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A Review of *Recursive Methods in Economic Dynamics**

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NANCY STOKEY AND ROBERT LUCAS, JR., and Ed Prescott have produced an exceptionally useful, thorough, and timely introduction to stochastic economic dynamics. Dynamic optimization techniques developed in Operations Research, formulated initially by Richard Bellman (1957), have been used extensively in economics, particularly in macroeconomics, finance, and public finance. Economic theorists have extended dynamic programming theory in several valuable directions. Of particular note for this book is the concept of *recursive equilibrium* introduced in Edward Prescott and Rajnish Mehra (1980). While these techniques have been used extensively, there has been no broad, unified, and comprehensive presentation of the concepts, tools, and applications of recursive dynamic techniques that is written for economists and demands no more mathematics than a typical student is exposed to in a good graduate program. This book succeeds marvelously in filling this need. Furthermore, given the depth of development, it is also a valuable reference for researchers.

Before describing the book's contents in detail, we should discuss what is distinctive and important about the recursive approach to dynamic economic problems. To do this, let's examine a simple problem and an alternative approach to its solution. The canonical problem for economic dynamics is the infinite horizon deterministic growth problem. Let k_t be the capital stock at the beginning of period t, $f(k_t)$ a neoclassical production function expressing period t production as a function of k_t, c_t consumption in period t chosen at the end of the period, $u(c)$ a concave utility function, and β the discount factor. Then a social planner for this infinitely lived economy will solve the problem

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad (1)$$

where the choice for the path c_t is constrained by production possibilities represented by the law of motion

$$k_{t+1} = f(k_t) - c_t \qquad (2)$$

where k_0 is the initial endowment of capital.

The approach taken in discrete-time control (corresponding to the calculus of variations techniques presented in Morton Kamien and Nancy Schwartz 1981) is to find the time path of consumption, c_t, which solves (1) subject to (2). The critical condition is the *Euler equation*, that is, the condition that the marginal value of consuming one dollar today is equal to the value of saving one dollar today and consuming the gross proceeds tomorrow. In this case, it reduces to the equation

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}). \qquad (3)$$

The Euler equation together with the law of motion, (2), and the initial endowment determine the optimal path of consumption choices, implying that the problem reduces to the differ-

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ence equations (2) and (3).[1] The literature on optimal control (Kenneth Arrow and Mordecai Kurz 1970; Kamien and Schwartz 1981) focuses on the time path of the shadow prices of the state, equal to $u'(c_t)$ in this case, as well as the time path of the state.

Economics, as opposed to operations research, is not solely concerned with social planning problems. Of more general concern is the computation of equilibrium under various market conditions. For the optimal growth problem, (1), the welfare theorems of general equilibrium theory tell us that the equilibrium of the complete market Arrow-Debreu model of general equilibrium will implement the social optimum. Therefore, with the optimal consumption path computed from (2) and (3), one can also compute the series of competitive equilibrium prices, p_t, for the economy defined by $u(c)$ and $f(k)$, where p_t is the price of period t consumption in terms of consumption at $t = 0$.

There has been a rich and extensive set of tools developed for deterministic optimal control models. Furthermore, they have been useful in computing equilibrium in economies where optimality and equilibria do not necessarily coincide, such as in monetary economies, economies with taxes, and environments with externalities. William Brock and Anastasios Malliaris (1989) is an excellent recent review of this literature. While these problems are interesting, their deterministic nature limits their generality. For example, in these deterministic models one cannot discuss asset prices, business cycles, or any other economic phenomena critically related to uncertainty and risk.

A more interesting planning problem, initially examined by Brock and Leonard Mirman (1972), is optimal growth with uncertainty in output. In this case, we assume that the production function in period t is hit with a multiplicative stochastic shock ϵ_t. For our purposes assume that the ϵ_t are independent and identically distributed, and that output in period t is $\epsilon_t f(k_t)$. Then (1) and (2) become

$$\max_{\tilde{c}_t} E\left[\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t)\right] \qquad (1')$$

$$\tilde{k}_{t+1} = \tilde{\epsilon}_t f(\tilde{k}_t) - \tilde{c}_t. \qquad (2')$$

Stochastic optimization problems present many difficulties when we turn to formalize them. If we proceed in this stochastic case as we did in the deterministic case, we index goods by both date and the history of the economy, as in the Arrow-Debreu approach to general equilibrium with time and uncertainty. This state-contingency approach can be useful, and was pioneered by Truman Bewley (1972) to prove existence of competitive equilibrium for such economies.

