

# THE IMPACT OF PORTFOLIO CONSTRAINTS IN INFINITE-HORIZON INCOMPLETE-MARKETS MODELS\*

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July 30, 1998

## Abstract

This paper examines a dynamic, stochastic exchange economy with two agents and two financial securities. Markets are incomplete and agents can have heterogeneous tastes and idiosyncratic income. We determine the impact of short-sale constraints and portfolio penalties on the equilibrium behavior of the model.

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\*The authors thank Robert Earle and an anonymous referee for detailed remarks on the first version. The paper was written while the third author was a visiting scholar at the Hoover Institution at Stanford University. He gratefully acknowledges the support of Deutsche Forschungsgemeinschaft.

# 1 Introduction

A paper by Heaton and Lucas (1996) is a prime example of a recently emerging literature which tries to determine whether the joint hypothesis of agents' heterogeneity and market incompleteness can explain observed prices in security markets. The motivation for this literature is the failure (Mehra and Prescott (1985)) of the representative consumer, consumption-based asset pricing model in explaining asset market prices. With the development of the general equilibrium theory with incomplete asset markets over the last two decades it is now well understood how to extend the Lucas-model to incorporate heterogeneous agents and market incompleteness (see Duffie et al. (1994), Magill and Quinzii (1996)). In addition to Heaton and Lucas (1996), Telmer (1993), Lucas (1994), Constantinides and Duffie (1996), Krusell and Smith (1997) and den Haan (1997) have examined various effects such as the impact of aggregate and idiosyncratic shocks, of borrowing and short-sale constraints, and of transactions costs. Without any additional special assumptions in these models there are no closed-form solutions for equilibrium prices, portfolios and consumption plans. Therefore, computational methods are needed to determine equilibria (see Heaton and Lucas (1996)).

Short-sale constraints are typically needed to ensure existence of equilibria by ruling out default or Ponzi-schemes. Zhang (1997) emphasizes that such constraints are often specified arbitrarily outside the economic model and gives endogenous borrowing constraints for a simple one-asset model. However, as soon as we consider models with more than one asset also containing long-lived assets no such endogenous wealth constraints can be obtained. Therefore, exogenously chosen short-sale constraints of some form are crucial to obtain equilibrium existence. However, it remains often unclear what impact these constraints have on the equilibrium. In this paper we analyze the impact of exogenous short-sale constraints in the model of Heaton and Lucas (1996) on the character of the equilibria using a new computational method by Judd et al. (1998).

Heaton and Lucas (1996) consider a model with two agents facing both aggregate and idiosyncratic risks. The agents can smooth consumption by trading in two securities. However, the extent of trade is limited by transaction costs and short-sale constraints on the securities. They calibrate dividends, endowments, and individual incomes to roughly match annual US data. Assuming that the exogenous shock together with agents' current period portfolio holdings form a sufficient statistic for the future evolution of the economy, Heaton and Lucas (1996) compute stationary equilibria and report results from a variety of model simulations. They impose tight short-sale constraints which ensure that the agents' portfolio holdings vary in a relatively small and compact space. They discretize this endogenous state space and develop a Gauss-Jacobi scheme which, in each state, finds a pattern of trades which nearly clears the market. Because of the discrete state space (they allow for only 30 different values for an agents' holdings in each security) this approach possibly yields large approximation errors. They report average (not maximum) errors in market clearance of up to 0.84 percent. Since Heaton and Lucas have a discrete endogenous state space they cannot easily improve their approximation. In particular, the discrete state space allows for only discrete jumps of agents portfolio holdings from period to period. When interpreting the results of their simulations, Heaton and Lucas (1996) claim that their short-sale constraints are rarely binding. In other words, they claim that the constraints are a mere technical artifact without any economic consequences.

In this paper we show that this claim of Heaton and Lucas appears to depend on their computational procedure. Indeed, we show that the algorithm by Judd et al. (1998) which yields a much better approximation results in an equilibrium in which the short-sale constraints are often binding. Consequently, the exogenously chosen short-sale constraints could potentially have a severe impact on the equilibrium and are not just a technicality without economic impact.

Surprisingly we find that at least with respect to asset prices the impact of various short-sale constraints appears to be identical. The choice of different bounds seems to be irrelevant and does not alter the behavior of the asset prices. Therefore, we also find that our results are similar to those of Heaton and Lucas.

For our calculations we use the algorithm developed by Judd et al. (1998) for the computation of equilibria in two-agent, two-asset models. Instead of discretizing the endogenous state space, Judd et al. (1998) allow agents to trade in the entire portfolio space. In particular, agents can trade in as small a quantity of an asset as they desire. The algorithm does not prohibit any kind of trading behavior and thereby does not introduce hidden barriers into the economic model. The algorithm computes polynomial approximations of the equilibrium functions on the entire state space through interpolation. Maximum relative errors are in the order of  $10^{-3}$  percent to  $10^{-5}$  percent of marginal utilities, which we interpret as agents' maximum Euler equation errors lying in the range of one dollar per \$100,000 to \$10,000,000 of consumption.

Because of the smooth approximation of trading strategies, the method of Judd et al. (1998) can be used to allow more frequent trading, an important improvement (with a discrete state space, the model cannot be calibrated to high-frequency data because per period trading becomes too small to be captured in a coarse grid). We illustrate this ability by computing the equilibrium for an economy with a discount factor of 0.99 instead of 0.95. Moreover, the algorithm of Judd et al. (1998) allows for heterogeneous levels of risk-aversion among agents and we illustrate their effect on asset prices as well.

We give the following intuitive explanation for the Heaton-Lucas procedure resulting in rarely binding short-sale constraints. Allowing only for fairly large discrete jumps of agents' portfolio holdings effectively introduces an implicit transaction cost. This transaction cost has a particular severe impact on the agents behavior when they are fairly poor. That is, they hold large short-positions in the assets and would like to trade only minimal amounts of the assets. This, however, is prohibited by the discrete portfolio space in which agents have to trade according to the numerical procedure. Therefore, the agents hardly trade at all anymore long before they hit the short-sale constraints. This behavior then leads to the observations made by Heaton and Lucas (1996).

There are at least two lessons to be learned from our exercise. First, we have to be very much aware of the fact that the type of numerical procedure we use might implicitly introduce additional constraints to the economic model leading to results which mostly depend on the algorithm but do not describe the economic model under consideration. For the specific model under consideration, our simulation results show that smooth approximations to the equilibrium functions are needed in order to discover important qualitative features of equilibrium. Second, for models with incomplete markets and heterogeneous agents, the short-sale constraints will typically be binding and thus could influence the equilibrium outcomes. This fact needs to be carefully considered when interpreting the results.

The remainder of this paper is organized as follows. Section 2 we present the model under con-

sideration. In Section 3 we use the model specifications of Heaton and Lucas (1996) and compute the impact of short-sale constraints on asset prices. Section 4 presents a variation of the model in which utility penalties for large short-positions lead to bounds on the portfolio space. In the Appendix we give a brief description of the algorithm of Judd et al. (1998) and report both running times and numerical errors for our computations.

## 2 The economic model

We consider an exchange economy where income and dividend shares as well as growth rates of aggregate consumption follow a Markov-chain (see Lucas (1994) and Heaton and Lucas (1996) for similar models).

