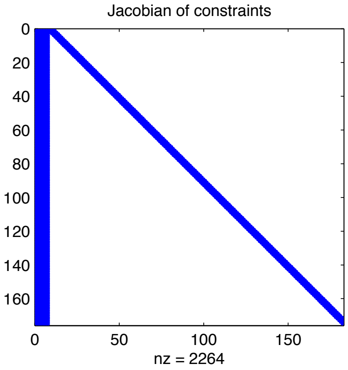
MPEC/MATLAB:

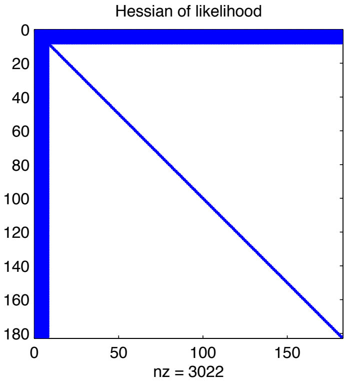
- User need to supply Jacobians, Hessian, and Sparsity Patterns
- Su and Judd do not supply analytical derivatives.
- ktrlink provides link between MATLAB and KNITRO solvers.

Sparsity patterns for MPEC

Two key factors in efficient implementations:

- Provide analytic-derivatives (huge improvement in speed)
- Exploit sparsity pattern in constraint Jacobian (huge saving in memory requirement)





Zurcher's Bus Engine Replacement Problem

Discretize the mileage state space x into n grid points

$$\hat{X} = \{\hat{x}_1, ..., \hat{x}_n\} \text{ with } \hat{x}_1 = 0$$

Mileage transition probability: for $j = 1, ..., J$

$$p(x'|\hat{x}_k, d, \theta_2) = \begin{cases} Pr\{x' = \hat{x}_{k+j}|\theta_2\} = \theta_{2j} \text{ if } d = 0 \\ Pr\{x' = \hat{x}_{1+j}|\theta_2\} = \theta_{2j} \text{ if } d = 1 \end{cases}$$

Mileage in the next period x' can move up at most J grid points. J is determined by the distribution of mileage.

Choice-specific expected value function for $\hat{x} \in \hat{X}$

$$EV_\theta(\hat{x}, d) = \hat{\Gamma}_\theta(EV_\theta)(\hat{x}, d)$$

$$= \sum_j^J \ln\left[\sum_{d' \in D(y)} \exp[u(x', d'; \theta_1) + \beta EV_\theta(x', d')]\right] p(x'|\hat{x}, d, \theta_2)$$

Bellman equation in matrix form

The choice specific expected value function can be found as fixed point on the Bellman operator

$$EV(d) = \hat{\Gamma}(EV) = \Pi(d) * \ln \left[\sum_{d' \in D(y)} \exp[u(d') + \beta EV(d')] \right]$$

where

$EV(d) = [EV(1, d), .., EV(n, d)]$ and $u(d) = [u(1, d), .., u(n, d)]$

$\Pi(d)$ is a $n \times n$ state transition matrix conditional on decision d

Transition matrix for mileage is sparse

Transition matrix conditional on keeping engine

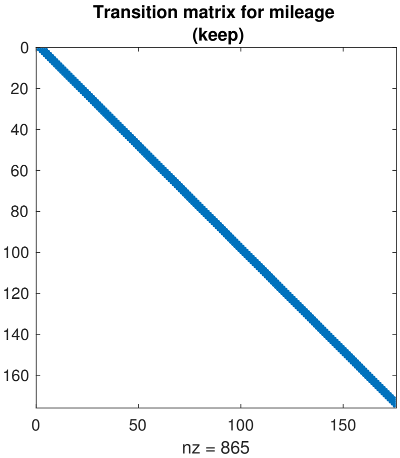
$$\Pi(d = \text{keep})_{nxn} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & & & & \pi_0 & \pi_1 & \pi_2 & 0 \\ 0 & & & & & \pi_0 & \pi_1 & \pi_2 \\ 0 & & & & & & \pi_0 & 1-\pi_0 \\ 0 & 0 & & & & & & 1 \end{pmatrix}$$

