NOTES AND COMMENTS

LOWER BOUNDS ON APPROXIMATION ERRORS TO NUMERICAL SOLUTIONS OF DYNAMIC ECONOMIC MODELS

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We propose a novel methodology for evaluating the accuracy of numerical solutions to dynamic economic models. It consists in constructing a lower bound on the size of approximation errors. A small lower bound on errors is a necessary condition for accuracy: If a lower error bound is unacceptably large, then the actual approximation errors are even larger, and hence, the approximation is inaccurate. Our lower-bound error analysis is complementary to the conventional upper-error (worst-case) bound analysis, which provides a sufficient condition for accuracy. As an illustration of our methodology, we assess approximation in the first- and second-order perturbation solutions for two stylized models: a neoclassical growth model and a new Keynesian model. The errors are small for the former model but unacceptably large for the latter model under some empirically relevant parameterizations.

Keywords: Approximation errors, error bound, forward error analysis, backward error analysis, Euler equation residuals, upper error bound, lower error bound, accuracy, numerical solution, approximate solution, new Keynesian model.

1. INTRODUCTION

Dynamic economic models do not typically admit closed-form solutions and must be studied with numerical methods. A numerical method approximates the exact solution up to some degree of accuracy. The control over the quality of approximation is critical if we want to get valid inferences from numerical experiments. That is, the constructed approximate solution must have a minimum acceptable quality for the questions studied; otherwise, it could happen that conclusions and policy implications are just driven by approximation errors.

Thus, an important question is: "How different is the approximate solution from the exact solution?" There is literature that focuses on an upper bound on approximation errors by assuming the worst-case scenario; see Peralta-Alva and Santos (2014) for a review. The present paper complements this literature by introducing lower-bound error analysis. A lower error bound delivers an optimistic, best-case scenario view about accuracy of an approximate solution. Our main insight is that it is generally quite easy to provide lower bounds on approximation errors by focusing on a strict subset of the model's equations and by determining minimal perturbations to the approximate solution that are necessary to solve the given subset of equations exactly.

The lower and upper error bounds are complementary and both of them are useful—they provide necessary and sufficient conditions for accuracy, respectively. Namely, if an upper error bound is small, then the actual approximation errors are even smaller, and we can conclude that a given approximate solution is sufficiently accurate. In turn, if a lower

error bound is unacceptably large, then the actual approximation errors are even larger, and hence, we conclude that a numerical solution is inaccurate.[2]

Our methodology of constructing lower error bounds is quite general: it is independent of a specific solution method and is applicable to both dynamic programming and equilibrium problems. In contrast, upper-error-bound analysis requires special assumptions and is limited to dynamic programming problems. It is not possible to conduct systematic upper-bound error analysis in many stochastic dynamic economic models that are used in practice.

As an illustration, we apply our methodology to assess the size of approximation errors in the first- and second-order perturbation solutions for two stylized models: a neoclassical optimal growth model and a variant of the new Keynesian model in line with Christiano, Eichenbaum, and Evans (2005), and Smets and Wouters (2007). The studied model features physical capital, adjustment costs to investment, variable utilization of capital, habit formation in consumption, as well as sticky wages and prices. For the growth model, we find that the approximation errors of the first-order perturbation solutions (linearization) are at most of order 0.1%, and they are even lower for the second-order perturbation solution. These errors are sufficiently small, and thus we cannot claim that perturbation methods are insufficiently accurate for the standard growth model. However, for a calibrated version of the new Keynesian model, the approximation errors exceed a hundred percent, which is unacceptably large; this is true even in the absence of an active zero lower bound (ZLB) on nominal interest rates. This is significant because linear perturbation methods are commonly used by central banks for solving their large-scale new Keynesian macroeconomic models for forming monetary policy and projections.

The related literature proposes several approaches to the accuracy evaluation. The forward error analysis poses the following question: "Given an economic model, how much must an approximate solution be modified to satisfy all model's conditions exactly?" The upper and lower error bounds are particular implementations of forward error analysis. There is also backward error analysis that inverts the question: "Given an approximate solution, how much must an economic model itself (in terms of parameters) be modified in order to make an approximate solution to satisfy all model's equations?" The backward error analysis was introduced in Wilkinson (1963); there was also mixed forward-backward error analysis introduced in Higham (1996); see also Sims (1990) and Kubler and Schmedders (2003, 2005) for related methods. Another common approach to the accuracy evaluation in the literature is the analysis of residuals in the model's equations; see Judd (1992). The existing approaches to accuracy evaluation have their pros and cons. The advantage of upper- and lower-bound error analysis is that they are direct approaches, namely, they explicitly show the size of the approximation errors in the model's variables. A potential shortcoming of the upper-bound error analysis is that it may be too pessimistic and may reject solutions that are sufficiently accurate. In turn, the lower-bound error analysis may be too optimistic and may fail to reject solutions that are insufficiently accurate. The other approaches do not suffer from these shortcomings but they are indirect: they provide some statistics related to accuracy but not the size of approximation errors itself.

The rest of the paper is organized as follows: In Section 2, we introduce the lower-error bound analysis and illustrate it with examples. In Section 3, we evaluate accuracy of

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[2]A low quality of numerical approximation can be possibly due to analytical / coding errors. Geweke (2004) constructed a statistical test that explicitly aims at detecting such errors in the context of Bayesian estimation. Our lower-bound error analysis does not allow us to tell what exactly accounts for a poor quality of approximation but just signals that the quality is poor.

perturbation solutions for the neoclassical growth model. In Section 4, we compare the lower-bound error analysis to other approaches to accuracy evaluation in the literature. In Section 5, we show accuracy results for the new Keynesian model. In Section 6, we conclude.

2. A LOWER BOUND ON APPROXIMATION ERRORS

In this paper, we follow a direct approach to accuracy evaluation which is known as *forward error analysis*. Forward error analysis poses the following question: "Given a system of equations, how much must an approximate solution be modified to satisfy all equations exactly?"

A conventional way of implementing forward error analysis consists in constructing an upper bound on the size of approximation errors (see, e.g., Bertsekas and Tsitsiklis (1996), Santos and Vigo-Aguiar (1998), Santos (2000), Schmitt-Grohé and Uribe (2004), and Santos and Peralta-Alva (2005), among others); see Peralta-Alva and Santos (2014) for a review of this literature. An upper error bound corresponds to a pessimistic—worst-case—scenario. Specifically, this literature asks: "What are the largest possible approximation errors that correspond to a given numerical solution?" The upper-bound error analysis provides a sufficient condition for accuracy: If an upper bound on approximation errors is small, we conclude that an approximate solution is accurate since the actual errors can never be larger than their upper bound.

We propose a complementary version of the forward error analysis that aims at constructing a lower bound on the size of approximation errors. The lower bound corresponds to an optimistic—best-case—scenario. Here, we ask: "How small can approximation errors potentially be made if we allow to violate some of the model's equations?" If the resulting lower error bound is still unacceptably large, we conclude that a numerical solution is inaccurate since the actual approximation errors can never be smaller than their lower bound. In this sense, our lower-bound error analysis provides a necessary condition for accuracy.

In the rest of the section, we formally introduce a framework for constructing lower bounds on approximation errors; we illustrate this framework in the two-dimensional case; and we discuss a relation between the lower and upper error bounds.

2.1. *A Framework for Constructing a Lower Error Bound*

We consider a system of n (possibly, nonlinear) equations with n unknowns:

$$G_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, n, \tag{1}$$

or in vector notations, we have $G(x) = 0$, where $G : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$. (This system represents a collection of the model's equations and may include a Bellman equation, Euler equations, market clearing conditions, budget constraints, and laws of motion for exogenous and endogenous shocks.) Here, we assume that there is a unique solution to (1), and in Section 2.3, we discuss some possible generalizations.

Let $x^* \in \mathbb{R}^n$ and $\widehat{x} \in \mathbb{R}^n$ be exact and approximate solutions to system (1), respectively (we assume that $\widehat{x} \neq 0$). We define an approximation error as a compensation $\delta^* \in \mathbb{R}^n$ that is needed to make an approximate solution $\widehat{x}$ to satisfy the model's equations exactly,

$$G\bigl(\widehat{x}(\mathbf{1} + \delta^*)\bigr) = 0, \tag{2}$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector of ones. Systems of equations studied in economics are often complex and finding an exact value of δ^* satisfying (2) is infeasible. (In fact, if we were able to find such a value, we would also be able to find an exact solution x^* using $x^* = \widehat{x}(\mathbf{1} + \delta^*)$.)

In the paper, we propose a technique for constructing a lower bound on δ^* for those complex cases. As a first step, let us remove $n - m$ equations from system (1), where $1 \le m < n$. As a result, we obtain a reduced system of m equations $g \equiv [g_1, \ldots, g_m]$:

$$g_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, m, \tag{3}$$

where g is a strict subset of equations from G. Consider now the problem of finding an approximation error δ that satisfies the reduced system of equations

$$g(\widehat{x}(\mathbf{1} + \delta)) = 0. \tag{4}$$

By construction, the reduced system (3) is underdetermined (rank-deficient): it contains n equations and m unknowns, $m < n$, and thus, it has multiple solutions (effectively a solution δ to (3) is a manifold). Consequently, there are multiple compensations δ that make an approximation $\widehat{x}$ to satisfy (3) exactly.

Let us denote by Ω^g a set of all possible compensations satisfying (3) for a given approximate solution $\widehat{x}$, that is,

$$\Omega^g \equiv \left\{ \delta \in \mathbb{R}^n : g(\widehat{x}(\mathbf{1} + \delta)) = 0 \right\}, \tag{5}$$

where superscript g refers to a specific set of equations $g \subset G$ used to form the reduced system (3). (For the original system G, the set $\Omega^G = \delta^*$ is a singleton.)