Sometimes this state contingent approach to representing the problem can be useful in going beyond existence analysis. One way to proceed is to determine the sequence of consumption decisions that satisfy the infinite series of first-order necessary conditions

$$u'(\tilde{c}_t) = \beta E[u'(\tilde{c}_{t+1})\tilde{\epsilon}_{t+1} f'(\tilde{k}_{t+1}) | I_t] \qquad (4)$$

where I_t is the information available to the social planner at time t. This stochastic Euler equation has been the foundation of much intertemporal econometric work since introduced by Robert Hall (1978). In particular, it has been intensively used in generalized method of moments tests of asset pricing theories, as in Lars Hansen and Kenneth Singleton (1982). In the case of linear-quadratic preferences and technology, these stochastic Euler equations can be juggled to yield representations for the equilibrium and optimal consumption choices at any time, which are linear functions of the infinite history of productivity shocks. This approach, exposited in Thomas Sargent (1987), is extensively used in the rational expectations macroeconomics literature. It can even be used to examine general equilibrium problems, as shown in Hansen and Sargent (1990).

Unfortunately this state-contingent approach becomes intractable once we leave the liner-quadratic world. In general, it is impractical to express current decisions as functions of history. While existence problems can be solved and some empirical work is possible, it is very difficult to do anything more detailed, such as

[1] Mathematically inclined readers will realize that this is not quite true because a terminal condition, called the *transversality condition at infinity*, is often another condition. A wonderful aspect of recursive techniques is that concerns about TVC_∞ seem to disappear once one makes the correct choice about the topological space where one looks for a solution. The details are not suitable material for this review article.

comparative dynamic exercises, in dynamic models with uncertainty, even after focusing on reasonable combinations of utility and production functions.

Recursive methods greatly simplify analysis of these dynamic models without making strong functional form assumptions. The key insight is that the state-contingency scheme of distinguishing goods by their date and complete history is excessive in many cases because current decisions often depend on the past in very limited ways. Consider the optimal growth problem with productivity shocks. The current capital stock alone determines the set of feasible consumption paths. Also, past consumption choices do not affect how the social planner ranks alternative future consumption paths. Therefore, the current consumption decisions depend on the past solely through history's effect on the current capital stock. Furthermore, the same logic indicates that the competitive equilibrium in such models will also have current prices and that allocations depend solely on the current capital stock, even in similar environments where distortions prevent equilibria from implementing Pareto allocations.

This observation leads us to an alternative focus, the recursive approach, for the analysis of dynamic economic models. When we take the recursive approach to the optimal growth problem, (1), we do not try directly to determine the stochastic consumption path, $\tilde{c}_t$, but rather focus on the *policy function*, $h(k)$, which expresses consumption at any time as a function of the capital stock at that time. This substantially alters the analysis of many economic models. In general, we turn away from looking for a sequence of prices and allocations that satisfy equilibrium conditions, and instead look for a collection of policy functions, independent of time, which express current decisions and prices as functions of the state variables, which in turn are sufficient statistics of the past. In the linear-quadratic case, one can easily move from representations of decisions that are functions of the infinite history of stochastic shocks to representations that are linear functions of appropriate state variables. In the linear-quadratic world, the distinctions between state-contingent solutions and recursive solutions revolve around different representations of linear stochastic processes, and are not as crucial. However, in nonlinear worlds, there are no easy ways to synthesize a policy rule from a state-contingent representation.

Computing the equilibrium policy function in a recursive model is valuable because it is a sufficient description of equilibrium, and from it one can derive any economic quantity. For example, the price of a one-period bond yielding one dollar tomorrow equals the expected marginal rate of substitution between today's consumption and consumption tomorrow, a quantity directly computable once we have the equilibrium policy function. Composing the policy function with the stochastic law of motion, one could compute the distribution of the state of the economy several periods into the future given the current state. For example, if taxes were a state variable, one could compute the distribution of the capital stock ten years hence as a function of the current tax rate. Econometricians could use the policy function to compute the variances and covariances necessary to estimate the structural parameters.