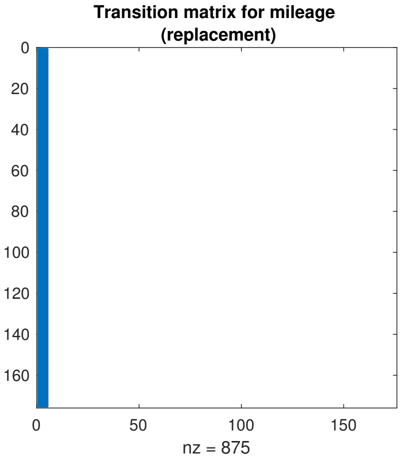
Transition matrix for mileage is sparse

Transition matrix conditional on replacing engine

$$\Pi(d = \text{replace})_{nxn} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \pi_0 & \pi_1 & \pi_2 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

Transition matrix is sparse





Monte Carlo: Rust's Table X - Group 1,2, 3

- Fixed point dimension: $n = 175$
- Maintenance cost function: $c(x, \theta_1) = 0 : 001 * \theta_1 * x$
- Mileage transition: stay or move up at most $J = 4$ grid points
- True parameter values:
  - $\theta_1 = 2 : 457$
  - $RC = 11.726$
  - $(\theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}) = (0.0937, 0.4475, 0.4459, 0.0127)$
- Solve for EV at the true parameter values
- Simulate 250 datasets of monthly data for 10 years and 50 buses

Is NFXP a dinosaur method?

Su and Judd (Econometrica, 2012)

TABLE II

NUMERICAL PERFORMANCE OF NFXP AND MPEC IN THE MONTE CARLO EXPERIMENTS[a]

β	Implementation	Runs Converged (out of 1250 runs)	CPU Time (in sec.)	# of Major Iter.	# of Func. Eval.	# of Contraction Mapping Iter.
0.975	MPEC/AMPL	1240	0.13	12.8	17.6	–
	MPEC/MATLAB	1247	7.90	53.0	62.0	–
	NFXP	998	24.60	55.9	189.4	134,748
0.980	MPEC/AMPL	1236	0.15	14.5	21.8	–
	MPEC/MATLAB	1241	8.10	57.4	70.6	–
	NFXP	1000	27.90	55.0	183.8	162,505
0.985	MPEC/AMPL	1235	0.13	13.2	19.7	–
	MPEC/MATLAB	1250	7.50	55.0	62.3	–
	NFXP	952	43.20	61.7	227.3	265,827
0.990	MPEC/AMPL	1161	0.19	18.3	42.2	–
	MPEC/MATLAB	1248	7.50	56.5	65.8	–
	NFXP	935	70.10	66.9	253.8	452,347
0.995	MPEC/AMPL	965	0.14	13.4	21.3	–
	MPEC/MATLAB	1246	7.90	59.6	70.7	–
	NFXP	950	111.60	58.8	214.7	748,487

[a]For each β, we use five starting points for each of the 250 replications. CPU time, number of major iterations, number of function evaluations and number of contraction mapping iterations are the averages for each run.

NFXP survival kit

Step 1: Read NFXP manual and print out NFXP pocket guide
Step 2: Solve for fixed point using Newton Iterations
Step 3: Recenter Bellman equation
Step 4: Provide analytical gradients of Bellman operator
Step 5: Provide analytical gradients of likelihood
Step 6: Use BHHH (outer product of gradients as hessian approx.)

STEP 1: NFXP documentation

Main references

- Rust (1987): "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher" *Econometrica* 55-5, pp 999-1033.
- Rust (2000): "Nested Fixed Point Algorithm Documentation Manual: Version 6" `https://editorialexpress.com/jrust/nfxp.html`
- Iskhakov, F. , J. Rust, B. Schjerning, L. Jinhyuk, and K. Seo (2015): "Constrained Optimization Approaches to Estimation of Structural Models : Comment." *Econometrica* 84-1, pp. 365-370.