The model of the economy  $\mathcal{E}$  features an infinite horizon and discrete time. Time is indexed by  $t \in \mathbb{N}_0$ . A time-homogeneous Markov process of exogenous shocks  $(y_t)_{t \in \mathbb{N}_0}$  is valued in a discrete set  $Y = \{1, 2, \dots, S\}$ . The underlying probability space is denoted by  $(\Omega, \mathcal{F}, Q)$  and the transition matrix by  $\mathbf{P}$ . A tribe  $\mathcal{F}_t \subset \mathcal{F}$  generated by  $\{y_0, \dots, y_t\}$  summarizes the information available at each time  $t$ . Finally, the filtration  $\mathbf{F}_t = \{\mathcal{F}_0, \dots, \mathcal{F}_t\}$  depicts how information is revealed through time  $t$ .

There are two types of infinitely lived agents indexed by  $h = 1, 2$ , and there is a single perishable consumption good in each state. Agent  $h$ 's individual endowment in period  $t$  given income state  $y \in Y$  is  $e_t^h = e^h(y_t) \in \mathbb{R}_{++}$ . Note that  $e_t^h$  depends only on  $y_t$ . In order to transfer wealth across time and states agents trade in securities. There is a (short-lived) riskless bond in each period  $t$  with price  $q_t^b$ . The bond's payoff  $d_{t+1}^b$  in units of the good in each state of the next period  $t + 1$  is specified below. Furthermore, in each period a long-lived asset in unit net supply, called stock, paying a dividend  $d^s : Y \rightarrow \mathbb{R}_+$  is traded at an after-dividend price  $q_t^s$ . We denote agent  $h$ 's portfolio in period  $t$  by  $\theta_t^h = (\theta_t^{hb}, \theta_t^{hs}) \in \mathbb{R}^2$  and his initial endowment of the stock by  $\theta_{-1}^{hs}$ . The initial endowment of the bond is always  $\theta_{-1}^{hb} = 0$  for both agents  $h = 1, 2$ .

Each agent  $h$  has von Neumann-Morgenstern preferences which are defined by a strictly monotone  $C^2$ , concave utility function  $u_h : \mathbb{R}_{++} \rightarrow \mathbb{R}$  possessing the Inada property, that is,  $\lim_{x \rightarrow 0} u_h'(x) = \infty$ , and a discount factor  $\beta_h \in (0, 1)$ . For any  $t$ -adapted consumption sequence  $c = (c_0, c_1, c_2, \dots)$  the associated utility for agent  $h$  is therefore:

$$U_h(c) = E \left\{ \sum_{t=0}^{\infty} \beta_h^t u_h(c_t) \right\}.$$

Note that the consumption sequence  $c$  is a stochastic process depending on the exogenous income states  $(y_t)_{t \in \mathbb{N}_0}$ . The expectation in the utility function is taken with respect to the probability space  $(\Omega, \mathcal{F}, Q)$ .

### Endowment and dividend processes

The aggregate endowment of the economy in period  $t$  is denoted by  $e_t(y_t) = e_t^1(y_t) + e_t^2(y_t) + d_t^s(y_t)$ . The economy is stationary in the growth rate of the aggregate endowment  $e_t$ . Specifically, we define  $\nu_t = e_t/e_{t-1}$  as the growth rate of  $e_t$ . Moreover, let  $\delta_t = d_t/e_t$  be the dividend share of the aggregate endowment. So, the sum of the individual endowments equals  $(1 - \delta_t)e_t$ . Of this

endowment agent  $h$  receives a portion  $\eta_t^h$ , that is, his endowment in period  $t$  equals  $e_t^h = \eta_t^h(1-\delta)e_t$ , where  $\eta_t^1 + \eta_t^2 = 1$ . The process of the growth rate  $\nu_t$ , the dividend share  $\delta_t$ , and the endowment shares  $\eta_t^h, h = 1, 2$ , follows a time-homogeneous Markov chain depending on the exogenous shock alone, that is,  $(\nu_t, \delta_t, \eta_t^1) = (\nu(y_t), \delta(y_t), \eta^1(y_t))$ .

The riskless payoff  $d_{t+1}^b$  of the bond in each state at period  $t+1$  equals the aggregate endowment  $e_t$  of the previous period. This definition of the bond payoff is necessary to obtain a stationary equilibrium while maintaining a compact portfolio space for the agents.

### Transaction costs on the financial markets

At each date  $t$  an agent  $h$  pays transaction costs of  $\omega(\theta_{t-1}^h, \theta_t^h)$ . We assume that  $\omega$  has the functional form

$$\omega(\theta_{t-1}, \theta_t) = \tau^b q_t^b (\theta_t^b)^2 + \tau^s q_t^s (\theta_t^s - \theta_{t-1}^s)^2,$$

where  $\tau^b, \tau^s$  are constants. Since the bond is short-lived in our model, transaction costs are paid on total bond holdings. The assumption of quadratic convex costs is needed to ensure that agents face a differentiable and convex programming problem.

### Short-sale constraints

In order to obtain an equilibrium one has to rule out the possibility of an indefinite postponement of debt and introduce constraints on agents' net wealth holdings. Furthermore, with long-lived stocks and incomplete markets one faces the usual existence problem which is due to a discontinuity in the demand function. Magill and Quinzii (1996) and Hernandez and Santos (1996) describe possible debt constraints and give proofs of generic existence. For computational purposes it is much more useful to obtain a bounded set of portfolios by imposing short-sale constraints on the assets. This assumption, while often thought of being a pure technicality which does not influence equilibrium outcomes, is not without problems. In Section 2.3 we provide a discussion of this matter.

Short sales can be constrained through a priori specified fixed exogenous lower bounds on the portfolio variables, that is,

$$\theta_t^{hb} \geq -B^{hb} \text{ and } \theta_t^{hs} \geq -B^{hs},$$

where  $B^{hb}, B^{hs} \geq 0$ . Note that we define the bounds on short sales as agent dependent since it is certainly realistic to assume that an agent's income influences how much he can borrow.

Heaton and Lucas (1996) define the bond to pay off one unit of the consumption good in each state over all time periods. With such a definition the borrowing constraint on the bond must grow with the aggregate endowment for an equilibrium to be stationary. Otherwise, the amount of relative wealth that can be transferred across time would tend to zero as time progresses.

In contrast, we can define the short-sale constraints on both the stock and the bond to be time- and state-independent. Since both the stock's dividends and the bond payoffs grow along with the aggregate endowment, the amount of wealth traded increases over time even when the number of shares traded remains the same. While Heaton and Lucas (1996) have to treat the stock and the

bond differently, our definitions allow symmetric conditions on both assets.

## 2.1 Competitive equilibrium

The notion of a competitive equilibrium for an economy  $\mathcal{E}$  is now defined as follows:

**Definition 1** *A competitive equilibrium for an economy  $\mathcal{E}$  is a collection of  $\mathbf{F}_t$ -measurable portfolio holdings  $\{(\theta_t^1, \theta_t^2)\}$  and asset prices  $\{q_t\}$  satisfying the following conditions:*

(1)  $\theta_t^{1b} + \theta_t^{2b} = 0$  and  $\theta_t^{1s} + \theta_t^{2s} = 1$  for all  $t$ .