Our next step is to choose the smallest possible compensation $\widehat{\delta}^g \in \Omega^g$ with respect to a given norm $\| \cdot \|$ that satisfies the reduced system of equations g, that is,

$$\min_{\delta \in \Omega^g} \|\delta\| \quad \text{s.t. } g(\widehat{x}(\mathbf{1} + \delta)) = 0. \tag{6}$$

We next establish the following useful relation between $\widehat{\delta}^g$ and δ^*.

PROPOSITION 1: *For a given $\widehat{x}$ and a given norm $\| \cdot \|$, we have $\|\widehat{\delta}^g\| \le \|\delta^*\|$, where δ^* and $\widehat{\delta}^g$ are defined by* (2) *and* (6), *respectively.*

PROOF: First, we have $\Omega^G \subseteq \Omega^g$ by (5), that is, any compensation δ^* satisfying (2) in the unrestricted system G must be also a possible compensation for the restricted system g, and hence, we have $\delta^* \in \Omega^g$. Second, by definition (6), we have $\widehat{\delta}^g = \arg\min_{\delta \in \Omega^g} \|\delta\|$. These two results together imply the statement of the proposition.   *Q.E.D.*

Proposition 1 shows that the smallest possible compensation $\widehat{\delta}^g$ in the reduced system can never be larger than the compensation $\delta^* \in \Omega$ in the original system. That is, $\widehat{\delta}^g$ is a lower bound of δ^*.

The lower error bound provides a simple way to discard numerical approximations that are insufficiently accurate. Namely, if a lower bound on approximation errors happens to be unacceptably large, the numerical approximation is clearly inaccurate since the actual approximation error δ^* can never be smaller than its lower bound $\widehat{\delta}^g$. If, on the other hand, they are low, we cannot say anything—this is why the lower error bound is a necessary but not sufficient condition for accuracy.

The constructed lower error bound $\widehat{\delta}^g$ depends on a specific subset of equations $g \subset G$ used for forming the reduced system (3): each different subset $g \subset G$ leads to a different lower error bound. Hence, it is important to make the procedure of the equations selection systematic and to give some theoretical foundations to the detailed choice of unknowns versus equations.

To have the best chance for detecting and discarding inaccurate approximations, we must select a reduced system g that leads to the largest possible lower error bound $\widehat{\delta}$, that is,

$$\max_{g \subseteq \mathbf{G}} \left\{ \min_{\delta \in \Omega^g} \|\delta\| \text{ s.t. } g\big(\widehat{x}(\mathbf{1} + \delta)\big) = 0 \right\}, \tag{7}$$

where $\mathbf{G}$ is a collection of all possible subsets of the original system G. Clearly, the largest possible lower error bound satisfying (7) is $\widehat{\delta}^G = \delta^*$, and it is obtained when we focus on the original system $g = G$ without removing any equation.

However, by assumption, it is computationally infeasible to solve G with a sufficiently high degree of accuracy, so the corresponding lower error bound $\widehat{\delta} = \delta^*$ cannot be reliably constructed. We must restrict attention to those subsets $g \subset G$ that can be solved either analytically or with negligible approximation errors (otherwise, non-negligible approximation errors may distort the lower error bound and may invalidate our inferences about accuracy).

The trade-off is the following: From one side, we want to remove as few equations as possible (since removing equations potentially increases the fit and reduces the lower error bound and hence, it reduces our chance to discard an inaccurate numerical approximation); and from the other side, we must remove all equations that cannot be solved either exactly or with negligible approximation errors (again, non-negligible approximation errors may distort the lower error bound and may invalidate our inferences about accuracy).

Potentially, many subsets of G can be solved accurately, but the following result allows us to reduce the number of subsets that must be considered.

PROPOSITION 2: *Let g' and g'' be two subsets of G such that $g' \subset g''$. Then, for a given $\widehat{x}$ and a given norm $\|\cdot\|$, we have $\|\widehat{\delta}^{g'}\| \leq \|\widehat{\delta}^{g''}\|$, where $\widehat{\delta}^{g'}$ and $\widehat{\delta}^{g''}$ are defined by (6).*

PROOF: First, we have $\Omega^{g''} \subseteq \Omega^{g'}$ by assumption $g' \subset g''$, that is, any compensation δ satisfying (6) under g'' must also satisfy it under a strict subset g', and hence, we have $\widehat{\delta}^{g'} \in \Omega^{g''}$. Second, by definition (6), we have $\widehat{\delta}^{g''} = \arg\min_{\delta \in \Omega^{g''}} \|\delta\|$. These two results together imply the statement of the proposition. *Q.E.D.*

The result of the proposition means that we do not need to consider all computationally feasible subsets of G but only those with the largest cardinality. That is, whenever we have two computationally feasible nested subsets $g' \subset g''$, only a subset with the larger number of equations g'' needs to be analyzed. In Section 3, we show a systematic procedure for constructing reduced systems of equations and the corresponding lower error bounds in dynamic economic models.

A convenient choice for the problem of constructing lower error bound (6) is an L_2 norm since it allows us to use first-order conditions (FOC), namely, we find the smallest compensation $\widehat{\delta}$ by solving the following least-squares problem:

$$\min_{\delta \in \mathbb{R}^n} \delta^\top \delta \quad \text{s.t. } g\big(\widehat{x}(\mathbf{1} + \delta)\big) = 0. \tag{8}$$

A necessary condition for the existence of a local minimum $\widehat{\delta}$ in (8) follows by a version of the well-known Theorem of Lagrange: (i) $g(\widehat{x}(\mathbf{1} + \widehat{\delta}))$ must be full ranked in a neighborhood of $\widehat{x}(\mathbf{1} + \widehat{\delta})$; and (ii) $\widehat{x}(\mathbf{1} + \widehat{\delta})$ must be a critical point of the Lagrange function, $\delta^\top \delta + \lambda g(\widehat{x}(\mathbf{1} + \delta))$, where $\lambda \in \mathbb{R}^m$ is a vector of Lagrange multipliers, that is,

$$2\widehat{\delta} + \lambda \nabla g(\widehat{x}(\mathbf{1} + \widehat{\delta}))\widehat{x} = 0, \tag{9}$$

where ∇g denotes a gradient of g. Furthermore, a sufficient condition for a local minimum is that the Lagrangian function is convex on a subset of $\mathbb{R}^n$ defined by $Z(\widehat{\delta}) = \{z \in \mathbb{R}^n : \nabla g(\widehat{x}(\mathbf{1} + \widehat{\delta}))z = 0\}$; see, for example, Sundaram (1996, Theorems 5.1 and 5.4) for proofs of these results.[3]

2.2. *Two-Dimensional Example*

We now illustrate the construction of a lower bound on approximation errors in a two-dimensional case. Let (x_1^*, x_2^*) and $(\widehat{x}_1, \widehat{x}_2)$ denote, respectively, the exact and approximate solutions to a two-dimensional version of system (1), namely, $G_i(x_1, x_2) = 0$, $i = 1, 2$ (again, we assume that $(\widehat{x}_1, \widehat{x}_2) \neq 0$). Following (2), we define an approximation error $(\delta_{x_1}^*, \delta_{x_2}^*)$ by $G_i(\widehat{x}_1(1 + \delta_{x_1}^*), \widehat{x}_2(1 + \delta_{x_1}^*)) = 0$, $i = 1, 2$.

To construct a lower bound on approximation errors, we remove equation $G_2(x_1, x_2) = 0$, and we focus on the reduced system composed of just one equation $g(x_1, x_2) \equiv G_1(x_1, x_2) = 0$. Following (5), we define a set of compensations Ω that are consistent with a restriction g

$$\Omega \equiv \big\{ (\delta_{x_1}, \delta_{x_2}) \in \mathbb{R}^2 : g\big(\underbrace{\widehat{x}_1(1 + \delta_{x_1})}_{=x_1^*}, \underbrace{\widehat{x}_2(1 + \delta_{x_2})}_{=x_2^*} \big) = 0 \big\}. \tag{10}$$

As we mentioned earlier, the reduced system of equations g is underdetermined and there are multiple compensations δ_{x_1} and δ_{x_2} that are consistent with (10). As an illustration, consider a special case when g is linear, that is,

$$g(x_1, x_2) = a_1 x_1 + a_2 x_2, \tag{11}$$

where a_1 and a_2 are constant coefficients. To describe all compensations satisfying (10), we can fix any δ_{x_1}, and we can find δ_{x_2} from (10) using (11) as follows:

$$\delta_{x_2} = \frac{a_1 \widehat{x}_1}{a_2 \widehat{x}_2}(1 + \delta_{x_1}) - 1. \tag{12}$$

From all possible compensations satisfying (12), we select the smallest one with respect to the least-squares norm by solving a two-dimensional version of the least-squares problem (8)

$$\min_{\delta_{x_1}, \delta_{x_2}} \delta_{x_1}^2 + \delta_{x_2}^2 \tag{13}$$

$$\text{s.t. } g\big(\widehat{x}_1(1 + \delta_{x_1}), \widehat{x}_2(1 + \delta_{x_2}) \big) = 0. \tag{14}$$

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[3]Instead of L_2, we can use other norms for measuring compensations, for example, a least absolute deviation L_1 or a maximum error L_∞. Furthermore, in some economic applications, we can possibly tolerate large approximation errors in some variables but we need very accurate solutions in other variables. In this case, approximation errors can be weighted by a measure of their economic significance in the objective function. For example, the objective function in (8) can be modified to $\delta^\top W \delta$, where W is an $n \times n$ matrix of weights (this case is similar to a weighted least-squares in econometrics).