Nested Fixed Point Algorithm

NFXP Documentation Manual version 6, (Rust 2000, page 18):

Formally, one can view the nested fixed point algorithm as solving the following constrained optimization problem:

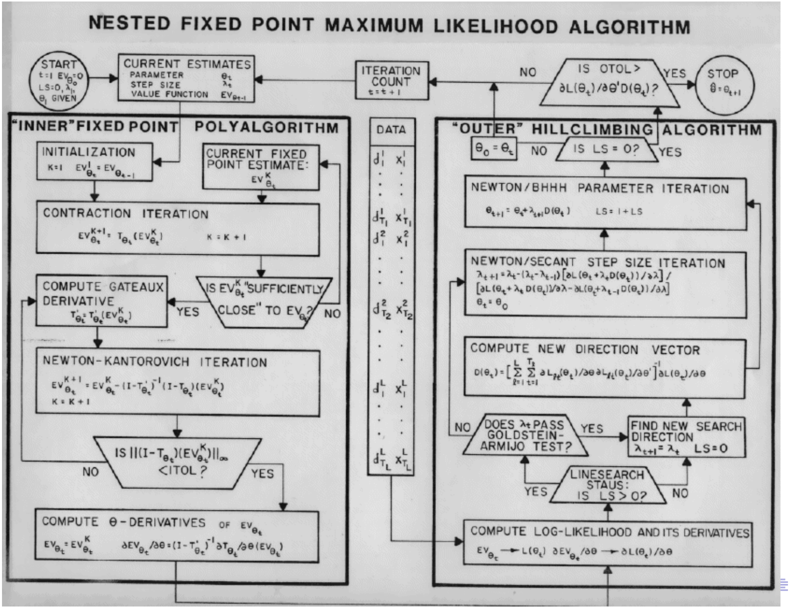
$$\max_{\theta, EV} L(\theta, EV) \text{ subject to } EV = \Gamma_\theta(EV) \tag{5}$$

Since the contraction mapping Γ *always has a unique fixed point, the constraint* $EV = \Gamma_\theta(EV)$ *implies that the fixed point* EV_θ *is an implicit function of* θ*. Thus, the constrained optimization problem (5) reduces to the unconstrained optimization problem*

$$\max_{\theta} L(\theta, EV_\theta) \tag{6}$$

where EV_θ *is the implicit function defined by* $EV_\theta = \Gamma(EV_\theta)$*.*

NFXP pocket guide



STEP 2: Newton-Kantorovich Iterations

- Problem: Find fixed point of the contraction mapping

$$EV = \Gamma(EV)$$

- Error bound on successive contraction iterations:
  $||EV_{k+1} - EV|| \leq \beta ||EV_k - EV||$
  linear convergence $\rightarrow$ slow when β close to 1
- Newton-Kantorovich:
  Solve $F = [I - \Gamma](EV_\theta) = 0$ using Newtons method
  $||EV_{k+1} - EV|| \leq A ||EV_k - EV||^2$
  quadratic convergence around fixed point, EV

STEP 2: Newton-Kantorovich Iterations

Convert the problem of finding a fixed point $EV_\theta = \Gamma(EV_\theta)$ into the problem of finding a zero of the nonlinear operator $F_\theta(EV_\theta)$

$$F_\theta(EV_\theta) = (I - \Gamma_\theta)(EV_\theta) = 0$$

where I is the identity operator on B, and 0 is the zero element of B (i.e. the zero function).

Newton-Kantorovich iteration:

$$EV_{k+1} \quad = \quad EV_k - (I - \Gamma')^{-1}(I - \Gamma)(EV_k)$$

The nonlinear operator $F_\theta = I - \Gamma_\theta$ has a Fréchet derivative $I - \Gamma'_\theta$ which is a bounded linear operator on B with a bounded inverse.