(2) For each agent  $h$ :

$$(c^h, \theta^h) \in \arg \max U_h(c^h) \text{ s.t. } \begin{cases} c_t^h = e_t^h + \theta_{t-1}^{hb} e_{t-1} + \theta_{t-1}^{hs} (q_t^s + d_t^s) - \theta_t^h q_t - \omega(\theta_{t-1}^h, \theta_t^h) \\ \theta_t^{hb} \geq -B^{hb}, \theta_t^{hs} \geq -B^{hs} \end{cases}$$

Note that by Walras' law condition (1) ensures that good markets clear, that is, aggregate consumption equals aggregate endowments minus the resources 'burned' for transactions.

For the marginal transaction costs we define the notation

$$\begin{aligned} \omega_{s1} &= \frac{\partial \omega(\theta_{t-1}, \theta_t)}{\partial \theta_{t-1}^s} \\ \omega_{s2} &= \frac{\partial \omega(\theta_{t-1}, \theta_t)}{\partial \theta_t^s} \\ \omega_b &= \frac{\partial \omega(\theta_{t-1}, \theta_t)}{\partial \theta_t^b} \end{aligned}$$

Denoting by  $\mu_t^{hb}$  and  $\mu_t^{hs}$  the Lagrange multipliers for the short-sale constraint on the bond and stock, respectively, at time  $t$  we obtain the following first-order conditions for agent  $h$ :

$$\begin{aligned} (q_t^b + \omega_b) u'_h(c_t) &= \beta_h E_t(e_t u'_h(c_{t+1})) + \mu_t^{hb} \\ \mu_t^{hb} (\theta_t^b + B^{hb}) &= 0 \\ \mu_t^{hb} &\geq 0 \\ \theta_t^b + B^{hb} &\geq 0 \\ (q_t^s + \omega_{s2}) u'_h(c_t) &= \beta_h E_t \{ (q_{t+1}^s + d_{t+1}^s - \omega_{s1}) u'_h(c_{t+1}) \} + \mu_t^{hs} \\ \mu_t^{hs} (\theta_t^s + B^{hs}) &= 0 \\ \mu_t^{hs} &\geq 0 \\ \theta_t^s + B^{hs} &\geq 0 \end{aligned}$$

The expectations  $E_t$  in the first-order conditions are taken with respect to the one-period transition probabilities from time  $t$  to time  $t+1$  given a state  $y_t$  and using the transition matrix  $\mathbf{P}$ .

Under our assumptions on preferences the first-order conditions of both agents together with the market-clearing conditions are necessary and sufficient for equilibrium.

## 2.2 Recursive equilibria

In order to compute an equilibrium for an infinite horizon model it is necessary to focus on equilibria which are dynamically simple. That is, it must be possible to describe the state of the system by a small number of parameters which provide a sufficient statistic for the evolution of the system. In contrast to the complete-markets model, the state space for our models will also include endogenous variables since they obviously influence the evolution of security prices in equilibrium. In particular, one expects the distribution of agents' portfolio holdings to influence equilibrium prices. We denote the set of agent 1's possible equilibrium portfolio holdings by  $\Theta \subset \mathbb{R}^2$ .

The usual assumption in the applied literature (see for example Telmer (1993) or Heaton and Lucas (1996)) is that the exogenous state and the agents' portfolio holdings alone constitute a sufficient state space for the evolution of the infinite horizon economy. Moreover, the existence of continuous policy functions  $f$  and price functions  $g$  is postulated, which map last period's portfolio holdings and the current exogenous shock into the current period portfolio holdings and asset price, respectively. For our model these assumptions mean that  $\Theta$  is the endogenous state space and that there exists a continuous function  $f : Y \times \Theta \rightarrow \Theta$  which determines agent 1's optimal portfolio choice in the current period given an exogenous shock  $y \in Y$  and portfolio holding  $\theta_- \in \Theta$ . Similarly, a continuous price function  $g : Y \times \Theta \rightarrow \mathbb{R}_{++}^2$  maps the current period exogenous shock  $y \in Y$  and agent 1's portfolio holding  $\theta_- \in \Theta$  into the current period prices of the securities.

While the assumptions underlying a recursive equilibrium are intuitive, there are no known conditions on the fundamentals of the economy which ensure existence of such an equilibrium. Judd et al. (1997 and 1998) gives a detailed discussion of the problems underlying the existence of recursive equilibria. We follow the standard assumption in the applied literature and assume the existence of a recursive equilibrium. Similar to Judd et al. (1998) we find that in models with short-sale constraints but without transaction costs, there is a one-to-one equilibrium relationship between the agents' wealth levels and their portfolio holdings. This allows us to plot equilibrium outcomes as functions of only one variable making the corresponding graphs much clearer.

## 2.3 Exogenous bounds on the endogenous state space

To both prove the existence of a recursive equilibrium and approximate such an equilibrium it is important to ensure that the set of portfolios that can be part of an optimal solution for the agents is bounded. Duffie et al. (1994) discuss the role of compactness for existence<sup>1</sup>; for computations boundedness is important because we compute equilibrium prices and portfolios as a function of some state variables and we want to approximate this function with splines with a finite number of nodes. Obviously, we cannot expect to approximate the equilibrium price and portfolio functions well on the endogenous state space  $\Theta$  if this set is unbounded.

From a theoretical point of view, it suffices for a proof of existence of a competitive equilibrium to assume that agents face some debt constraint so that Ponzi-schemes are ruled out, see Magill and Quinzii (1996) and Hernandez and Santos (1996). In the applied literature, however, typically borrowing and short-sale constraints are used to rule out Ponzi-schemes and default. Zhang (1997)

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<sup>1</sup>Note that Duffie et al. (1994) consider a different model since their economy is stationary in levels as opposed to growth rates. However, it is fairly straightforward to generalize their proof to incorporate a growth economy where all agents have homothetic preferences.

provides a detailed analysis of the effect of different specifications of debt and short-sale constraints for a model with a single asset and shows that the specifications affect the results considerably. In particular, he is able to derive endogenous borrowing constraints for his simple model. However, for the Heaton and Lucas model it is easy to see that a debt constraint cannot imply restrictions on the norm of agents' portfolios. Consider the easiest case where agents are not allowed to hold negative wealth, that is,  $e_t^h + \theta_{t-1}^{hs}(q_t^s + d_t^s) + \theta_{t-1}^{hb}e_{t-1} > 0$ . For each  $\theta^{hs} \in \mathbb{R}$  it is possible to find a  $\theta^{hb}$  such that the debt constraint is satisfied. For this reason we have to introduce some portfolio restriction in order to obtain a bounded region for which we need to evaluate the policy and price functions. Short-sale constraints immediately result in a compact feasible portfolio set for the agents. We emphasize that these constraints are more than just a technical or computational artifact helping with the analysis. Indeed, as our simulations in Section 3 show, one of the short-sale constraints is often binding. Henceforth, these constraints do influence the character of the economic equilibrium. Most likely, the character will be altered in comparison to the equilibrium of an economy without any portfolio restriction. It is important to keep these facts in mind when analyzing the computational results.

### 3 Incomplete markets and short-sale constraints

It is commonly thought that in a Markovian framework like ours the effects of market incompleteness on prices and allocations are very small. In this section we briefly discuss this view in order to illustrate our algorithm with some examples. While we do not intend to provide any new insight to the question to what extent the joint hypothesis of market incompleteness and consumer heterogeneity enriches the pricing implications of the model, we do point out the effects of incomplete markets where they are relevant to the results.