The reduction of a dynamic model to a recursive model must be done carefully. The critical step is defining the state variable. While this is not a trivial step, introspection will generally suffice. State variables are, from the point of view of any moment, predetermined variables, and may include lagged endogenous variables. For example, if there were habit formation in utility, then current decisions would depend on past consumption as well as on current wealth, implying that past consumption must be included in the state variable. When in doubt, one should include a variable in the state vector. Sometimes reducing a problem to a finite set of state variables may not be possible, such as when part of the stochastic process underlying taste or productivity shocks is not Markovian. However, for numerous economic problems, the Markov assumption is a reasonable one to make in order to obtain a tractable model amenable to empirical analysis and comparative dynamics.

Note that this reviewer's emphasis on the policy function differs somewhat from the literature and this book. The recursive approach often focuses on computing a *value* function, that is, a function that tells, given the current state, the supremum over all possible policies of the present values of current and future utility. In dynamic programming, the existence of a value function is trivial as long as there is an upper bound on the present value. The existence of a policy function that actually achieves the value function is more problematic in the abstract, but is generally not a problem in many economic contexts. However, I prefer to make the policy function the focus because all of the positive economics is contained in the policy function whereas the value function is an unobservable, subjective concept. Therefore, the value function has utility for economic analysis only to the extent it facilitates the characterization and computation of the policy function. Sometimes, the value functions can be eliminated from the positive analysis, as will be the case below.

Once we decide to focus on policy functions, we need to find conditions that determine the optimal (or equilibrium) policy functions. In the deterministic growth case, we derive the key equation by modifying the Euler condition and imposing feasibility. That is, if today's capital stock is k and today's consumption is $h(k)$, then tomorrow's capital stock is $f(k) - h(k)$, tomorrow's output is $f[f(k) - h(k)]$ and tomorrow's consumption is $h\{f[f(k) - h(k)]\}$. Furthermore, if the policy is optimal, then the Euler equation must hold:

$$u'[h(k)] = \beta u' [\![h\{f[f(k) - h(k)]\}]\!] f'\{f[f(k) - h(k)]\}. \quad (5)$$

A reader may be initially unimpressed by these developments because they appear to be just another way to compute Pareto efficient competitive equilibria. Furthermore, policy functions themselves are not novel because dynamic programming has always focused on policy functions, as did Brock and Mirman (1972). What is more recent, and more important to economic applications, is the realization that the recursive approach can be used to compute equilibria in many economic environments, even where equilibria do not correspond to solutions of a planner's problem. A simple example of this is where net income is taxed at a rate τ and returned lump-sum to individuals. Assuming the tastes and technology assumed in (1) above, the law of motion of k is unchanged and the critical Euler equation for the *equilibrium* policy function becomes

$$u'[h(k)] = \beta u'[\![h\{f[k - h(k)]\}]\!] [\![(1 - \tau)\{f'[k - h(k)] - 1\} + 1]\!]. \quad (6)$$

In David Bizer and Kenneth Judd (1989), recursive equilibrium techniques based on (6) are used to show existence of equilibrium and to analyze that equilibrium. Notice that the only difference between (6) and the Euler equation for the optimal growth problem, (5), is the introduction of the constant $(1 - \tau)$. From a mathematical perspective, the two functional equations are very similar. This example is a particularly clear one showing that from a formal point of view the critical equations for recursive equilibrium differ little from the critical equations for dynamics programming, but recursive equilibria can cover a much broader range of economic environments.

It may still seem that we have not gained anything except a different way to express difficult problems. Instead of finding infinite sequences of real numbers satisfying (2) and (3), we instead look for a policy function $h(k)$ which solves the single equation, such as (5) in the deterministic case. This is called a *functional equation* because the unknown is a function, $h(k)$. Of course, in trying to solve the functional equation (5) we must search over all possible choices for $h(k)$, a set of infinite dimension. Initially, the choice between solving an infinite set of one-dimensional equations or a single infinite-dimensional equation is not a pleasant one. Showing that the recursive approach is tractable and superior is the task the authors have set for themselves and do accomplish with this book.

After an introductory chapter, chapter 2 begins the book with a very readable overview of the basic ideas and economic intuition for the book's subject. The initial nontechnical discussions also illustrate the broad scope of economic applications for recursive techniques. This is very helpful because a reader will be more willing to work through the technical material in later chapters after being treated to a sample of the final reward.

Part II begins the serious analysis with an examination of deterministic optimal control models. Chapter 3 introduces the reader to basic mathematical tools. We immediately see the general usefulness of the book. At one level, its material is rigorously presented, and it is never afraid of using the mathematics necessary to develop the economics. On the other hand, it does not assume that the reader has an advanced degree in mathematics, for all of the necessary mathematical tools are developed from basics.