The Fixed Point (poly) Algorithm

1. Successive contraction iterations (until EV is in domain of attraction)
2. Newton-Kantorovich (until convergence)

STEP 2: Newton-Kantorovich Iterations, $\beta = 0.9999$

Successive Approximations, VERY Slow

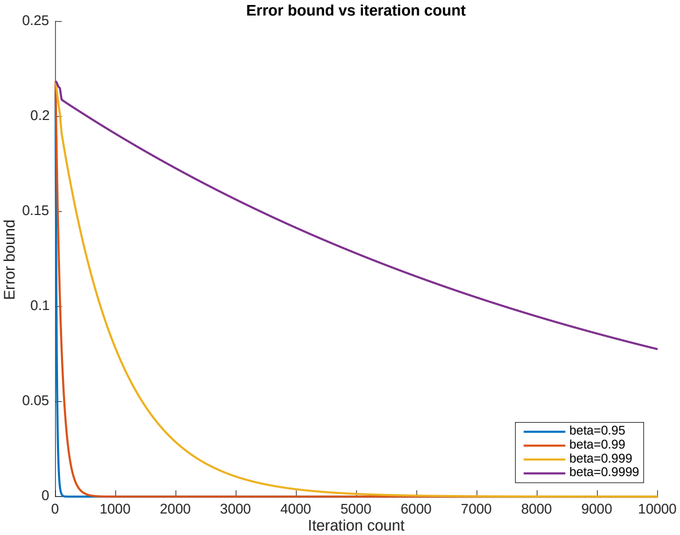
```
Begin contraction iterations
   j              tol          tol(j)/tol(j-1)
   1          0.24310300       0.24310300
   2          0.24307590       0.99988851
   3          0.24304810       0.99988564
   :               :                :
 9998         0.08185935       0.99990000
 9999         0.08185116       0.99990000
 10000        0.08184298       0.99990000
Elapsed time: 1.44752 (seconds)

Begin Newton-Kantorovich iterations
  nwt             tol
   1        9.09494702e-13
Elapsed time: 1.44843 (seconds)

Convergence achieved!
```

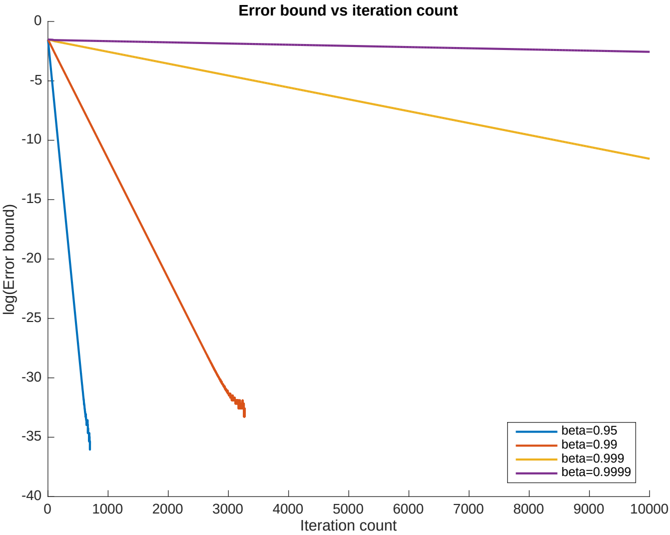
STEP 2: Newton-Kantorovich Iterations

Successive Approximations, VERY Slow



STEP 2: Newton-Kantorovich Iterations

Successive Approximations, Linear convergence



STEP 2: Newton-Kantorovich Iterations, $\beta = 0.9999$

Quadratic convergence!

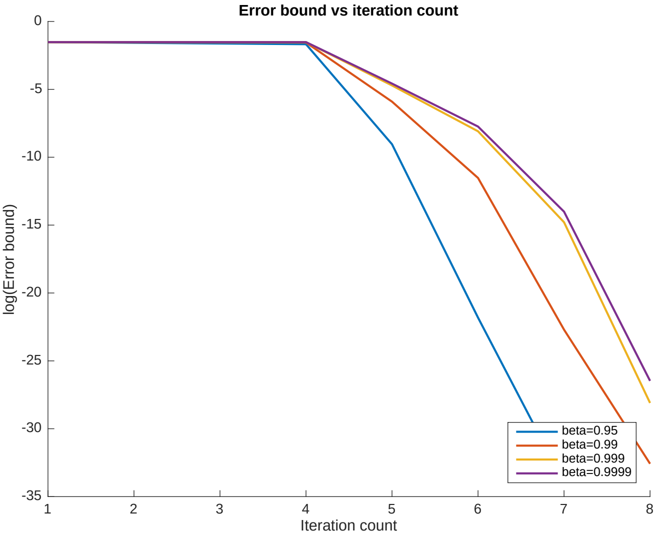
```
Begin contraction iterations
   j              tol         tol(j)/tol(j-1)
   1           0.21854635       0.21854635
   2           0.21852208       0.99988895
Elapsed time: 0.00056 (seconds)

Begin Newton-Kantorovich iterations
  nwt             tol
   1         1.03744352e-02
   2         4.40564315e-04
   3         8.45941486e-07
   4         3.63797881e-12
Elapsed time: 0.00326 (seconds)

Convergence achieved!
```

STEP 2: Newton-Kantorovich Iterations

NR: Quadratic convergence!