First note that as in Judd et al. (1998) there exists a one-to-one equilibrium relationship in models without transaction costs between the possible portfolio holdings agents might hold in equilibrium and their respective wealth levels. Although transaction costs typically destroy this relationship it continues to hold for many wealth levels, in particular when the transaction costs are small. Therefore, defining  $w = e^1 + \theta^{1b}d^b + \theta^{1s}(q^s + d^s)$  we can graph the current period equilibrium prices and equilibrium portfolio holdings as a function of wealth  $w$ . After computing the equilibrium prices and portfolios we simulated the economies for 10000 periods. The figures we show in the discussions below depict scatter plots of simulated prices and portfolios against the wealth levels of agent 1.

We assume that agents have constant relative risk aversion utility functions with a coefficient of relative risk aversion  $\gamma_h$  and identical time-preference factors  $\beta$ . The initial stock positions are always  $\theta_{-1}^1 = \theta_{-1}^2 = 0.5$ . We assume throughout that transaction costs are very low and given by  $\tau^b = 0.00001, \tau^s = 0.0001$ . In order to be able to display bond and stock holdings in the same figure in a meaningful fashion we slightly alter the base levels of the endowments and dividends to be  $e_{-1} = 10$  and the bond payoff to be  $e_{t-1}/e_{-1}$ . As a result, a borrowing constraint of 6 percent of aggregate endowment results in constraining the bond position of the agents to be not less than 0.6. We start the economy in state  $y_0 = 1$ . See Appendix C for a specification of the parameters for the endowment and dividend processes and the transition matrix of the Markov process of exogenous shocks.



### 3.1 The effect of portfolio constraints

In this subsection we assume that  $\gamma_1 = \gamma_2 = 1.5$  and that  $\beta = 0.95$ . We examine two different specifications for short-sale constraints. In the first specification agents are not allowed to short the stock and can at most promise to pay 6 percent of aggregate endowment in bonds. This specification is almost identical to the least tight constraints imposed in Heaton and Lucas (1996). Figure 1a depicts the equilibrium net portfolio holdings in state 1 as a function of agent 1's total wealth<sup>2</sup>. It is clear that the short-sale constraints do influence agents' portfolio holdings. In fact, simulations show that the constraint on bond holdings is binding for approximately two-thirds of the time. This finding, which contradicts the claim in Heaton and Lucas that short-sale constraints rarely bind (see the above discussion on differences of the computational strategies for an explanation of this contradiction), seems to indicate that the choice of portfolio bounds is an important determinant of equilibrium asset prices. Zhang (1997) shows that the expected return and volatility of a single asset vary significantly as the short-sale constraints on this asset vary. However, as Zhang points out, the real constraints faced by agents are the wealth constraints which are implied by the chosen short-sale constraints for one-asset models. In a framework with more than one asset there are situations where binding short-sale constraints on one asset do not imply that the agents are constrained in their borrowing since they have the possibility of trading in the other asset.

We relax both the constraint on stock holdings and the constraint on bond holdings significantly. Instead of assuming that agents can borrow 6 percent of average income we assume now that they can borrow up to 8 percent and that in addition they can short 0.1 units of the stock. Figure 1c is the analogue of Figure 1a for this case of less restrictive constraints. In this case simulations show that the constraint is approximately binding 55 percent of the time.

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<sup>2</sup>All functions shown look similar in all other states, we show the functions only conditional on state 1 to simplify the figures. The curve that is flat for wealth levels between 5 and 10 having the value -0.06 represents the bond holding of agent 1. Since agent 1 has an original position of  $\theta_{-1}^1 = 0.5$  a net portfolio position of -0.4 means that he is holding a stock portfolio of 0.1. The evolution of the economy over time depends very much on the state in which the economy starts at time  $t = 0$ . Therefore the portfolio curves in Figures 1a and 1c do not display symmetry around some medium wealth level.

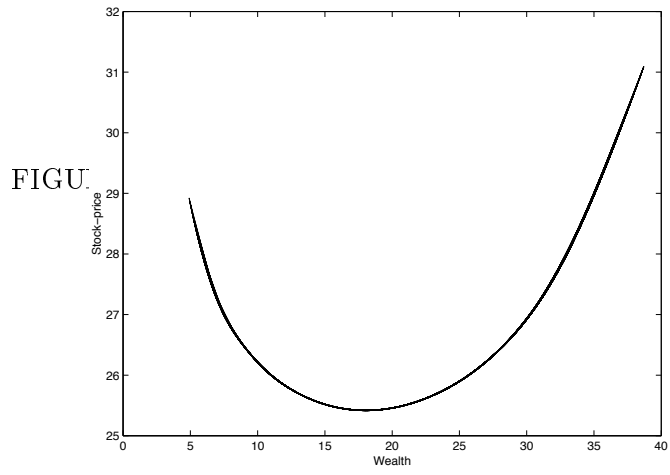
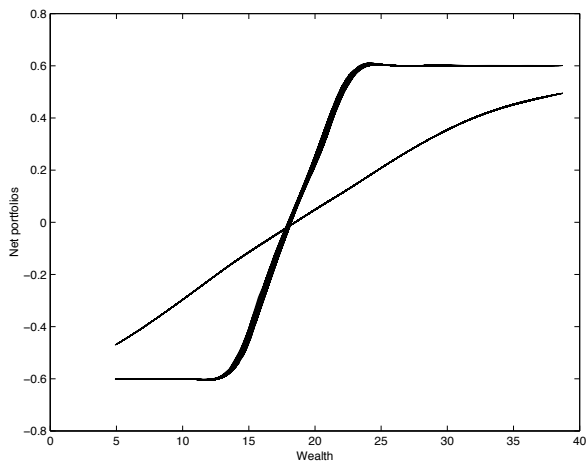


FIGURE 1b — Stock price.

Figure 1b depicts the equilibrium stock-price function in state 1 for the tight constraints, while Figure 1d shows the same function for the relaxed constraints. Surprisingly the figures appear to be quite similar. These figures show that a relaxation of short-sale constraints has no effect on equilibrium prices. While short-sale constraints often do bind in this model, and while short-sale constraints also affect the distribution of wealth, their effect on asset prices is negligible as long as the choice of short-sale constraints does not imply binding wealth constraints.

Do note, however, the effect of income heterogeneity among the agents and incomplete security markets. In a representative agent model, the equilibrium prices would depend on the exogenous shock alone, the shape of the equilibrium stock price graphs implies that asset price volatility is higher in the incomplete markets model than in a representative agent model with similar endowment processes.

### 3.2 Heterogeneous preferences

We now assume that agents have heterogeneous preferences. We assume that agent 2 is more risk averse than agent 1 and we set  $\gamma_1 = 1$  and  $\gamma_2 = 2$  for the agents' coefficient of relative risk aversion. We keep our specification of short-sale constraints at 8 percent of aggregate income for the bond and  $-0.1$  units for the stock. Figure 2a shows the analogue of Figures 1a and 1c. The tiny transaction costs we impose, do have an effect in this case. There is no longer a one-to-one relationship between portfolio holdings and wealth levels. But the functions of wealth are still a convenient and clear representation of the equilibrium. The heterogeneity in risk aversion does not seem to affect the shape of the portfolio graphs. However as Figure 2b shows, the less risk-averse agent does end up holding most of the wealth in the economy. The figure is a histogram of wealth levels which occurred on the equilibrium path during a simulation. Figure 2c and Figure 2d show the equilibrium prices for bond and stock, respectively, as functions of the wealth of the less risk-averse agent 1. The heterogeneity of the preferences changes the shape of the price function significantly.