An interior solution of (13), (14) satisfies

$$\frac{\delta_{x_1}}{\delta_{x_2}} = \frac{g_{x_1}\big(\widehat{x}_1(1+\delta_{x_1}), \widehat{x}_2(1+\delta_{x_2})\big)\widehat{x}_1}{g_{x_2}\big(\widehat{x}_1(1+\delta_{x_1}), \widehat{x}_2(1+\delta_{x_2})\big)\widehat{x}_2}, \tag{15}$$

where $g_{x_1}(\cdot)$ and $g_{x_2}(\cdot)$ denote first-order partial derivatives of $g(\cdot)$ with respect to the first and second arguments, respectively. Hence, to construct the smallest possible approximation errors, we must solve a system of two equations (14), (15) with respect to two unknowns δ_{x_1} and δ_{x_2}.

For the case of a linear equation (11), we can solve this system in a closed form,

$$\widehat{\delta}_{x_i} = -\frac{a_i\widehat{x}_i(a_1\widehat{x}_1 + a_2\widehat{x}_2)}{(a_1\widehat{x}_1)^2 + (a_2\widehat{x}_2)^2}, \quad i = 1, 2, \tag{16}$$

where to derive (16), we used the fact that $g_{x_1}(\cdot) = a_1$ and $g_{x_2}(\cdot) = a_2$.

However, for a general nonlinear restriction $g(x_1, x_2) = 0$, system (14), (15) does not admit a closed-form representation. If approximation errors are small, a sufficiently accurate solution to (14), (15) can be obtained by using a first-order Taylor expansion

$$g\big(\widehat{x}_1(1+\delta_{x_1}), \widehat{x}_2(1+\delta_{x_2})\big) \approx g(\widehat{x}_1, \widehat{x}_2) + g_{x_1}(\widehat{x}_1, \widehat{x}_2)\widehat{x}_1\delta_{x_1} + g_{x_2}(\widehat{x}_1, \widehat{x}_2)\widehat{x}_2\delta_{x_2}. \tag{17}$$

Combining (17) with FOC (15), evaluated in $(\widehat{x}_1, \widehat{x}_2)$, yields

$$\widehat{\delta}_{x_i} = -\frac{g_{x_i}(\widehat{x}_1, \widehat{x}_2)\widehat{x}_i g(\widehat{x}_1, \widehat{x}_2)}{\big[g_{x_1}(\widehat{x}_1, \widehat{x}_2)\big]^2(\widehat{x}_1)^2 + \big[g_{x2}(\widehat{x}_1, \widehat{x}_2)\big]^2(\widehat{x}_2)^2}, \quad i = 1, 2. \tag{18}$$

If approximation (18) is not sufficiently accurate, we need either to construct a Taylor expansion of a higher order or to find a nonlinear solution to (14), (15) using a numerical solver such as a Newton method. In that case, linear approximation (18) can be used as an initial guess for a numerical solver.

2.3. *Discussion*

The advantage of lower- and upper-bound error approaches is that they are direct approaches that assess the size of the approximation errors in the solution—our true object of interest. The limitations of these two approaches are typical for necessary and sufficient conditions, respectively. The lower-bound error analysis may be too optimistic and may fail to reject solutions that are insufficiently accurate. In turn, the upper-bound error analysis may be too pessimistic and may reject solutions that are sufficiently accurate. The lower and upper error bounds are complementary and both of them are useful. A combination of both is even more useful than either one individually because it shows a possible range for approximation errors.

There are approaches to accuracy evaluation in the literature that do not suffer from these shortcomings but they are indirect: they provide some numbers related to accuracy but do not assess the size of approximation errors directly. We describe the relation between direct and indirect approaches to accuracy evaluation in Section 4.

We must emphasize that our lower-bound error analysis does not provide a basis for claiming that some approximate solution is accurate but only for detecting and discarding inaccurate solutions. Indeed, even if the lower error bound $\widehat{\delta}$ is small, it could be that the actual approximation errors δ^* are large, so that the numerical approximation is still

inaccurate. Furthermore, the lower-bound error analysis does not allow us to discriminate between competing algorithms. The fact that one algorithm has a smaller lower bound than the other does not necessarily mean that the former algorithm is more accurate than the latter one. The goal of the lower-bound error analysis is limited to tracing some of those algorithms that do not guarantee a minimally acceptable quality of approximation necessary for the questions studied.

Finally, let us mention some possible extension of our lower-bound error analysis. In the benchmark case, we focus on the version of system (1) which is exactly identified, that is, that the number of equations n is equal to the number of unknowns k, that is, $n = k$. However, our construction is also applicable to the case of an underidentified system with $n < k$, except that in this case, we will have not just one but multiple compensations δ^* satisfying (2) and for each possible exact compensation, a separate lower error bound needs to be constructed. Furthermore, our analysis can be extended to the cases when system (1) is overidentified, that is, $n > k$, in particular, overidentified systems of equations are commonly used in econometrics, for example, generalized method of moments; see Hansen (1982). In this case, there is no compensation δ^* satisfying (2) exactly but there is one that maximizes the fit according to a given norm, and we can still assess its lower bound by removing different subsets of the model's equations.

3. ASSESSING APPROXIMATION ERRORS IN THE OPTIMAL GROWTH MODEL

In Section 2, we developed the lower-error-bound framework for the usual (finite-dimensional) system of equations in which the unknowns are variables. However, economic models typically lead to systems of functional equations, in which the unknowns are functions. Any numerical analysis of functional equations requires some kind of discretization (since it is impossible to evaluate numerically functions in every point of a continuous domain). Once a system of functional equations is discretized, we can construct the lower error bound as described in Section 2. In this section, we show how to construct lower bounds on approximation errors for dynamic economic models characterized by infinite-dimensional systems of equations. As a main example, we consider the standard neoclassical stochastic growth model and we assess the error bounds for numerical solutions produced by first- and second-order perturbation methods. We choose this model because it is simple, well-known, and provides a convenient framework for explaining, illustrating, and testing the methodology of lower-bound error analysis in the context of functional equations. In Section 5, we consider our second more interesting and novel application—a stylized new Keynesian model.

3.1. *Discretizing a System of Functional Equations*

We formulate the model, and we discretize its optimality conditions.

3.1.1. *The Model*

The representative agent solves

$$\max_{\{k_{t+1}, c_t\}_{t=0,\dots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{19}$$

$$\text{s.t. } c_t + k_{t+1} = (1 - d)k_t + \exp(\theta_t) A f(k_t), \tag{20}$$

$$\theta_{t+1} = \rho \theta_t + \varepsilon_{t+1}, \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2), \tag{21}$$

where (k_0, θ_0) is given; E_t is the conditional expectation operator; c_t, k_t, and θ_t are consumption, capital, and productivity level, respectively; $\beta \in (0, 1)$, $d \in (0, 1]$, $\rho \in (-1, 1)$, $\sigma \geq 0$, $A > 0$ are the parameters; u and f are strictly increasing, continuously differentiable, and concave; u' and f' denote the first derivatives of u and f, respectively.

The Euler equation of (19)–(21) is

$$u'(c_t) = \beta E_t\{u'(c_{t+1})[1 - d + \exp(\theta_{t+1})Af'(k_{t+1})]\}. \tag{22}$$

A solution to the model is policy functions $c_t = C(k_t, \theta_t)$ and $k_{t+1} = K(k_t, \theta_t)$ that satisfy (20)–(22) for all (k_t, θ_t) within the relevant domain.

3.1.2. *State-Contingent Approximation Errors*

Let us first define exact approximation errors in the sense (2). We consider an approximate numerical solution to (19)–(21) in the form of approximate consumption and capital functions, $\widehat{C} \approx C$ and $\widehat{K} \approx K$, respectively.

We define approximation errors δ_C and δ_K as state-contingent compensation functions that make an approximate solution to satisfy the model's equations (20) and (22) exactly:

$$\begin{aligned}
\widehat{C}(k, \theta)&\big(1 + \delta_C(k, \theta)\big) + \widehat{K}(k, \theta)\big(1 + \delta_K(k, \theta)\big) \\
&= (1 - d)k + \exp(\theta)Af(k),
\end{aligned} \tag{23}$$

$$\begin{aligned}
u'\big(\widehat{C}(k, \theta)&\big(1 + \delta_C(k, \theta)\big)\big) \\
&= \beta E_t\big\{u'\big(\widehat{C}\big(k', \theta'\big)\big(1 + \delta_C\big(k', \theta'\big)\big)\big) \\
&\quad \times \big[1 - d + \exp(\theta')Af'\big(\widehat{K}(k, \theta)\big(1 + \delta_K(k, \theta)\big)\big)\big]\big\},
\end{aligned} \tag{24}$$

where $(k, \theta) \in D \subseteq \mathbb{R}_+^2$ and $\theta' = \rho\theta + \varepsilon'$ with $\varepsilon' \sim \mathcal{N}(0, \sigma^2)$.