STEP 2: When to switch to Newton-Kantorovich

Observations:

- $tol_k = \|EV_{k+1} - EV_k\| < \beta \|EV_k - EV\|$
- tol_k quickly slow down and declines very slowly for β close to 1
- Relative tolerance tol_{k+1}/tol_k approach β

When to switch to Newton-Kantorovich?

- Suppose that $EV_0 = EV + k$.
  (Initial EV_0 equals fixed point EV plus an arbitrary constant)
- Another successive approximation does not solve this:

$$
\begin{aligned}
tol_0 &= \|EV_0 - \Gamma(EV_0)\| = \|EV + k - \Gamma(EV + k)\| \\
&= \|EV + k - (EV + \beta k)\| = (1 - \beta)k \\
tol_1 &= \|EV_1 - \Gamma(EV_1)\| = \|EV + \beta k - \Gamma(EV + \beta k)\| \\
&= \|EV + \beta k - (EV + \beta^2 k)\| = \beta(1 - \beta)k \\
tol_1/tol_0 &= \beta
\end{aligned}
$$

- Newton will immediately “strip away” the irrelevant constant k
- Switch to Newton whenever tol_1/tol_0 is sufficiently close to β

STEP 3: Recenter to ensure numerical stability

Logit formulas must be reentered.

$$P_i = \frac{\exp(V_i)}{\sum_{j \in D(y)} \exp(V_j)} = \frac{\exp(V_i - V_0)}{\sum_{j \in D(y)} \exp(V_j - V_0)}$$

and "log-sum" must be recenteret too

$$EV_\theta = \int_y \ln \sum_{j' \in D(y)} \exp(V_j) p(dy|x, d, \theta_2) = \int_y \left(V_0 + \ln \sum_{j' \in D(y)} \exp(V_j - V_0) \right) p(dy|x, d, \theta_2)$$

If V_0 is chosen to be $V_0 = \max_j V_j$ we can avoid numerical instability due to overflow/underflow

STEP 4: Analytical Fréchet derivative of Bellman operator

Fréchet derivative

- For NK iteration we need Γ'

$$EV_{k+1} = EV_k - (I - \Gamma')^{-1}(I - \Gamma)(EV_k)$$

- In terms of its finite-dimensional approximation, Γ'_θ takes the form of an $N \times N$ matrix equal to the partial derivatives of the $N \times 1$ vector $\Gamma_\theta(EV_\theta)$ with respect to the $N \times 1$ vector EV_θ
- Γ'_θ is simply β times the transition probability matrix for the controlled process $\{d_t, x_t\}$
- Two lines of code in MATLAB

STEP 1-4: MATLAB implementation of Γ_θ and Γ'_θ

```
function [ev1, pk, dbellman_dev]=bellman_ev(ev, mp, P)
    cost=0.001*mp.c*mp.grid;                    % Cost function
    vK=-cost             + mp.beta*ev;          % Value off keep
    vR=-cost(1)-mp.RC + mp.beta*ev(1);          % Value of replacing

    % Need to recenter logsum by subtracting max(vK, vR)
    maxV=max(vK, vR);
    V=(maxV + log(exp(vK-maxV)  +  exp(vR-maxV)));
    ev1=P{1}*V;

    % If requested, also compute choice probability
    if nargout>1
        pk=1./(1+exp((vR-vK)));
    end
    if nargout>2 % compute Frechet derivative
        dbellman_dev=mp.beta*bsxfun(@times, P{1}, pk');
        % Add additional term for derivative wrt Ev(1)
        % since Ev(1) enter logsum for all states
        dbellman_dev(:,1)=dbellman_dev(:,1)+mp.beta*P{1}*(1-pk);
    end
end % end of ZURCHER.bellman_ev
```

Bellman operator can also be written in terms of the smoothed value function

Define the smoothed value function $V_\sigma(x) = \int V(x,\epsilon)g(\epsilon|x)d\epsilon$ where σ represents parameters that index the distribution of the $\epsilon's$.

Under our assumptions so far, the smoothed value function, V_σ is a fixed point on the mapping

$$V_\sigma = \hat{\Gamma}_\sigma(V_\sigma) = \ln\left[\sum_{d'\in D(y)} \exp[u(d') + \beta\Pi(d') * V_\sigma]\right]$$

where $V_\sigma = [V_\sigma(1), .., V_\sigma(n)]$ and $u(d) = [u(1,d), .., u(n,d)]$

Easy to implement to implement Fréchet derivative.