### 3.3 The time-discount factor

The average interest rate produced by the specification of growth rates, dividends, and income distribution is unrealistically high. Heaton and Lucas (1996) point out that this can be corrected by assuming larger value for  $\beta$ , but do not do so in order to improve the performance of their algorithms. It is well known, however, that the choice of  $\beta$  affects the equilibrium outcome beyond the level of average stock and bond return (see for example Calvet (1998)). We compute the equilibrium for  $\beta = 0.99$  and  $\gamma_1 = \gamma_2 = 1.5$ . Figure 3b shows a simulation of the equilibrium interest rate for 200 periods. While the average interest rate is still slightly above the U.S. historical average there are time periods where the interest rate is almost -2 percent (after 110 time periods).

As Figures 3c-d show, the equilibrium price graphs differ significantly from the ones with a low beta. Besides the difference in absolute levels (which is to be expected and which is obviously needed in order to drive the interest rate to a realistic level) one also notes that prices are much more volatile in this case.

## 4 Portfolio bounds through penalties

In this section we propose a different approach to obtain bounded portfolios. Agents are allowed to hold portfolios of any size but get penalized for large portfolio holdings; the intuition behind such a model assumption is that there are certain costs associated with large short positions which are not explicitly modeled. To capture these effects we introduce penalties to agents' utilities. When these penalties get sufficiently large, agents have no interest to inflate their portfolios resulting in bounded asset positions. The advantage of utility penalties on large short positions is that this restrictions does not constitute an a priori exogenous constraint on short sales. Instead, the penalties lead to endogenous avoidance of large short positions.

For this approach to bounded portfolios we compute first and second moments of security returns and compare our results to the results obtained by Heaton and Lucas (1996).

### 4.1 Penalties on portfolios

We define this penalty on portfolios in such a fashion that small short positions are not punished. Only if the short sales become sufficiently large a penalty is imposed. Moreover, we allow for a larger short position in an asset to be unpenalized in the presence of a long position of sufficient size in the other asset. Modeling the penalty in this fashion is motivated by the presence of collateral; agents are typically allowed to take on large short positions if they are able to provide sufficient collateral in form of another asset. A penalty constraint could have the following impact for example: if agent  $h$  does not hold the stock at all, he can borrow up to  $-B^{hb} = -1$  on the bond market without being penalized; however, if he holds one unit of the stock, he can borrow up to  $-B^{hb} = -1.5$  on the bond market before the penalty sets in.

In order to define such a penalty constraint, we slightly tilt the rectangular region given through short-sale constraints, so that we end up with a region as depicted in Figure 4. For our computational procedure we define sets  $K^h$  by four parameters  $\alpha_1^{hb}$ ,  $\alpha_2^{hb}$ ,  $\alpha_1^{hs}$  and  $\alpha_2^{hs}$ . We define  $K^h$  as the set of  $\theta^h$  satisfying three inequalities:

$$\begin{aligned}\theta^{hs} &\geq \alpha_1^{hb}/\alpha_1^{hs}\theta^{hb} + \psi_2^h \\ \theta^{hs} &\geq -\alpha_1^{hs}/\alpha_1^{hb}\theta^{hb} + \alpha_1^{hs} \\ \theta^{hs} &\leq \alpha_2^{hb}/\alpha_2^{hs}\theta^{hb} + \psi_1^h\end{aligned}$$

where  $\psi_1^h = (3(\alpha_2^{hs})^2 + (\alpha_2^{hb})^2)/(2\alpha_2^{hs})$  and  $\psi_2^h = 1.5\alpha_1^{hs} + 0.5(\alpha_1^{hb})^2/\alpha_1^{hs}$ . Figure 4 shows the resulting set  $K = K^1 \cap K^2$  of possible portfolio holdings for agent 1 for  $\alpha_1^{hb} = \alpha_1^{hs} = -1$  and for  $\alpha_2^{hb} = \alpha_2^{hs} = 1$ . Note that both agents are only restricted with respect to taking short positions in one or both assets. Therefore we impose only three constraints per agent. Nevertheless the set  $K$  is a box since the lower constraints on agent 2 in equilibrium (by market clearing) result in upper constraints for agent 1.

Now we define for  $\delta \in \mathbb{R}_{++}$  the set  $K^{h\delta}$  as the subset of  $K^h$  whose points are at least a distance  $\delta$  away from the boundary. Let  $\phi$  denote the distance between a point  $\theta \notin K^{h\delta}$  and the boundary of  $K^{h\delta}$ , then the penalty function we use for our model is of the form where

$$\rho^h(\theta) = \begin{cases} \kappa^a(\phi)^4 & \text{for } \theta \notin K^{h\delta} \\ 0 & \text{otherwise} \end{cases}$$

where  $\kappa^a$ ,  $a \in \{b, s\}$ , is some positive constant. Note, there is no punishment for large long positions. If  $\kappa^a$  is sufficiently large the penalty function almost acts like a hard short-sale constraint on the corresponding asset  $a \in \{b, s\}$ . The portfolio penalties lead our agents to have utility functions over consumption and portfolio holdings of the form

$$V_h(c, \theta) = U_h(c) - E \left\{ \sum_{t=0}^{\infty} \beta^t \rho^h(\theta_t) \right\}.$$

The notion of a competitive equilibrium for an economy with portfolio penalties is now defined analogously to Definition 1. Using the notation

$$\begin{aligned} \rho_b^h &= \frac{\partial \rho^h(\theta_t)}{\partial \theta_t^b} \\ \rho_s^h &= \frac{\partial \rho^h(\theta_t)}{\partial \theta_t^s} \end{aligned}$$

for the marginal utility penalty the first-order conditions for agent  $h$  are:

$$\begin{aligned} (q_t^b + \omega_b)u'_h(c_t) - \rho_b^h &= \beta_h E_t(e_t u'_h(c_{t+1})) \\ (q_t^s + \omega_{s2})u'_h(c_t) - \rho_s^h &= \beta_h E_t \{ (q_{t+1}^s + d_{t+1} - w_{s1})u'_h(c_{t+1}) \} \end{aligned}$$

Note that we no longer have to deal with inequality constraints in the first-order conditions thereby simplifying the solution procedure for computing equilibria, see Judd et al. (1998).

## 4.2 Incomplete markets and transaction costs

In the framework of our model with portfolio penalties we evaluate the effects of transaction costs using the parametric data of Heaton and Lucas (1996). The main difference between our analysis and Heaton and Lucas' work is that we disentangle the effects of transaction costs on bonds and transaction costs on stocks and that we examine the effects of transaction costs on the distribution of asset holdings. Heaton and Lucas argue that because "portfolio balance is a second-order consideration relative to consumption smoothing" there have to be considerable transaction costs in both markets.

To compute first and second moments of returns and trading volume we perform 100 simulations, each for 1000 periods. We then take the average values as estimates for the moments. We first examine the case where the agents are only allowed to sell short a small percentage of the stock without holding bonds as collateral. We allow them to hold a short position in bonds alone which amount to around ten percent of aggregate income. That is we set  $\alpha_1^{hs} = -0.1$  and  $\alpha_1^{hb} = -0.1 \cdot e_t$  (and  $\alpha_2^h$  appropriately).