3.1.3. *Discretizing the Optimality Conditions*

We next discretize the system of functional equations (20) and (22) for a numerical treatment. We discretize the system along two dimensions: First, we choose a finite set of points that covers the continuous domain $(k, \theta) \in D \subseteq \mathbb{R}_+^2$ in which the accuracy is evaluated. Second, we construct a finite set of integration nodes that represent the future states $(k', \theta') \in D \subseteq \mathbb{R}_+^2$ in which integrals (expectation functions) in the right side of Euler equation (22) are evaluated.[4] The discretized budget constraint (20) and Euler equation (22) are

$$\underbrace{\widehat{c}_t(1 + \delta_{c_t})}_{=c_t} + \underbrace{\widehat{k}_{t+1}(1 + \delta_{k_{t+1}})}_{=k_{t+1}} = \exp(\theta_t)Af(k_t) + (1 - d)k_t, \tag{25}$$

[4]Our analysis can also be applied directly to models with a finite number of shocks. In such models, future exogenous states are known exactly and need not be approximated with quadrature or other numerical integration methods.

$$u'\big(\underbrace{\widehat{c}_t(1+\delta_{c_t})}_{=c_t}\big) = \beta \sum_{j=1}^{J}\big\{u'\big(\underbrace{\widehat{c}_{t+1,j}(1+\delta_{c_{t+1,j}})}_{=c_{t+1,j}}\big)$$

$$\times \big[1 - d + \exp(\theta_{t+1,j})Af'\big(\underbrace{\widehat{k}_{t+1}(1+\delta_{k_{t+1}})}_{=k_{t+1}}\big)\big]\big\}, \tag{26}$$

where $\{\varepsilon_{t+1,j}\}_{j=1,\dots,J}$ is a set of integration nodes that determines the future exogenous states $\theta_{t+1,j} = \rho\theta_t + \varepsilon_{t+1,j}$; δ_{c_t}, $\delta_{k_{t+1}}$, and $\delta_{c_{t+1,j}}$ are approximation errors that show how much an approximate solution $\widehat{c}_t$, $\widehat{k}_{t+1}$, and $\widehat{c}_{t+1,j}$ must be modified to become an exact solution c_t, k_{t+1}, and $c_{t+1,j}$, respectively.

Furthermore, since the exact approximation errors are state contingent, it must be the case that δ_{c_t} and $\delta_{c_{t+1,j}}$ are generated by the same function of the state variables, that is,

$$\delta_{c_t} = \delta_C(k_t, \theta_t) \quad \text{and} \quad \delta_{c_{t+1,j}} = \delta_C(k_{t+1}, \theta_{t+1,j}) \quad \text{for all } t, j. \tag{27}$$

With an additional restriction (27), equations (25), (26) are a discretized version of the state-contingent representation of approximation errors (23) and (24).

Let us assume that it is infeasible to construct state-contingent error functions (27) with a high degree of accuracy. Again, if we could construct δ_C and δ_K accurately, we would also be able to infer an accurate solution $C = \widehat{C}(1+\delta_C)$ and $K = \widehat{K}(1+\delta_K)$. In turn, if δ_C and δ_K are constructed with errors themselves, we would not be able to tell whether such functions measure the errors in $\widehat{C}$ and $\widehat{K}$ or they measure the errors in their own computation $\widehat{\delta}_C$ and $\widehat{\delta}_K$. That is, having non-negligible approximation errors in approximation errors would contaminate the analysis of approximation errors and invalidate the accuracy inferences. Given that a construction of state-contingent error functions that satisfy (25), (26), and (27) is infeasible, we focus on constructing their lower bounds.

3.2. *A Lower Bound on Approximation Errors*

We define lower error bounds, and we discuss the implementation details.

3.2.1. *Defining a Lower Error Bound*

Our benchmark implementation of the lower-bound error analysis for a system of functional equations is as follows: We drop the equations that require approximation errors to be state-contingent functions (27), and we construct approximation errors δ_{c_t}, $\delta_{k_{t+1}}$, and $\delta_{c_{t+1,j}}$, $j = 1, \dots J$ satisfying (25), (26) only. Since we ignore (27), we can construct a solution to (25), (26) in the point-by-point manner. Such a construction involves no function approximation but only a numerical resolution of the usual system of nonlinear equations and can be performed very accurately using a numerical solver.

After removing (27), system (25), (26) is underdetermined and does not identify δ_{c_t}, $\delta_{k_{t+1}}$, and $\delta_{c_{t+1,j}}$, $j = 1, \dots, J$, uniquely (we have two equations with $2 + J$ unknowns, i.e., the solution to (25), (26) is a manifold). To construct a lower error bound, we solve a least-squares problem of type (8):

$$\min_{\delta_{c_t}, \delta_{k_{t+1}}, \delta_{c_{t+1,1}}, \dots, \delta_{c_{t+1,J}}} \delta_{c_t}^2 + \delta_{k_{t+1}}^2 + \delta_{c_{t+1,1}}^2 + \cdots + \delta_{c_{t+1,J}}^2 \tag{28}$$

$$\text{s.t. (25), (26).}$$

Problem (28) produces $\{\delta_{c_t}, \delta_{k_{t+1}}, \delta_{c_{t+1,1}}, \ldots, \delta_{c_{t+1,J}}\}_{t=1}^{T}$ that solve (28) in each point $t \in 1, \ldots, T$. By construction, the resulting approximation errors are smaller than the exact state-contingent approximation errors also satisfying (27) in each point $t \in 1, \ldots, T$. That is, the solution to (28) is a lower bound on the exact state-contingent approximation errors satisfying (25), (26), and (27).

3.2.2. *Implementation Details of the Lower-Bound Error Analysis*

A numerical construction of the lower error bound requires us to make several choices including those of a domain in which the accuracy is evaluated, a specific discretization of that domain, a specific numerical method for approximating integrals, and a specific norm for measuring the size of approximation errors. Below, we discuss these choices.

Domain for Evaluating the Accuracy of Solutions.   While the solution to the model (19)–(21) is defined on a very large domain $(k, \theta) \subseteq \mathbb{R}_+^2$, a vast majority of states in that domain has practically zero probability of occurrence. The related literature requires a numerical approximation to be accurate only in a relatively small fraction of this domain where the probability of visiting the states is bounded away from zero. We consider two alternative schemes for distinguishing and discretizing the relevant domain for accuracy evaluation: one is a set of simulated points, and the other is a set of uniformly spaced points in a rectangle around the steady state. Both schemes have their advantages and shortcomings: The simulated points represent a high probability area of the state space in which the solution "lives" but there is a chance that a relatively bad approximation takes us to the region where the required compensation may not be very informative (one way or another). In turn, a rectangular domain may include a large fraction of low-probability states in which high accuracy is not essential.[5] In low-dimensional problems, we can cover the rectangle with a tensor-product grid, while in problems with high dimensionality, we can populate a hyper-rectangular domain with a set of uniformly-spaced low discrepancy points.[6]

To determine the range of state variables for constructing a rectangular domain, we use the results from simulation, namely, we choose the rectangular domain to exactly enclose a given set of simulated points. Under this construction, approximation errors on a stochastic simulation provide a lower bound on approximation errors on a larger rectangular domain in the following sense: if an approximate solution is inaccurate on a given set of simulated points, it cannot be accurate on a larger domain that encloses this set of simulated points. To produce a set of simulated points for accuracy evaluation, we use first-order perturbation solutions, which are numerically stable in simulation.

Numerical Integration.   To approximate the expectation function, we need sufficiently accurate numerical integration methods that do not distort our accuracy analysis. In economic models with smooth decision functions, like our optimal growth model, deterministic integration methods such as Gauss Hermite quadrature and monomial rules deliver very accurate approximation to the expectation functions. For example, in the studied model, even the simplest Gauss Hermite rule with just two quadrature nodes delivers six

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[5]For example, the high-probability set has a shape of ellipse for this and other similar models with normally distributed shocks; see Judd, Maliar, and Maliar (2011a, 2011b) for a discussion and graphical illustration.

[6]This technique was introduced to economics by Rust (1997); see Niederreiter (1992) for a survey of low discrepancy sequences. Also, see Maliar and Maliar (2014, 2015) for further examples of applications of low discrepancy sequences in the context of economic problems.

digits of precision in the numerical solutions; see Judd, Maliar, and Maliar (2011a). In our present analysis, we use even more accurate Gauss Hermite rule with 10 quadrature nodes.

Norm for Measuring Approximation Errors. Problem (28) produces a set function of minimal approximation errors $\{\delta_{c_t}, \delta_{k_{t+1}}, \delta_{c_{t+1,1}}, \ldots, \delta_{c_{t+1,J}}\}_{t=1}^T$ on a given discretized domain. To aggregate the resulting errors over the domain into a unique lower error bound, we can use any standard norms. In the paper, we report L_1 and L_∞ norms, which are the average and maximum absolute approximation errors across all variables and all points in the domain. Our preferred choice is an L_∞ norm which insures that an approximate solution is accurate everywhere in both present and future states, that is, uniformly accurate. Also, an L_∞ norm provides the lower error bound with respect to the "fineness" of the domain discretization. Namely, if we were able to construct approximation errors in all points of the continuous domain (or use a very fine discretization), this would only increase the lower error bound under an L_1 norm, relatively to the bound obtained under a more coarse discretization.

3.3. *Numerical Experiments*

We describe the calibration and solution procedures and construct the lower error bound numerically.

3.3.1. *Calibration and Solution Procedure*

We use Dynare to compute the first- and second-order perturbation solutions, referred to as PER1 and PER2, respectively; for a description of this software, see Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011). We parameterize the model (19)–(21) by assuming $u(c_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$ with $\gamma \in \{\frac{1}{10}, 1, 10\}$ and $f(k_t) = k_t^\alpha$ with $\alpha = 0.33$. We set $\beta = 0.99$, $d = 0.025$, $\rho = 0.95$, and $\sigma = 0.01$, and we normalize the steady state of capital to 1 by assuming $A = \frac{1/\beta-(1-d)}{\alpha}$. We simulate the model for $T = 10{,}200$ periods (we disregard the first 200 observations to eliminate the effect of initial conditions.)