STEP 1-4: MATLAB implementation based on smoothed value function

```
function [V1, pk, dBellman_dV]=bellman_integrated(V0, mp, P)
    cost=0.001*mp.c*mp.grid;                          % Cost function
    vK=-cost              + mp.beta*P{1}*V0;          % Value off keep
    vR=-mp.RC-cost(1)     + mp.beta*P{2}*V0;          % Value of replacing
    maxV=max(vK, vR);
    V1=(maxV + log(exp(vK-maxV)  +  exp(vR-maxV)));

    % If requested, also compute choice probability
    if nargout>1
        pk=1./(1+exp((vR-vK)));
    end

    if nargout>2 % compute Frechet derivative
        dBellman_dV=mp.beta*(P{1}.*pk + P{2}.*(1-pk));
    end
end % end of ZURCHER.bellman_integrated
```

STEP 5: Provide analytical gradients of likelihood

Gradient similar to the gradient for the conventional logit

$$\partial \ell_i^1(\theta)/\partial\theta = [d_{it} - P(d_{it}|x_{it},\theta)] \times \partial(v_{repl.} - v_{keep})/\partial\theta$$

- Only thing that differs is the inner derivative of the choice specific value function that besides derivatives of current utility also includes $\partial EV_\theta/\partial\theta$ wrt. θ
- By the implicit function theorem we obtain

$$\partial EV_\theta/\partial\theta = [I - \Gamma'_\theta]^{-1}\partial\Gamma/\partial\theta'$$

- By-product of the N-K algorithm: $[I - \Gamma'_\theta]^{-1}$

STEP 5: MATLAB implementation of scores

```
cost=0.001*mp.c*mp.grid;
dc=0.001*mp.grid;

% step 1: compute derivative of contraction operator wrt. parameters
dbellman_dmp=zeros(mp.n,2);
dbellman_dmp(:, 1)=(1-pk)*(-1); % Derivative wrt. RC
dbellman_dmp(:, 2)=pk.*(-dc);   % Derivative wrt. c

% step 2: compute derivative of ev wrt. parameters
devdmp=F\dbellman_dmp;

% step 3: compute derivative of log-likelihood wrt. parameters
score=bsxfun(@times, (data.d-pxR), ...
    [-ones(N,1) dc(data.x,:)] + (devdmp(ones(N,1),:)-devdmp(data.x,:)))
```

STEP 6: BHHH

- Recall Newton-Raphson

$$\theta^{g+1} = \theta^{g} - \lambda \left(\Sigma_i H_i\left(\theta^{g}\right)\right)^{-1} \Sigma_i s_i\left(\theta^{g}\right)$$

- Berndt, Hall, Hall, and Hausman, (1974):
  Use *outer product of scores* as approx. to Hessian

$$\theta^{g+1} = \theta^{g} + \lambda \left(\Sigma_i s_i s_i'\right)^{-1} \Sigma_i s_i$$

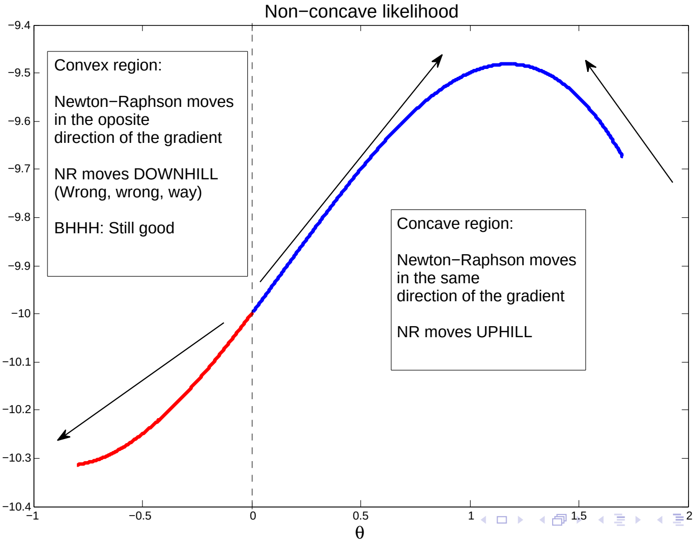
- Why is this valid? Information identity:

$$-E\left[H_i\left(\theta\right)\right] = E\left[s_i\left(\theta\right) s_i\left(\theta\right)'\right]$$