We consider 5 different specifications of transaction costs:  $(\tau_b^1, \tau_s^1) = (0.0005, 0.01)$ ,  $(\tau_b^2, \tau_s^2) = (0.005, 0.05)$ ,  $(\tau_b^3, \tau_s^3) = (0.01, 0.1)$ ,  $(\tau_b^4, \tau_s^4) = (0.1, 0.1)$ , and  $(\tau_b^5, \tau_s^5) = (0.5, 1.0)$ . The effects of these costs on returns are summarized in Table I. The effects on trading are summarized in Table II. The table reports average volume of trading as a percentage of per capita income and its standard deviation as well as the fraction of transaction costs on total trading activity.

TABLE I  
Asset Returns

	Av. Bond Return	Av. Stock Return	S.D. Bond Return	S.D. Stock Return
$\tau^1$	6.4 %	7.0 %	4.6 %	2.6 %
$\tau^2$	6.3 %	7.0 %	4.5 %	2.6 %
$\tau^3$	6.3 %	7.0 %	4.4 %	2.6 %
$\tau^4$	5.8 %	7.0 %	4.3 %	2.7 %
$\tau^5$	4.8 %	7.0 %	5.4 %	3.1 %
$\tau_{C2}^3$	6.7 %	7.0 %	2.0 %	2.6 %
$\tau_{C2}^4$	6.2 %	7.1 %	2.5 %	2.6 %

TABLE II  
Trading Volume and Average Cost per Unit Volume

	Av. Bond Volume	Av. Stock Volume	S.D. Bond Volume	S.D. Stock Volume	Av. Bond Tr. Costs	Av. Stock Tr. Costs
$\tau^1$	2.544 %	19.687 %	2.728 %	9.502 %	0.0068 %	0.0371 %
$\tau^2$	2.012 %	19.862 %	2.137 %	9.441 %	0.0536 %	0.1874 %
$\tau^3$	2.030 %	19.621 %	2.162 %	9.297 %	0.1082 %	0.371 %
$\tau^4$	0.459 %	20.566 %	0.518 %	10.197 %	0.239 %	0.387 %
$\tau^5$	0.557 %	16.971 %	0.593 %	8.348 %	1.4472 %	3.173 %
$\tau_{C2}^3$	2.152 %	19.932 %	2.257 %	9.042 %	0.115 %	0.382 %
$\tau_{C2}^4$	0.415 %	20.941 %	0.465 %	9.992 %	0.217 %	0.399 %



We perform the same simulations for the case where the specifications of the short-sale constraints are more realistic. We assume  $\alpha_1^{hs} = -0.5$  and  $\alpha_1^{hb} = -0.15 \cdot e_t$ . With this specification agent 1 is allowed to hold  $-0.17 \cdot e_t$  bonds when holding 0.12 stocks or  $-1.0$  stocks when holding  $0.75 \cdot e_t$  bonds. For this case we only consider the specifications  $\tau^3$  and  $\tau^4$ , and they are reported in Tables I and II under  $\tau_{C2}^3$  and  $\tau_{C2}^4$ .

### 4.3 A few observations

First, similar to the findings of Heaton and Lucas (1996), the equity premium is unrealistically low for all realistic specifications of transaction costs. As Heaton and Lucas point out, although markets are incomplete and it is costly to trade the existing securities, the agents are nevertheless able to smooth out a substantial part of their idiosyncratic risks. Even for unreasonably high transaction costs ( $\tau^4$  and  $\tau^5$ ) where this behavior becomes very costly, no realistic values of the equity premium can be achieved. As we relax the penalty, that is, as we increase the set  $K^\delta$ , the equity premium decreases even further. This conforms with the intuition that more trading opportunities allow more risk-sharing and hence increase consumption smoothing. In this case the riskless rate is almost as high as the expected return on equity.

Second, the second moments of returns are unrealistic for all specifications of the transaction costs. This is a well-known result for both the representative agent model and for the case with incomplete markets. For a given specification of the model which produces realistic values of the equity premium either the resulting volatility of bond returns is too high or the resulting stock return volatility is too low compared with actually observed volatility in financial markets. Surprisingly as we relax the penalty, the second moments become more realistic. With the second specification of the penalty the volatility of both stock and bond returns decreases, however, the bond return is now less volatile than the stock return.

Third, the trading volume is sensitive to transaction costs. In particular, one can observe that for specifications of transaction costs which produce more realistic equity premia, the amount of bonds held compared with the agents' stock holdings is unrealistic. This is caused by the fact that we modeled the bond as being short-term riskless borrowing and we did not allow for the possibility to roll over the bond each period without transaction costs. Note that overall average transaction expenditures tend to be quite small even for cases where portfolio holdings are unrealistic. This is caused by the assumption of convex transaction costs.

It is difficult to assess how to extend the model to produce a realistic trading volume. Observed trading volume (especially for shorter time periods than the ones considered here) might be caused in large parts by the arrival of new information. It is not clear what percentage of observed trading volume is caused by incomplete equitization of risk (i.e. the failure to trade claims on future income streams) and what is caused by informational price moves. Furthermore, empirical studies often examine daily trading volume as opposed to the yearly volume we model. Note however, that our average trading volume differs substantially from than that reported in Heaton and Lucas (1996). This is likely due to the fact that they use a discrete endogenous state space and therefore essentially look at a different model. The discrete-state aspect of their analysis introduces another transaction cost, one not motivated by economic considerations but rather introduced solely for

reasons of computational tractability. The differences point out the value of using a continuous state space approach.

In summary, convex transaction costs do not offer a realistic explanation for the observed high US-equity premium. The only way they seem to affect the equity premium is through lower bond returns. It is very unrealistic, however, to have a higher volatility in bond return than in stock returns.

## Appendix

### A A spline collocation algorithm

We give a brief description of the main elements of the algorithm of Judd et al. (1998) which we used for the computations in this paper.

For each exogenous state the equilibrium functions  $f$  and  $g$  (see Section 2.2) are approximated through piecewise cubic polynomials. These two-dimensional polynomials are represented through B-splines in a tensor-product approach. For the determination of the coefficients of the spline functions a collocation grid is chosen of as many points in the endogenous state space as there are unknown coefficients. At each of these points and for every exogenous shock the equilibrium-describing system of equations of the agents' first-order conditions and the market-clearing conditions is solved using a Gauss-Jacobi approach, that is, starting with some initial guess the approximate equilibrium functions are computed in an iterative process. In each iteration the equilibrium equations are solved – via a homotopy or Newton method – at each collocation point and for all exogenous shocks using the previous iterate as the policy functions governing the model's behavior in the subsequent period. The coefficients of the new iterate are determined through interpolation. The algorithm finally terminates when a stopping criterion is reached, such as that the maximum difference of the approximate equilibrium functions over all states and collocation points falls below some prespecified error tolerance.

Note that the algorithm computes smooth approximations to the equilibrium functions. Therefore, in the subsequent simulations agents are allowed to trade arbitrary small amounts of the assets and are not confined to a discrete grid of portfolio positions.