We solve minimization problem (28) numerically for each given state (k_t, θ_t). To compute expectation in (26), we use a 10-point Gauss Hermite quadrature integration rule. To find initial guesses for $\delta_{c_t}, \delta_{k_{t+1}}, \delta_{c_{t+1,j}}, j = 1, \ldots, J$, we compute first-order Taylor expansions of (25), (26), and we solve the resulting linear-quadratic programming; see Appendix A.1 of the Supplemental Material (Judd, Maliar, and Maliar (2017)) for details. We then employ a quasi-Newton solver to compute a highly accurate nonlinear solution to (28) using the first-order approximation as an initial guess; see Appendix A.2 of the Supplemental Material for details. Our hardware is Intel® Core™ i7-2600 CPU @ 3.400 GHz with RAM 12.0 GB. Our software is written in MATLAB 2012a.

3.3.2. *Numerical Results on the Lower Error Bound*

The results for an accuracy test on a stochastic simulation are provided in the upper panel of Table I. To save on space, we report only the smallest and largest approximation errors for the future state, that is, $\delta_{c_{t+1}}^{\min} \equiv \min_{j\in J}\{\delta_{c_{t+1,1}}, \ldots, \delta_{c_{t+1,J}}\}$ and $\delta_{c_{t+1}}^{\max} \equiv \max_{j\in J}\{\delta_{c_{t+1,1}}, \ldots, \delta_{c_{t+1,J}}\}$, respectively.

Across all the cases, highest maximum approximation errors are $10^{-2.62} \approx 0.25\%$ and $10^{-3.65} \approx 0.025\%$ for PER1 and PER2, respectively, which corresponds to the case of the

TABLE I

APPROXIMATION ERRORS IN THE CURRENT AND FUTURE VARIABLES IN THE NEOCLASSICAL STOCHASTIC GROWTH MODEL[a]

Method	Norm	$\gamma=\frac{1}{10}$				$\gamma=1$				$\gamma=10$			
		δ_{c_t}	$\delta_{k_{t+1}}$	$\delta_{c_{t+1}}^{\min}$	$\delta_{c_{t+1}}^{\max}$	δ_{c_t}	$\delta_{k_{t+1}}$	$\delta_{c_{t+1}}^{\min}$	$\delta_{c_{t+1}}^{\max}$	δ_{c_t}	$\delta_{k_{t+1}}$	$\delta_{c_{t+1}}^{\min}$	$\delta_{c_{t+1}}^{\max}$
Stochastic simulation of 10,000 points													
PER1	L_1	−3.76	−4.05	−9.07	−3.86	−4.54	−4.11	−8.77	−4.63	−4.18	−3.75	−8.72	−4.32
	L_∞	−3.00	−2.99	−6.56	−2.80	−3.71	−3.03	−7.43	−3.75	−3.53	−2.62	−7.08	−3.65
PER2	L_1	−5.43	−5.72	−10.86	−5.55	−6.10	−5.68	−10.72	−6.27	−5.41	−4.75	−9.56	−5.53
	L_∞	−4.33	−4.38	−9.02	−4.44	−4.89	−4.43	−8.84	−4.85	−4.18	−3.65	−6.82	−3.76
Tensor product grid of 10,000 points													
PER1	L_1	−2.38	−3.64	−7.88	−2.99	−2.48	−5.49	−8.22	−4.01	−2.25	−5.71	−8.20	−4.19
	L_∞	−1.48	−2.72	−6.11	−2.12	−1.82	−4.50	−6.89	−3.37	−1.68	−4.10	−6.65	−3.40
PER2	L_1	−3.96	−5.05	−9.68	−4.41	−3.76	−6.12	−9.63	−5.29	−3.16	−5.78	−9.14	−4.86
	L_∞	−3.17	−4.10	−8.40	−3.53	−3.05	−4.67	−8.36	−4.52	−2.33	−3.73	−7.21	−3.91
Low discrepancy sequence of 10,000 points													
PER1	L_1	−2.39	−3.65	−7.88	−3.00	−2.49	−5.50	−8.23	−4.02	−2.26	−5.71	−8.21	−4.20
	L_∞	−1.49	2.73	−5.97	−2.12	−1.82	−4.51	−6.98	−3.38	−1.68	−3.70	−6.74	−3.22
PER2	L_1	−3.98	−5.06	−9.69	−4.43	−3.77	−6.12	−9.58	−5.30	−3.17	−5.78	−9.14	−4.88
	L_∞	−3.18	−4.12	−8.23	−3.54	−3.07	−4.67	−8.36	−4.54	−2.34	−3.75	−7.53	−3.87

[a]Notes: PER1 and PER2 denote the first- and second-order perturbation solutions; δ_{c_t} and $\delta_{k_{t+1}}$ are t-period absolute values of approximation errors in consumption and capital; and $\delta_{c_{t+1}}^{\min}$ and $\delta_{c_{t+1}}^{\max}$ are the largest and smallest absolute value of approximation errors in future consumption across $J=10$ integration nodes; L_1 and L_∞ are, respectively, the average and maximum of absolute values of the corresponding approximation errors across test points (in log10 units); and γ is the coefficient of risk aversion.

largest risk aversion coefficient considered, $\gamma=10$. These numbers are sufficiently small, meaning that under an optimistic view, the approximation errors are acceptable in size. Again, our test is a necessary condition for accuracy and does not allow us to conclude that perturbation solutions are accurate. We can only say that our numerical experiments do not provide a basis for claiming that perturbation methods are insufficiently reliable in the context of the standard optimal growth model.

We also evaluate the accuracy on a rectangular domain. We fix the range of values for state variables (k, θ) using the simulation results from the previous experiment. In our first experiment, we covered the resulting rectangular domain by a tensor-product grid with 100 by 100 points, which is 10,000 points in total. In our second experiment, we populated this domain with 10,000 low discrepancy, Sobol points. These experiments are reported in the two lower panels of Table I, respectively. As expected, the constructed lower error bounds are somewhat larger on a rectangular domain than on the benchmark simulation-based domain (essentially because the accuracy is evaluated on a larger domain) but our qualitative conclusions stay the same.[7]

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[7]There are other possible implementations of the lower-bound error analysis. In Appendix A.3, we discuss one such implementation that solves for just one approximation error δ_{E_t} in the expectation function $(1+\delta_{E_t})\widehat{E}_t = E_t[\cdot]$, instead of solving for state-contingent approximation errors in consumption $\delta_{c_{t+1,1}}, \ldots, \delta_{c_{t+1,J}}$ in J integration nodes. This alternative method is easier to implement but the accuracy results are more difficult to interpret.

4. RELATION OF LOWER-BOUND ERROR ANALYSIS TO OTHER ACCURACY MEASURES IN THE LITERATURE

There are three main approaches to accuracy evaluation in the economic literature: forward error analysis, backward error analysis, and analysis of residuals. The forward error analysis assesses an approximation error in the solution of a given model. (Hence, our lower-bound error analysis is a variant of forward error analysis.) Backward error analysis proceeds in a reverse manner: it takes an approximate solution as given and asks how much the model itself must be modified to make an approximate solution to satisfy all the model's equations. Finally, analysis of residuals consists in evaluating residuals in the model's equations for a given approximate solution. Below, we discuss a relation of these three approaches to our lower-bound error analysis and we illustrate these alternative accuracy measures in the context of the studied model.

4.1. *Conventional Forward Error Analysis for the Growth Model*

Conventional forward error analysis for the standard growth model was carefully implemented in Santos (2000). His construction relies on the fact that the standard growth model (19)–(21) can be reformulated as a dynamic programming problem. The contraction mapping property of the Bellman operator makes it possible to derive an upper bound on approximation errors analytically. The analysis of Santos (2000) shows that a worst-case scenario can often be too pessimistic and may lead to a rejection of numerical solutions that are sufficiently accurate. For example, under the standard calibration of a similar optimal growth model, Santos (2000, Table I) obtained an upper error bound on policy functions of order 10^3. Consequently, he also showed that this error bound can be reduced by about three orders of magnitude by using some additional information from a specific numerical solution, so that the upper bound is of order 10^0 or 10^{-1}.

In turn, our optimistic lower bound on approximation error is much smaller in size, namely, 10^{-2} or 10^{-3}; see Table I. However, our lower-bound error analysis can understate the size of the approximation errors and thus, may fail to reject inaccurate solutions. But the two bounds together show us the relevant range for the size of approximation errors. This is why the upper and lower error bounds are complementary and they are both even more useful than each of them individually.

An important limitation of conventional upper-bound error analysis is that it is restricted to dynamic programming problems. It is generally not possible to conduct a systematic upper-bound error analysis in many economic models that are used in practice. In particular, this kind of error analysis is not directly applicable to non-optimal equilibrium problems such as a new Keynesian model studied in the previous section. In turn, our methodology of constructing lower error bounds is quite general: it is independent of a specific solution method and applicable to both dynamic programming and equilibrium problems.

4.2. *Backward and Mixed Forward-Backward Error Analysis*

A backward error analysis was introduced in Wilkinson (1963) who posed the following question: "How much must the parameters of a model be modified in order to make an approximate solution to satisfy the model's equation exactly?" Higham (1996) introduced a mixed forward-backward analysis which is an extension of backward analysis that allows for changes in both equilibrium quantities and the model's parameters. Sims (1990) proposed an accuracy test which is similar in spirit to the backward error analysis: he

measured accuracy by how far the true distribution of stochastic shocks is situated from the distribution of stochastic shocks implied by the approximate solution. Kubler and Schmedders (2005) showed how a backward and a mixed backward-forward analysis can be used to evaluate the accuracy of numerical solutions in a life-cycle model with incomplete markets and heterogeneous agents. Finally, Kogan and Mitra (2013) proposed to measure the quality of approximation in terms of a welfare loss that results from inaccuracy of an approximate solution. They constructed a supplementary model with perfect foresight and assessed the difference in welfare between that supplementary model and the true stochastic model with an approximate solution; this provides an upper bound on the welfare loss.