(only valid for MLE and CMLE)

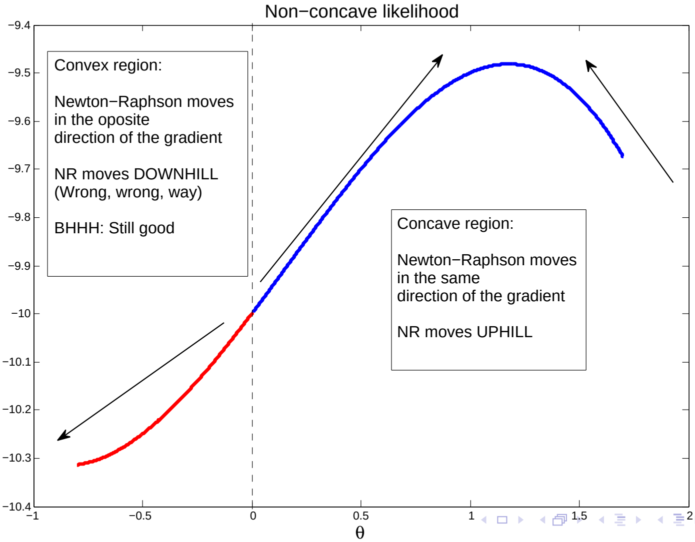
STEP 6: BHHH

Some times linesearch may not help Newtons Method



STEP 6: BHHH

Some times linesearch may not help Newtons Method



STEP 6: BHHH

Advantages

- $\Sigma_i s_i s_i'$ is always positive definite
  I.e. it always moves uphill for λ small enough
- Does not rely on second derivatives

Disadvantages

- Only a good approximation
  - At the true parameters
  - for large N
  - for well specified models (in principle only valid for MLE)
- Only superlinear convergent - not quadratic

We can always use BHHH for first iterations and the switch to BFGS to update to get an even more accurate approximation to the hessian matrix as the iterations start to converge.

STEP 6: BHHH

Advantages

- $\Sigma_i s_i s_i'$ is always positive definite
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We can always use BHHH for first iterations and the switch to BFGS to update to get an even more accurate approximation to the hessian matrix as the iterations start to converge.

STEP 6: BHHH

"The road ahead will be long. Our climb will be steep. We may not get there in one year or even in one term. But, America, I have never been more hopeful than I am tonight that we will get there. I promise you, we as a people will get there." (Barack Obama, Nov. 2008)

STEP 6: Ooups, new sheriff in town

Use BHHH!

Convergence!

β=0.9999

```
---------------------------------------------------------------------
***                     Convergence Achieved
***
---------------------------------------------------------------------
                           _
                           \`\
                           |= |
                          /-  ;.---.
                    _ __.'    (____)
                     `        (_____)
                     _'  ._ .' (____)
                      `       (___)
                     __`'-----'`
Number of iterations: 9
grad*direc          0.00003
Log-likelihood   -276.74524

    Param.          Estimates          s.e.         t-stat
---------------------------------------------------------------------
    RC                11.1525        0.9167        12.1655
    c                  2.3298        0.3288         7.0856
---------------------------------------------------------------------
```