While this approach results in a long book, it is efficient in that the authors have taken from the mathematics literature just what is needed for serious economic analysis. It is very helpful that the mathematics is presented in intuitive economic contexts. For example, nonlinear functional equations of the sort (5) are not standard fare even in graduate mathematics (which instead focus on continuous-time functional equations), but appear very naturally in discrete-time optimal growth problems. In general, this book is a very good place to learn the necessary mathematics of functional equations and discrete-time stochastic processes, far better than dense mathematical treatises which have different agendas and are meant for different audiences. While I hope that students using Stokey and Lucas would learn basic measure theory in a mathematics course, even that would not be necessary. On the other hand, if one needs to learn some mathematics in greater detail, this book adequately prepares a reader for the mathematics literature, and assists the reader in this regard with frequent references to the standard mathematical sources.

Chapter 4 develops dynamic programming in infinite horizon contexts. It covers both bounded and unbounded utility functions. The development follows the approach of David Blackwell (1965) and Eric Denardo (1967). It also shows how to use dynamic programming to derive deterministic Euler equations. Chapter 5 discusses a wide range of applications of the deterministic dynamic programming paradigm. These applications include growth, technical progress, learning, human capital, investment, growth with intertemporally nonseparable preferences, and monopoly pricing with unknown demand.

If the policy function is the appropriate focus for analysis, it should contain all properties of the solution, and once it has been computed, all other questions can be addressed. Chapter 6 shows how the policy function can be used to analyze the dynamics of the solution. The first concern in a deterministic model is whether the optimal policy function causes the system to converge to a particular point, known as the *steady state*. This leads one to study the stability of a system, the first topic of this chapter. If one has a convergent system around a steady state, and the policy function is smooth at that point, then one can learn much about a model's asymptotic dynamics by *"linearizing"* the policy function around the steady state. Such a linearization technique replaces the nonlinear policy function by a linear policy function which is the best approximation near the steady state. The appropriate way to do this is the next topic of chapter 6. Linearization techniques are powerful because they reduce a generally intractable nonlinear problem to a tractable linear problem. These techniques are very useful in economic models because they allow us to do comparative dynamics exercises, such as determining the effect of a monetary injection, a tax increase, or a persistent but decaying productivity shock. Because these exercises typically summarize the interesting positive economics of a model, the value of linearization techniques is clear.

Part III moves to stochastic models and is the core of the book. Chapters 7 and 8 are very accessible presentations of the basic measure theory and Markov process theory which is needed for extending the deterministic theory of Part II to stochastic models. Chapters 9 and 10 return to dynamic programming and economic applications. Formalizing dynamic programming in stochastic environments is difficult because of all the bookkeeping any notation must do. This presentation provides all the details without getting lost in cumbersome notation. We see some familiar examples from the deterministic sections, optimal growth and inventories, but we are also treated to some new ones, such as asset pricing and job search.

The book next turns to ways of describing the long-run behavior of a stochastic dynamic model. In deterministic growth models, we do this by studying the steady states. This cannot be done in stochastic models because the stochastic shocks prevent steady convergence to any particular point. We would still like to have an analogous concept of long-run tendencies. This is accomplished in an uncertain environment by studying how the probability distribution of the capital stock evolves over time. Let

$$G_t(k) = \text{prob}(k_t \leq k \mid k_0) \quad (7)$$

be the probability distribution function for $\tilde{k}_t$ given the initial capital stock. With the recursive approach, the dynamic evolution of G_t can be expressed simply, and the convergence of G_t to a long-run distribution can be examined. Chapters 11, 12, and 14 develop the mathematics defining the concepts of convergence needed for empirical analysis. Chapter 13 then applies these concepts to inventory theory, optimal growth, and monetary economies.