### B Errors and running times

In order to evaluate the quality of our approximations to the true equilibrium transition functions, we compute the residuals of the Euler equations. In order to obtain relative errors we divide them by the price times the agent's current period marginal utility. We evaluate these errors at 100 times 100 points in our state space and report the maximum error, the average errors usually lie around one to two orders of magnitude below these maximum errors. This way of reporting errors is similar to the errors reported in Heaton and Lucas (1996). For each agent's Euler equation they compute the price which makes it hold exactly and then take the difference of these prices. All running times refer to our computations on a PentiumPC233.

### B.1 Errors and running times for Section 3

TABLE III  
Running Times and Errors

$\beta$	$\gamma_1$	$\gamma_2$	Maximum error in %	Running time in hours.minutes
0.95 (tight)	1.5	1.5	$2.11 \cdot 10^{-6}$	8.36
0.95 (loose)	1.5	1.5	$5.09 \cdot 10^{-6}$	8.52
0.95	1	2	$1.02 \cdot 10^{-5}$	9.31
0.99	1.5	1.5	$7.98 \cdot 10^{-5}$	33.50

### B.2 Errors and running times for Section 4

TABLE IV  
Running Times and Errors

	Maximum error in %	Running time in hours.minutes
$\tau^1$	$2.19 \cdot 10^{-3}$	8.25
$\tau^2$	$7.65 \cdot 10^{-4}$	8.16
$\tau^3$	$1.29 \cdot 10^{-4}$	7.50
$\tau^4$	$9.44 \cdot 10^{-5}$	7.33
$\tau^5$	$2.85 \cdot 10^{-5}$	7.09
$\tau_{C2}^3$	$5.02 \cdot 10^{-4}$	3.02
$\tau_{C2}^4$	$8.13 \cdot 10^{-5}$	2.54

## C Parameters for the Heaton-Lucas model

Heaton and Lucas (1996) assume that aggregate growth rates follow an 8-state Markov chain and calibrate their model using the PSID (Panel Study of Income Dynamics) and NIPA (National Income and Product Accounts). We use their calibration for the ‘Cyclical Distribution Case.’

TABLE V  
Markov Chain Model for Exogenous State Variables

State	$\nu$	$\delta$	$\eta^1$
1	0.9904	0.1403	0.3279
2	1.0470	0.1437	0.4405
3	0.9904	0.1562	0.3279
4	1.0470	0.1600	0.4405
5	0.9904	0.1403	0.6721
6	1.0470	0.1437	0.5595
7	0.9904	0.1562	0.6721
8	1.0470	0.1600	0.5595

TABLE VI  
Transition Probability Matrix  $\mathbf{P}$

.4365	.2343	.0881	.0473	.1515	.0098	.0306	.0020
.2416	.3461	.0337	.0483	.1698	.1201	.0237	.0168
.0555	.0458	.3977	.3281	.0193	.0019	.1381	.0137
.0360	.0792	.1783	.3924	.0253	.0275	.1253	.1362
.1515	.0098	.0306	.0020	.4365	.2343	.0881	.0473
.1698	.1201	.0237	.0168	.2416	.3461	.0337	.0483
.0193	.0019	.1381	.0137	.0555	.0458	.3977	.3281
.0253	.0275	.1253	.1362	.0360	.0792	.1783	.3924

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## References

- [1] Constantinides, G.M. and D.Duffie (1996): "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104, 219–240.
- [2] Calvet, L.E. (1998): "Incomplete Markets and Volatility," working paper, Yale University.
- [3] Duffie, D., J. Geanakoplos, A. Mas-Colell, and A. McLennan (1994): "Stationary Markov Equilibria," *Econometrica*, 62, 745–781.
- [4] den Haan, W. (1997): "Solving Dynamic Models with Aggregate Shocks and Heterogeneous Agents," *acroeconomic Dynamics* 1, 355–386.
- [5] Heaton, J. and D.J. Lucas (1996): "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," *Journal of Political Economy*, 104, 443–487.
- [6] ——— (1997): "Market Frictions, Savings Behavior, and Portfolio Choice," *Macroeconomic Dynamics*, 1, 76–101.
- [7] Hernandez, A. and M.S.Santos (1996): "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory*, 71, 102–130.
- [8] Judd, K.L., F. Kubler and K. Schmedders (1997): "Computing Equilibria in Infinite Horizon Finance Economies I: The Case of One Asset," submitted for publication.
- [9] ——— (1998): "Incomplete Markets with Heterogenous Tastes and Idiosyncratic Income," submitted for publication.
- [10] Krusell, P. and A.A. Smith, Jr. (1997): "Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns," *Macroeconomic Dynamics*, 1, 387–422.
- [11] Lucas, D.J. (1994): "Asset pricing with undiversifiable income risk and short-sale constraints. Deepening the equity premium puzzle," *Journal of Monetary Economics*, 34, 325–241.
- [12] Lucas, R.E.Jr. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.
- [13] Magill, M. and M. Quinzii (1996): "Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles," *Journal of Mathematical Economics*, 26, 133–170.
- [14] Mehra, R. and E.C.Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–161.
- [15] Telmer, C.I. (1993): "Asset-Pricing Puzzles in Incomplete Markets," *Journal of Finance*, 48, 1803–1832.
- [16] Zhang, H.H. (1997): "Endogenous Borrowing Constraint with Incomplete Markets," *Journal of Finance*, 52, 2187–2209.

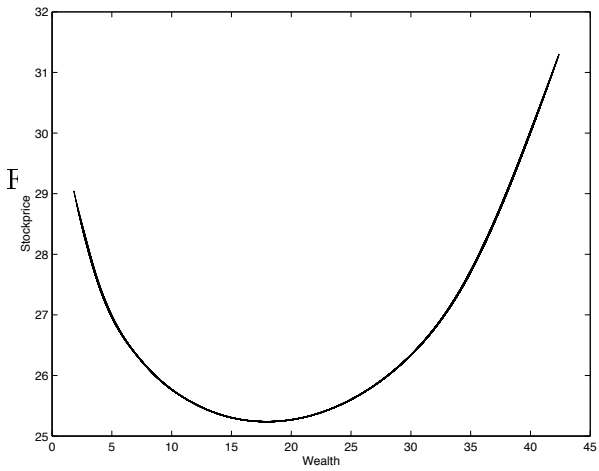
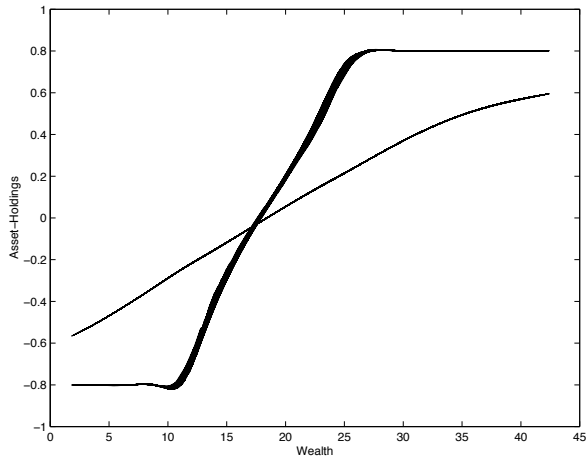
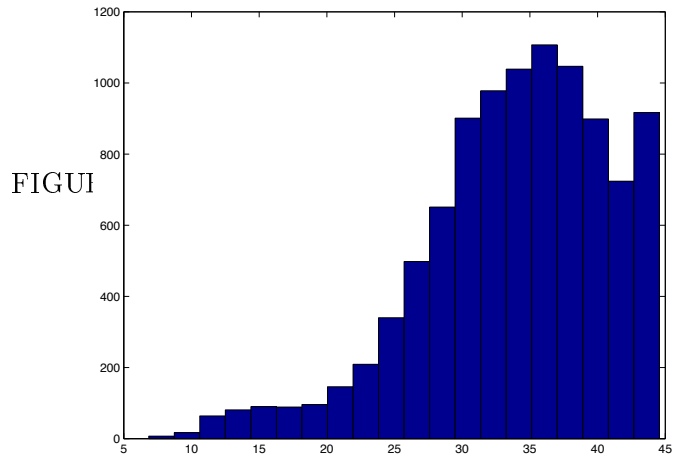
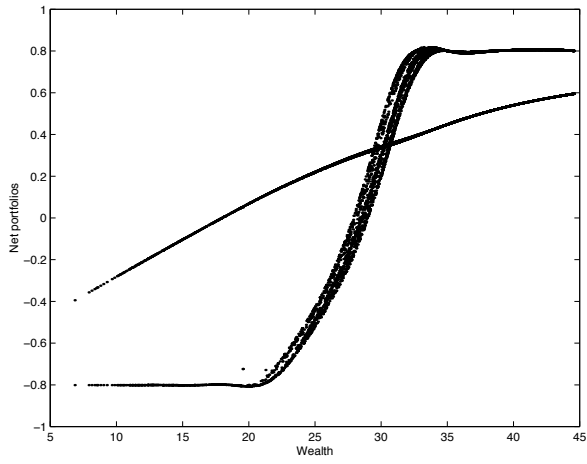