There are many possible ways to implement a backward error analysis for the optimal growth model (19)–(21). We choose one such way by measuring the accuracy in the Euler equation (22) and budget constraint (20) by the implied values of the parameters β and d, denoted by $\beta(k_t, \theta_t)$ and $d(k_t, \theta_t)$, respectively,

$$\beta(k_t, \theta_t) = E_t \left\{ \frac{u'(\widehat{c}_{t+1})}{u'(\widehat{c}_t)} \left[1 - d + \exp(\rho \theta_t + \varepsilon_{t+1}) A f'(\widehat{k}_{t+1}) \right] \right\}^{-1}, \qquad (29)$$

$$d(k_t, \theta_t) \equiv \left\{ 1 - \frac{\widehat{c}_t + \widehat{k}_{t+1} - \exp(\theta_t) A f(k_t)}{k_t} \right\}. \qquad (30)$$

We compute $\beta(k_t, \theta_t)$ and $d(k_t, \theta_t)$ on the same set of simulated points as was used for computing all our previous statistics; see Section 3.3. The results are provided in Table II. The accuracy implications here are similar to those in Tables I and III. The least accurate solution is obtained under $\gamma = 10$, in particular, PER1 implies that $\beta(k_t, \theta_t)$ and $d(k_t, \theta_t)$ range within $[0.9870, 0.9894]$ and $[0.0225, 0.0261]$, respectively, which correspond to up to 0.3% and 10% deviations from their exact values $\beta = 0.99$ and $d = 0.025$, respectively. PER2 is more accurate than PER1, in particular, under $\gamma \in \{\frac{1}{10}, 1\}$, the parameters values implied by PER2 coincide with their exact values at least up to four digits.

The backward and mixed forward-backward accuracy measures are also indirect measures of accuracy and are generally subject to the same critique as the analysis of residuals. Namely, they do not show the distance between the exact and approximate solutions but

TABLE II

THE DISTRIBUTION OF THE PARAMETERS VALUES IN THE NEOCLASSICAL STOCHASTIC GROWTH MODEL[a]

Parameters	$\gamma = \frac{1}{10}$		$\gamma = 1$		$\gamma = 10$	
	$\beta(k_t, \theta_t)$	$d(k_t, \theta_t)$	$\beta(k_t, \theta_t)$	$d(k_t, \theta_t)$	$\beta(k_t, \theta_t)$	$d(k_t, \theta_t)$
PER1						
mean	0.9900	0.0251	0.9900	0.0251	0.9892	0.0249
min	0.9897	0.0249	0.9857	0.0248	0.9870	0.0225
max	0.9901	0.0259	0.9900	0.0259	0.9894	0.0261
PER2						
mean	0.9900	0.0250	0.9900	0.0250	0.9900	0.0250
min	0.9900	0.0250	0.9900	0.0250	0.9892	0.0248
max	0.9900	0.0250	0.9900	0.0250	0.9909	0.0251

[a]Notes: PER1 and PER2 denote the first- and second-order perturbation solutions; "mean," "min," and "max" are, respectively, the average, minimum, and maximum of the values of the corresponding model's parameter on a stochastic simulation of 10,000 observations; γ is the coefficient of risk aversion.

the distance between the parameters or some mixture of the parameters and approximate solutions. It is not always clear how to relate the implied deviations in the parameters to the accuracy of the solutions. For example, we know that the equilibrium quantities are typically very sensitive to β and that they are less sensitive to d, so it could be that 0.3% deviation in β implies a larger accuracy decline than a 10% deviation in d. Hence, we must have some knowledge of how sensitive the model's variables are to the parameters. The results of backward error analysis would be even more difficult to interpret for a new Keynesian economy. Furthermore, it is not clear whether or not, for any given model and for any approximate solution, one can find a supplementary model (a parameters vector) that leads to zero approximation errors.

4.3. *Analysis of Residuals in the Model's Equations*

A commonly used accuracy measure in the literature is the size of residuals in the model's equations (such as a Bellman equation, Euler equations, market clearing conditions, budget constraints, and laws of motion for exogenous and endogenous shocks); see, for example, Judd (1992), Jin and Judd (2002), Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006), Juillard and Villemot (2011), Judd, Maliar, and Maliar (2011a); also, see a statistical test of residuals of Den Haan and Marcet (1994). Furthermore, Kubler and Schmedders (2003) proposed a notion of epsilon equilibrium and introduced an accuracy measure that requires that the residuals in all model's equations are smaller than a given target level; an epsilon equilibrium may exist even if the true equilibrium does not.

4.3.1. *Analysis of Residuals in the Neoclassical Growth Model*

Let us analyze the residuals in the studied growth model. We define unit-free residuals in a point (k_t, θ_t) by rewriting (22) and (20) as follows:

$$\mathcal{R}^1(k_t, \theta_t) = \frac{u'^{-1}\big[\beta E_t\{u'(\widehat{c}_{t+1})[1 - d + \exp(\rho\theta_t + \varepsilon_{t+1})Af'(\widehat{k}_{t+1})]\}\big]}{\widehat{c}_t} - 1, \quad (31)$$

$$\mathcal{R}^2(k_t, \theta_t) = \frac{\exp(\theta_t)Af(k_t) + (1 - d)k_t - \widehat{c}_t}{\widehat{k}_{t+1}} - 1, \quad (32)$$

where $\widehat{c}_{t+1} = \widehat{C}(k_{t+1}, \theta_{t+1}) = \widehat{C}(\widehat{K}(k_t, \theta_t), \rho\theta_t + \varepsilon_{t+1})$, $\widehat{c}_t = \widehat{C}(k_t, \theta_t)$, and $\widehat{k}_{t+1} = \widehat{K}(k_t, \theta_t)$. Here, we express $\mathcal{R}^1$ and $\mathcal{R}^2$ in terms of consumption and capital units, respectively, which is parallel to the definitions of approximation errors δ_{c_t} and $\delta_{k_{t+1}}$ in our lower-bound error analysis. Namely, $\mathcal{R}^1(k_t, \theta_t)$ is the same as δ_{c_t} if we assume that c_{t+1} and k_{t+1} are computed without errors (i.e., we set $\delta_{k_{t+1}} = \delta_{c_{t+1,j}} = 0$, $j = 1, \ldots, J$) and $\mathcal{R}^2(k_t, \theta_t)$ is the same as $\delta_{k_{t+1}}$ if we assume that c_t is computed without errors (i.e., we set $\delta_{c_t} = 0$).

We compute $\mathcal{R}^1(k_t, \theta_t)$ and $\mathcal{R}^2(k_t, \theta_t)$ on a set of simulated points (for the details of the simulation procedure, see Section 3.3). The results are provided in Table III. As we can see, the maximum residuals across the two equilibrium conditions are below $10^{-2.56} \approx$ 0.28% for PER1 and about $10^{-3.65} \approx 0.025\%$ for PER2.

4.3.2. *Advantages and Shortcomings of the Analysis of Residuals*

The advantage of this accuracy measure is that the residuals are very easy to compute: we just need to plug an approximate solution into the model's equations and to see how far the residuals are from zero. However, this measure has also important limitations.

TABLE III

RESIDUALS OF THE MODEL'S EQUATIONS IN THE NEOCLASSICAL STOCHASTIC GROWTH MODEL[a]

Residuals	$\gamma = \frac{1}{10}$		$\gamma = 1$		$\gamma = 10$	
	$\mathcal{R}^1$	$\mathcal{R}^2$	$\mathcal{R}^1$	$\mathcal{R}^2$	$\mathcal{R}^1$	$\mathcal{R}^2$
PER1						
L_1	−3.61	−4.12	−4.40	−4.12	−4.09	−3.74
L_∞	−2.56	−3.02	−3.55	−3.04	−3.52	−2.61
PER2						
L_1	−5.29	−5.80	−5.96	−5.69	−5.30	−4.75
L_∞	−4.20	−4.41	−4.74	−4.44	−4.05	−3.65

[a]Notes: PER1 and PER2 denote the first- and second-order perturbation solutions; L_1 and L_∞ are, respectively, the average and maximum of absolute values of residuals in the model's equations across optimality condition and test points (in log10 units) on a stochastic simulation of 10,000 observations; and γ is the coefficient of risk aversion.

First, it is an indirect accuracy measure: the residuals provide some statistics related to accuracy but they do not directly show the size of approximation errors in the model's variables. The relation between the residuals and approximation errors is established in the literature only for a special case of strongly concave, infinite-horizon optimization problems by Santos (2000). He showed that approximation errors in policy functions are of the same order of magnitude as the size of the Euler equation residuals. In general, such relations are not known.

Second, small residuals in a model's equation do not necessarily imply small approximation errors in the model's variables that enter this equation. To see the point, let us consider the following illustrative example. Consider a numerical approximation to capital $\widetilde{k}_{t+1}$ in the neoclassical growth model (19)–(21) and assume that $\widetilde{c}_t$ is computed to satisfy the budget constraint (20) exactly as is done by global solution methods, that is, $\widetilde{c}_t = \exp(\theta_t) A f(k_t) + (1-d)k_t - \widetilde{k}_{t+1}$. By construction, the residual in this equation is zero, that is,

$$\widetilde{\mathcal{R}}^2(k_t, \theta_t) = \frac{\exp(\theta_t) A f(k_t) + (1-d)k_t - \widetilde{c}_t}{\widetilde{k}_{t+1}} - 1 = 0.$$

This does not mean that the approximation errors in $\widetilde{k}_{t+1}$ and $\widetilde{c}_t$ are zeros but that the two errors simply offset one another to make the residual equal to zero, $\widetilde{k}_{t+1}\delta_{k_{t+1}} + \widetilde{c}_t\delta_{c_t} = 0$, where, as before, we assume that the exact and approximate solutions are related by $k_{t+1} = \widetilde{k}_{t+1}(1 + \delta_{k_{t+1}})$ and $c_t = \widetilde{c}_t(1 + \delta_{c_t})$. In this example, arbitrary large approximation errors in the model's two variables are consistent with a zero residual in this model's equation.