The applications in parts II and III are generally optimal control problems in standard economic settings. Much of the real business cycle literature, such as John Long and Charles Plosser (1983), and the asset pricing literature, such as Lucas (1978) and Mehra and Prescott (1985), is based on using dynamic programming to compute competitive equilibria. Dynamic programming techniques are also becoming increasingly valuable because recent developments in economic theory have shown that many dynamically complex problems can be reformulated as dynamic programming problems. Kydland and Prescott (1980) formulated a dynamic optimal taxation problem as a dynamic programming problem. The same was shown by Dilip Abreau, David Pearce, and Ennio Stacchetti (1986) for optimal cartel behavior, and by Edward Green (1987), and Stephen Spear and Sanjay Srivastava (1987) for dynamic contracting problems under asymmetric information. In all of these cases, the critical step was to add a state variable expressing a promise by the planner to the agent(s) about the future evolution of the planner's policy, and to add incentive compatibility constraints that force the planner to respect each agent's choices of unobservable actions. The key point of these papers is not that a problem can be approached as an optimization problem in *some* way, but that one can derive a *low-dimensional* statistic, summarizing past history, such that the dynamic programming solution using it as the state variable will do as well as any other more complex, dynamic solution. This dimensionality reduction makes tractable seemingly intractable dynamic problems. As we become more adept in identifying recursive structures, dynamic programming will become an even more useful tool in analyzing dynamic problems and the techniques developed in Stokey and Lucas more valuable.

Part IV moves away from control problems and to a competitive equilibrium focus. The competitive equilibrium analysis builds on the preceding developments because, even when equilibrium does not solve a social planner's problem, dynamic programming provides us with the demand and supply relationships necessary for equilibrium analysis. Chapter 15 examines the definition of equilibrium and Pareto optimality. Here the necessary mathematical tools are different. Stokey and Lucas develop a version of Bewley (1972) which is adequate for the tastes and technologies examined here. This treatment allows them to connect standard general equilibrium concepts such as competitive equilibrium and Pareto optimality with the recursive equilibrium solutions, demonstrating that recursive techniques do find the competitive equilibria.

Equilibrium theory is further developed and applied in chapters 16 and 17. The examples of dynamic competitive equilibria are wide-ranging, including industry investment, heterogeneous consumers, and overlapping generations models of money. The examples from the overlapping generations literature point out that recursive equilibrium analysis is not confined to infinitely lived representative agent models. There is also a presentation of Lucas (1972), one of the first and most important papers to use recursive equilibrium techniques. Chapter 18 closes the book with the recursive formulation of dynamic equilibrium with taxes. These last examples show us that recursive equilibrium is not confined to the analysis of problems that are also solutions to planning problems.

In using the recursive approach, one should be aware of its limitations. The recursive approach has some limitations in studying models where there is not a match between optimality and equilibrium, such as in the sunspot literature and in dynamic game theory. The theme of the sunspot literature, as in David Cass and Karl Shell (1983), is that variables that "should not" be important can play a role in equlibrium. A recursive equilibrium analysis that excludes supposedly extraneous variables will miss sunspot equilibria. This is not a crucial limitation because this problem can often be handled by appropriately expanding the definition of the state. In the case of sunspots, for example, one can always increase the set of state varaibles and see whether there are equilibria that depend critically on the supposedly extraneous variables. The other limitation of recursive techniques arises in strategic contexts. In dynamic games, past actions may be used to facilitate cooperation. Specifications of the state variable that limit memory may eliminate interesting equilibria. While these limitations may be important for some problems, they have no effect for many others.

While this book is concerned largely with techniques leading to formulating recursive equilibria and proving existence of optimal programs and equilibria, the recursive methods developed here form the foundation for further analysis of these models, and will be useful in much future research on these and other economic problems. I shall now turn to aspects of recursive equilibrium analysis, both applications and quantitative techniques, which were, quite understandably, not covered in this book.

While the book gives numerous examples of recursive equilibria, there are others that will probably be more extensively developed in the future. For example, much of the work in competitive security markets assumes complete state-contingent securities. Because this assumption is of questionable accuracy in its extreme form, it would be valuable to know how equilibrium is affected when the collection of traded securities is limited. Recursive equilibrium is the natural way to formulate the problem, as Milton Harris (1987) shows in an example of recursive equilibrium with incomplete security markets. There has also been a recent surge in interest in economic growth and the role of externalities, with Paul Romer (1986) and Lucas (1988) being early examples. Stochastic extensions of this work will be done with recursive equilibrium. Another class of problems amenable to recursive analysis is noncooperative game theory. In fact, the dynamic game literature, for example, Avner Friedman (1972), formulates nonzero sum continuous-time games as recursive equilibria.

Finally, I would like to discuss extensions of recursive equilibrium techniques that will allow one to probe more deeply in analyzing a dynamic model. In particular, when we study economic models, we generally want to do comparative statics exercises. We would like to address dynamic questions in a similar comparative fashion. For example, if the risk aversion of the representative agent increases, is there more or less saving? What happens to the variance of output? of consumption? Many of these questions were addressed using a recursive equilibrium approach by Jean-Pierre Danthine and John Donaldson (1981). John Laitner (1990) develops general methods for computing how parameter changes alter equilibrium policy functions and moments of economic time series.