FIGURE 1d — Stock price.





FIGUI

FIGURE 2b — Histogram of wealth levels.

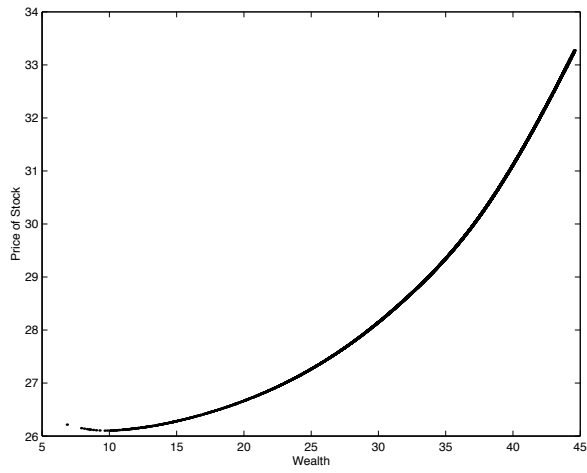
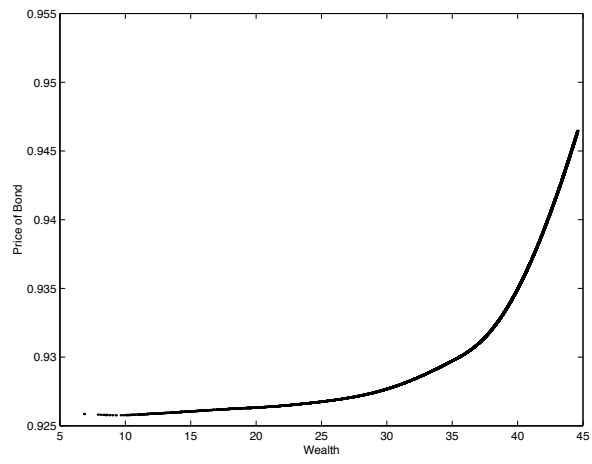


FIGURE 2d — Stock price.

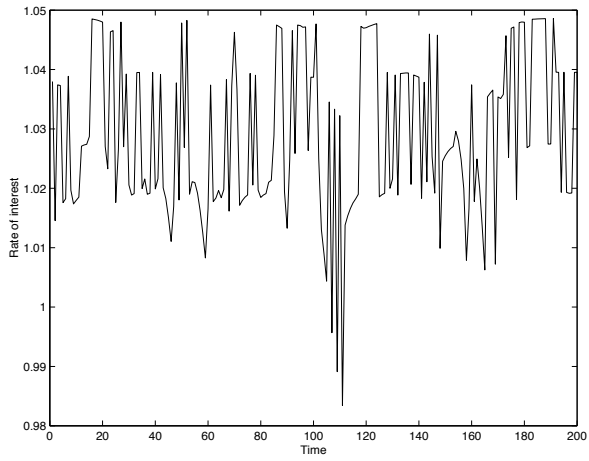
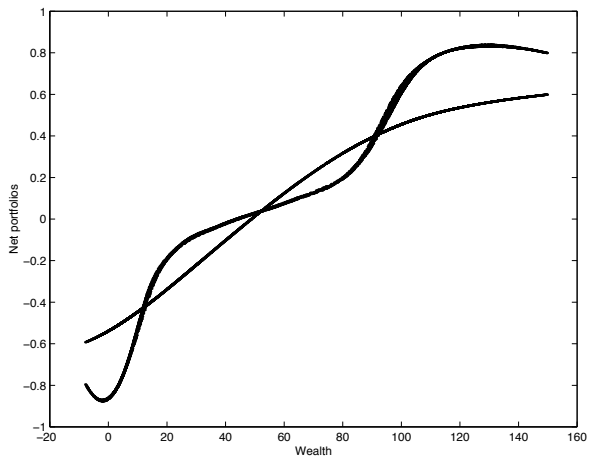


FIGURE 3b — Simulated interest rates.

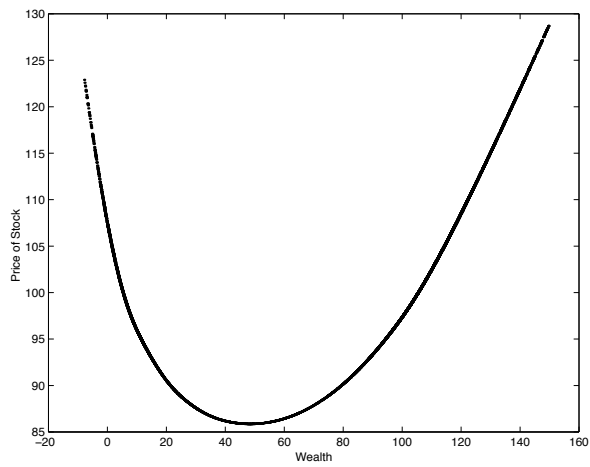
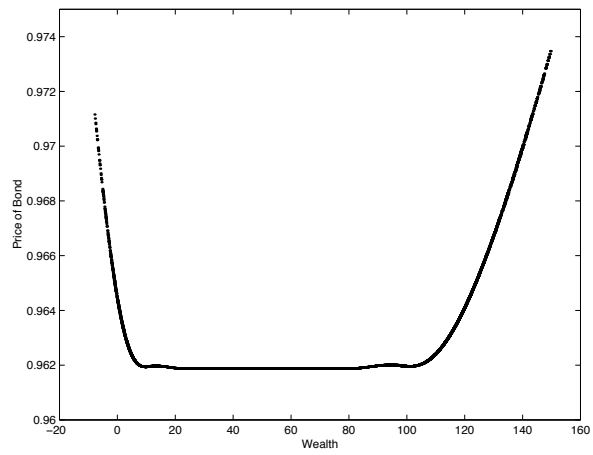


FIGURE 3d — Stock price.