Third, the size of residuals in the model's equations is affected by a specific way in which the residuals are constructed. For example, instead of $\mathcal{R}^2(k_t, \theta_t)$, given by (32), we can construct the following unit-free residual $\overline{\mathcal{R}}^2(k_t, \theta_t)$:

$$\overline{\mathcal{R}}^2(k_t, \theta_t) \equiv \frac{\exp(\theta_t) A f(k_t) + (1-d)k_t}{\widehat{k}_{t+1} + \widehat{c}_t} - 1.$$

To make appropriate accuracy inferences, we must take into account the value of the variable with respect to which the residual is constructed, namely, $\mathcal{R}^2(k_t, \theta_t)$ shows the residuals relative to $\widehat{k}_{t+1}$, while $\overline{\mathcal{R}}^2(k_t, \theta_t)$ shows the residuals relative to $\widehat{k}_{t+1} + \widehat{c}_t$. In

the optimal growth model, a specific way of constructing residuals does not affect the qualitative implications about the accuracy but it might be important for more nonlinear models; for example, it is important for the new Keynesian model, studied in the next section.

5. A NEW KEYNESIAN MODEL

We now assess approximation errors in a stylized new Keynesian model with Calvo-type price frictions and a Taylor (1993) rule. The conventional upper-bound error analysis builds on dynamic programming approaches and thus, is not directly applicable to non-optimal equilibrium problems like the studied new Keynesian model. However, our lower-bound error analysis can be applied to equilibrium problems in exactly the same way as to the optimal growth model of Section 3. To save on space, in the main text, we limit ourselves to summarizing the key findings, and we describe the implementation details of the lower-bound error analysis for new Keynesian models in Appendix B of the Supplemental Material.

The previous literature on new Keynesian models shows that perturbation methods produce large residuals in the presence of zero lower bound (ZLB) on the nominal interest rate; see, for example, Aruoba, Cuba-Borda, and Schorfheide (2013), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), Maliar and Maliar (2015). However, the model we consider here does not have an active ZLB. Thus, our analysis shows that approximation errors can be very large for perturbation solutions even in the absence of an active ZLB.

We consider a variant of Christiano, Eichenbaum, and Evans (2005), and Smets and Wouters (2007). The model features physical capital, adjustment costs to investment, variable utilization of capital, habit formation in consumption, as well as sticky wages and prices. There are four types of stochastic shocks, namely, to monetary policy, neutral productivity, investment-specific productivity, and government spending. The economy is populated by labor packers, households, final-good firms, intermediate-good firms, monetary authority, and government; see Appendix B.1 for the model's description.

We consider three alternative parameterizations: a benchmark parameterization, which is in line with the estimates of Sims (2014), and two sensitivity experiments in which we increase the (net) inflation target from 0 to 2 percent, and we decrease the elasticities of substitution in the production functions of final-good producers and labor packers from 10 to 5; see Appendix B.6.1. Under the benchmark parameterization, a lower error bound reaches almost 130 percent, which corresponds to (net) price inflation. Under the second parameterization, approximation errors are of the same order; however, the largest errors are in (net) price inflation of reoptimizing firms. Finally, under the last parameterization, the lower error bound is of order 40 percent, which corresponds to labor input; see Appendix B.6.2. Thus, we can observe large approximation errors in different variables depending on specific parameterization. The fact that errors are so huge even under the most optimistic scenario makes these numerical solutions unacceptable for any application!

The lower-bound error analysis provides us with an insight into which variables are approximated insufficiently accurately and are likely to be a bottleneck for the overall accuracy (although these results are only suggestive because the lower error bound may not be interpreted as actual approximation errors). For the studied model, these are inflation variables, investment variables (gross investment, nominal interest rate, capital utilization), as well as price dispersion and labor; see Table SIII in Appendix B.6.2.

There are techniques in the literature that can selectively increase the accuracy of approximation of some variables under perturbation methods. One is a change of variables of Judd (2003): it constructs many locally equivalent Taylor expansions and chooses the one that is most accurate globally; see also Fernández-Villaverde and Rubio-Ramírez (2006) for extensions of this method. The other is a hybrid of local and global solutions of Maliar, Maliar, and Villemot (2013) that combines local solutions produced by a perturbation method with global solutions constructed to satisfy the model's equations exactly. These techniques can possibly increase the accuracy of plain perturbation methods.

We next assess the residuals in the model's equations. We find that in most equations, the residuals are small; however, there are equations in which residuals are as large as 50 percent, for example, Taylor rule, the law of motion for price dispersion, and the price index; see Table SIV in Appendix B.6.3. Given that some residuals are so large, we can see why the perturbation method does so poorly in terms of lower bounds on approximation errors: Even if we distribute approximation errors among variables in the way that is most favorable for the overall accuracy (best-case scenario for accuracy), some of the approximation errors will necessarily be large to accord with large residuals.

Finally, we also demonstrate that the way in which we construct the residuals might be important for the results. For example, in Appendix B.6.3, we show two different unit-free representations of the residuals in one of the model's equations that produce the mean residuals of about 30 percent and 1400 percent (and the maximum residual is even larger)! In these two cases, the residuals are small (large) because they are evaluated relative to a variable whose value is large (small). Hence, to make meaningful qualitative inferences about accuracy from the analysis of residuals, it is important to take into account the size of variables with respect to which residuals are evaluated. In turn, our lower error bounds are not subject to this shortcoming: they are independent of the way in which the model's equations are written.

Our results are economically significant. Perturbation methods are commonly used in the literature on new Keynesian models, and they are generally viewed as acceptable methods. Moreover, linear perturbation methods are currently used by all central banks for solving their large-scale new Keynesian macroeconomic models for forming monetary policy and projections, for example, the International Monetary Fund's Global Economy Model, GEM (Bayoumi, Laxton, and Pesenti (2001)), the U.S. Federal Reserve Board's SIGMA model (Erceg, Guerrieri, and Gust (2006)), the Bank of Canada Terms of Trade Economic Model, ToTEM (Dorich, Johnston, Mendes, Murchison, and Zhang (2013)), the European Central Bank's New Area-Wide Model, NAWM (Coenen, McAdam, and Straub (2008)), the Bank of England COMPASS model (Burgess, Fernandez-Corugedo, Groth, Harrison, Monti, Theodoridis, and Waldron (2013)), and the Swedish Riksbank's Ramses II model (Adolfson, Laséen, Christiano, Trabandt, and Walentin (2013)). The accuracy of perturbation solutions is not assessed in these applications. Our analysis suggests that accuracy evaluations are important, and that alternative solution methods are needed that can deliver more accurate solutions to this important class of economic models.

6. CONCLUSION

Conventional upper-bound error analysis focuses on worst-case scenarios and provides sufficient conditions for accuracy of numerical solutions. In this paper, we introduce a complementary lower-bound error analysis that focuses on certain best-case scenarios and provides a necessary condition for accuracy of numerical solutions. We specifically

construct the smallest possible (optimistic) approximation errors that are consistent with a strict subset of the model's equations. If even these optimistic errors are too large, we conclude that a numerical solution is inaccurate. Although the two applications studied in the paper come from macroeconomics, dynamic models are currently used in virtually all fields in economics, so that the proposed lower-bound error analysis has a broad range of potential applications.

A potential shortcoming of our accuracy test is that it may fail to reject inaccurate solutions because some inaccurate solutions may appear to be sufficiently accurate under best-case scenarios. But one of the two studied models—a stylized new Keynesian model—failed to pass even this relatively undemanding test under some empirically relevant parameterizations. Upper error bounds are unknown for new Keynesian models but they are also unnecessary in those cases when an approximate solution fails to satisfy even necessary conditions for accuracy. Thus, our simple accuracy test is powerful enough to detect and to discard inaccurate solutions in practically relevant applications. We hope that the lower-bound error analysis proposed in the paper can be automated and integrated in software for solving dynamic economic models such as Dynare.

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SUPPLEMENT TO "LOWER BOUNDS ON APPROXIMATION ERRORS TO NUMERICAL SOLUTIONS OF DYNAMIC ECONOMIC MODELS": ONLINE APPENDICES

By Kenneth L. Judd, Lilia Maliar, and Serguei Maliar

In Appendices A and B, we describe additional details of the lower-bound error analysis in the neoclassical stochastic growth model and in the new Keynesian model studied in the main text.

Appendix A: Neoclassical Stochastic Growth Model

In this section, we focus on the neoclassical stochastic growth model. In Appendix A.1, we derive a lower error bound by using linearized model's equations; in Appendix A.2, we construct a more accurate lower error bound by using nonlinear model's equations; and in Appendix A.3, we discuss alternative implementations of the lower-bound error analysis.