Unfortunately, there is a limit as to how much one can learn from qualitative manipulation of the recursive equilibrium equations. In fact, there are few precise results in general equilibrium analysis because of the tendency of income effects to make anything possible. This does not mean that it is futile to ask comparative dynamic questions. The more relevant question to address is what happens when we restrict ourselves to tastes and technologies that have some empirical plausibility. To do this, we need to compute more explicitly specific equilibria, a task which in itself is nontrivial.

Fortunately, recursive equilibrium formulations of economic problems are amenable to the whole range of approximation techniques. First, there are *asymptotic* methods, also called *perturbation*—techniques that approximate the true solution to a high degree as some parameters get close to zero. These methods are analytic in the sense that they are closed-form formulas, and often have a high degree of validity. Being closed forms, one can use them to prove theorems for an open set of cases. Stokey and Lucas do touch on these methods in their treatment of the linearization techniques for deterministic models around a steady state in chapter 6. This linear approximation of a recursive equilibrium is asymptotically valid in the sense that the error of the approximate solution is quadratic in the difference between the initial capital stock and the steady-state capital stock. This is only the first step in a perturbation analysis because, using only linear operations, one can go on to find good quadratic, or cubic, or higher-order approximations to the policy function near the steady state (see Alain Bensoussan 1988). These approximations would be useful in economic analyses. For example, a quadratic approximation would indicate how rapidly the linear approximation is decaying away from the steady state. More generally, such extended local approximations would be more accurate around a steady state, allow the approximation to capture a wider range of nonlinear effects, and generally increase the range over which the approximation fits well.

A related standard technique in stochastic dynamic programming is to take the policy function for a deterministic problem and use it as a basis for computing an asymptotically valid approximation for a "neighboring" stochastic problem (see Wendell Fleming 1971 and Bensoussan 1988), that is, a problem with the same structure except for a shock with small variance. Michael Magill (1977) showed how to use this technique to analyze macroeconomic issues, and the empirical real business cycle literature (such as Kydland and Prescott 1982) has applied it extensively. These approximations are easily computed once a problem is expressed in a recursive fashion.

These analytical asymptotic approximations will be somewhat limited because they generally make a "smallness" assumption—near a steady state, a small variance, and so on. There will always be some doubt as to their validity away from the center of the approximation. However, with the continuing increase in computing power and algorithmic efficiency, there has been and will continue to be more extensive use of numerical approaches to computing globally good approximations of equilibrium of economies and games. Here is an area where the recursive approach will be most valuable. Most numerical studies of dynamic models have taken a state-contingency approach; that is, they date the commodities and then compute the equilibrium dynamic path of prices and allocations. This is the case, for example, in Alan Auerbach and Laurence Kotlikoff's (1987) study of macroeconomic fiscal policy. While the state-contingency approach can be useful in numerical analysis of deterministic models, it is useless for numerical computation of dynamic models with uncertainty because computer memory would be quickly exhausted for any reasonable model.

In contrast, approximating recursive equilibria is quite tractable. Economic applications of numerical solutions of recursive equilibria go back to Robert Gustafson (1958) in the agricultural economics literature, and to Bellman and Dreyfus (1962) in the dynamic programming literature. The main task is finding some way to represent policy functions on the computer. Generally, one uses standard approximation techniques, such as piecewise linear interpolation, polynomials, and splines, to represent the policy function. These approximating functions depend on only a finite number of undetermined parameters. Once the form of the approximation is determined, the numerical task is to find that set of parameter values that will cause the functional equations describing the recursive equilibrium, such as (5), to be nearly satisfied. There is also a large and useful literature on solving such functional equations form physics, chemistry, fluid dynamics, and engineering (e.g., C. A. J. Fletcher 1984). Given the mathematical similarity between the equations defining recursive equilibrium and the equations arising in other fields of study, one expects that these standard methods will find numerical approximations to recursive equilibria. For example, (5) can be solved in a few seconds on a personal computer with an error less than .01 percent (see Judd 1989). By combining the recursive techniques discussed in Stokey and Lucas with standard approximation techniques, economists will be able to analyze economic problems previously beyond our grasp.

In summary, *Recursive Methods in Economic Dynamics* is an excellent review of the current state of the art in discrete-time dynamic economic analysis. It will long serve as a valuable teaching and reference tool.

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