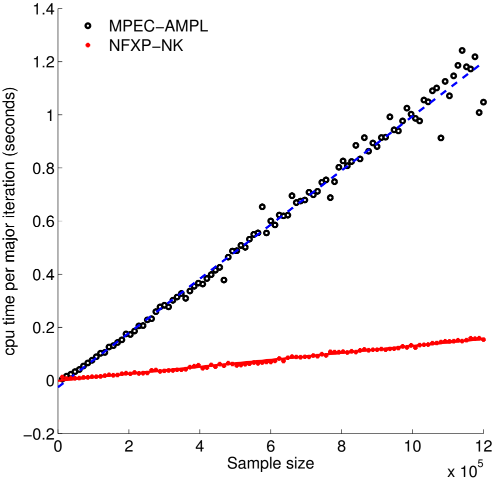
MPEC versus NFXP-NK: sample size 6,000

β	Converged (out of 1250)	CPU Time (in sec.)	# of Major Iter.	# of Func. Eval.	# of Bellm. Iter.	# of N-K Iter.
			MPEC-Matlab			
0.975	1247	1.677	60.9	69.9		
0.985	1249	1.648	62.9	70.1		
0.995	1249	1.783	67.4	74.0		
0.999	1249	1.849	72.2	78.4		
0.9995	1250	1.967	74.8	81.5		
0.9999	1248	2.117	79.7	87.5		
			MPEC-AMPL			
0.975	1246	0.054	9.3	12.1		
0.985	1217	0.078	16.1	44.1		
0.995	1206	0.080	17.4	49.3		
0.999	1248	0.055	9.9	12.6		
0.9995	1250	0.056	9.9	11.2		
0.9999	1249	0.060	11.1	13.1		
			NFXP-NK			
0.975	1250	0.068	11.4	13.9	155.7	51.3
0.985	1250	0.066	10.5	12.9	146.7	50.9
0.995	1250	0.069	9.9	12.6	145.5	55.1
0.999	1250	0.069	9.4	12.5	141.9	57.1
0.9995	1250	0.078	9.4	12.5	142.6	57.5
0.9999	1250	0.070	9.4	12.6	142.4	57.7

MPEC versus NFXP-NK: sample size 60,000

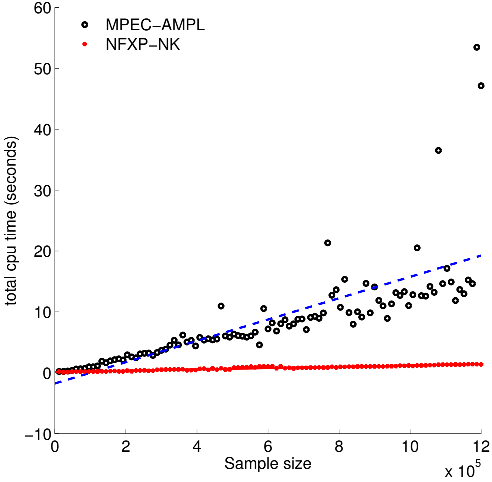
β	Converged (out of 1250)	CPU Time (in sec.)	# of Major Iter.	# of Func. Eval.	# of Bellm. Iter.	# of N-K Iter.
			MPEC-AMPL			
0.975	1247	0.53	9.2	11.7		
0.985	1226	0.76	13.9	32.6		
0.995	1219	0.74	14.2	30.7		
0.999	1249	0.56	9.5	11.1		
0.9995	1250	0.59	9.9	11.2		
0.9999	1250	0.63	11.0	12.7		
			NFXP-NK			
0.975	1250	0.15	8.2	11.3	113.7	43.7
0.985	1250	0.16	8.4	11.4	124.1	46.2
0.995	1250	0.16	9.4	12.1	133.6	52.7
0.999	1250	0.17	9.5	12.2	133.6	55.2
0.9995	1250	0.17	9.5	12.2	132.3	55.2
0.9999	1250	0.17	9.5	12.2	131.7	55.4

CPU time is linear sample size



$$T_{NFXP} = 0.001 + 0.13x \; (R^2 = 0.991), \; T_{MPEC} = -0.025 + 1.02x \; (R^2 = 0.988).$$

CPU time is linear sample size



$T_{NFXP} = 0.129 + 1.07x$ $(R^2 = 0.926)$, $T_{MPEC} = -1.760 + 17.51x$ $(R^2 = 0.554)$.

Summary remarks

Su and Judd (Econometrica, 2012) used an inefficient version of NFXP

- that solely relies on the method of successive approximations to solve the fixed point problem.

Using the efficient version of NFXP proposed by Rust (1987) we find:

- MPEC and NFXP-NK are similar in performance when the sample size is relatively small.
- NFXP does not slow down as $\beta \to 1$

Desirable features of MPEC

- Ease of use by people who are not interested in devoting time to the special-purpose programming necessary to implement NFXP-NK.
- Can easily be implemented in the intuitive AMPL language.

Inference

- NFXP: Trivial to compute standard errors by inverting the Hessian from the unstrained likelihood (which is a by-product of NFXP).
- MPEC: Standard errors can be computed inverting the *bordered Hessian* Reich and Judd (2019): Develop simple and efficient approach to compute confidence intervals.

MPEC does not seem appropriate when estimating life cycle models