A.1. *Constructing Lower Error Bound by Using Linearized Model's Equations*

Euler Equation

We first linearize the Euler equation. Let us assume a CRRA utility function $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$. For this utility function, Euler equation (26), expressed in terms of approximation errors, is

$$
\begin{aligned}
&\widehat{c}_t^{-\gamma}(1+\delta_{c_t})^{-\gamma} - \beta E_t\big\{\widehat{c}_{t+1}^{-\gamma}(1+\delta_{c_{t+1}})^{-\gamma} \\
&\qquad \cdot \big[1 - d + \alpha \exp(\theta_{t+1}) A \widehat{k}_{t+1}^{\alpha-1}(1+\delta_{k_{t+1}})^{\alpha-1}\big]\big\} = 0.
\end{aligned} \tag{A.1}
$$

One can view (A.1) as a function of δ's, that is, $f(\delta_{c_t}, \delta_{c_{t+1}}, \delta_{k_{t+1}}) = 0$. Finding a first-order Taylor expansion of f around $\delta_{c_t} \to 0$, $\delta_{c_{t+1}} \to 0$, $\delta_{k_{t+1}} \to 0$ (in particular, using $(1+x)^\alpha \simeq 1 + \alpha x$) and omitting a second-order term $\delta_{c_{t+1}}\delta_{k_{t+1}} \approx 0$, we have

$$
\begin{aligned}
&\widehat{c}_t^{-\gamma} - \gamma\delta_{c_t}\widehat{c}_t^{-\gamma} - \beta E_t\big\{\widehat{c}_{t+1}^{-\gamma}\big(1 - d + \alpha \exp(\theta_{t+1}) A \widehat{k}_{t+1}^{\alpha-1}\big)\big\} \\
&\quad + \beta E_t\big\{\widehat{c}_{t+1}^{-\gamma}\gamma\delta_{c_{t+1}}\big(1 - d + \alpha \exp(\theta_{t+1}) A \widehat{k}_{t+1}^{\alpha-1}\big)\big\} \\
&\quad - \beta E_t\big\{\widehat{c}_{t+1}^{-\gamma}\big(\alpha \exp(\theta_{t+1}) A \widehat{k}_{t+1}^{\alpha-1}(\alpha-1)\delta_{k_{t+1}}\big)\big\} = 0.
\end{aligned}
$$

By discretizing the future exogenous states into J integration nodes, we replace the state-contingent functions $\widehat{c}_{t+1}$ and $\delta_{c_{t+1}}$ by $\widehat{c}_{t+1,j}^{-\gamma}$ and $\delta_{c_{t+1},j}$, $j = 1, \dots, J$, respectively, which yields

$$
1 - \gamma\delta_{c_t} - h_1 + \gamma\sum_{j=1}^{J} m_j \delta_{c_{t+1},j} - (\alpha-1)\delta_{k_{t+1}}h_2 = 0,
$$

where

$$h_1 \equiv \beta \sum_{j=1}^{J} \left\{ \frac{\widehat{c}_{t+1,j}^{-\gamma}}{\widehat{c}_t^{-\gamma}} \left(1 - d + \alpha \exp(\theta_{t+1,j}) A \widehat{k}_{t+1}^{\alpha-1}\right) \right\},$$

$$h_2 \equiv \beta \sum_{j=1}^{J} \left\{ \frac{\widehat{c}_{t+1,j}^{-\gamma}}{\widehat{c}_t^{-\gamma}} \left(\alpha \exp(\theta_{t+1,j}) A \widehat{k}_{t+1}^{\alpha-1}\right) \right\},$$

$$m_j \equiv \beta \omega_j \frac{\widehat{c}_{t+1,j}^{-\gamma}}{\widehat{c}_t^{-\gamma}} \left(1 - d + \alpha \exp(\theta_{t+1,j}) A \widehat{k}_{t+1}^{\alpha-1}\right),$$

with $\theta_{t+1,j} = \rho \theta_t + \varepsilon_j$, and ε_j, ω_j denoting a jth integration node and weight. Combining the terms yields a linear equation in δ's,

$$a^{1,1} \delta_{c_t} + a^{1,2} \delta_{k_{t+1}} + \sum_{j=1}^{J} a_j^{1,3} \delta_{c_{t+1,j}} = b^1, \tag{A.2}$$

where

$$a^{1,1} \equiv -\gamma, \qquad a^{1,2} \equiv -(\alpha - 1)h_2, \qquad a_j^{1,3} \equiv \gamma m_j, \qquad b^1 \equiv h_1 - 1.$$

Budget Constraint

We next linearize the budget constraint. We rewrite the budget constraint (25) as

$$\widehat{c}_t + \delta_{c_t} \widehat{c}_t + \widehat{k}_{t+1} + \delta_{k_{t+1}} \widehat{k}_{t+1} - (1-d)k_t - \exp(\theta_t) A k_t^\alpha = 0. \tag{A.3}$$

Thus, we get

$$a^{2,1} \delta_{c_t} + a^{2,2} \delta_{k_{t+1}} = b^2, \tag{A.4}$$

where

$$a^{2,1} \equiv \widehat{c}_t, \qquad a^{2,2} \equiv \widehat{k}_{t+1}, \qquad b^{2,1} \equiv (1-d)k_t + \exp(\theta_t) A k_t^\alpha - \widehat{c}_t - \widehat{k}_{t+1}.$$

Minimization Problem

The minimization problem (28) in a point (period) t is given by

$$\min_{\delta_{c_t}, \delta_{k_{t+1}}, \{\delta_{c_{t+1,j}}\}_{i=1,\dots,J}} \delta_{c_t}^2 + \delta_{k_{t+1}}^2 + \sum_{j=1}^{J} \delta_{c_{t+1,j}}^2 \quad \text{s.t. (A.2), (A.4).} \tag{A.5}$$

To solve (A.5) numerically, we use quadratic programming software (we use a "quadprog" routine in MATLAB).

A.2. *Constructing Lower Error Bound by Using Nonlinear Model's Equations*

We now construct the lower error bound using the original nonlinear equations. Budget constraint (A.3) yields

$$\delta_{k_{t+1}} = \frac{(1-d)k_t + \exp(\theta_t) A k_t^\alpha - \widehat{c}_t(1 + \delta_{c_t})}{\widehat{k}_{t+1}} - 1. \tag{A.6}$$

From budget constraint (A.3), we also get

$$\left[\widehat{k}_{t+1}(1+\delta_{k_{t+1}})\right]^{\alpha-1} = \left[(1-d)k_t + \exp(\theta_t)Ak_t^{\alpha} - \widehat{c}_t(1+\delta_{c_t})\right]^{\alpha-1}.$$

Substituting the latter equation into Euler equation (A.1), we have

$$(1+\delta_{c_t})^{-\gamma} - \beta E_t\left[\frac{\widehat{c}_{t+1}^{-\gamma}}{\widehat{c}_t^{-\gamma}}(1+\delta_{c_{t+1}})^{-\gamma}(1-d)\right]$$

$$- \left[\frac{(1-d)k_t + \exp(\theta_t)Ak_t^{\alpha} - \widehat{c}_t(1+\delta_{c_t})}{\widehat{k}_{t+1}}\right]^{\alpha-1}$$

$$\times \beta E_t\left[\frac{\widehat{c}_{t+1}^{-\gamma}}{\widehat{c}_t^{-\gamma}}(1+\delta_{c_{t+1}})^{-\gamma}\alpha\exp(\theta_{t+1})A\widehat{k}_{t+1}^{\alpha-1}\right] = 0.$$

By discretizing the future exogenous states into J integration nodes, we replace the state-contingent functions $\widehat{c}_{t+1}$ and $\delta_{c_{t+1}}$ by $\widehat{c}_{t+1,j}^{-\gamma}$ and $\delta_{c_{t+1,j}}$, $j = 1, \ldots, J$, respectively, which yields

$$\delta_{c_t} = \left\{\beta\sum_{j=1}^{J}\omega_j\left[\frac{\widehat{c}_{t+1}^{-\gamma}}{\widehat{c}_t^{-\gamma}}(1+\delta_{c_{t+1,j}})^{-\gamma}(1-d)\right]\right.$$

$$+ \left[\frac{(1-d)k_t + \exp(\theta_t)Ak_t^{\alpha} - \widehat{c}_t(1+\delta_{c_t})}{\widehat{k}_{t+1}}\right]^{\alpha-1} \qquad \text{(A.7)}$$

$$\left.\times \beta\sum_{j=1}^{J}\omega_j\left[\frac{\widehat{c}_{t+1}^{-\gamma}}{\widehat{c}_t^{-\gamma}}(1+\delta_{c_{t+1,j}})^{-\gamma}\alpha\exp(\theta_{t+1,j})A\widehat{k}_{t+1}^{\alpha-1}\right]\right\}^{-1/\gamma} - 1$$

$$= 0.$$

Therefore, the least-squares problem (28) becomes

$$\min_{\delta_{c_t}, \delta_{k_{t+1}}, \{\delta_{c_{t+1,j}}\}_{i=1,\ldots,J}} \delta_{c_t}^2 + \delta_{k_{t+1}}^2 + \sum_{j=1}^{J}\delta_{c_{t+1,j}}^2 \quad \text{s.t. (A.7), (A.6).} \qquad \text{(A.8)}$$

The resulting minimization problem contains just $J + 1$ unknowns, given by δ_{c_t} and $\{\delta_{c_{t+1,j}}\}_{j=1,\ldots,J}$ that are constructed using a numerical solver. Note that δ_{c_t} appears both in the left and right side of (A.7) and we need to compute it numerically. To solve problem (A.8), we use MATLAB's nonlinear optimization routine "fminsearch."

A.3. *Alternative Implementations of Lower-Bound Error Analysis*

There are many possible ways of defining approximation errors. First, we could consider approximation errors in in model's different variables, for example, the errors in the investment or output functions instead of those in capital or consumption functions. This will affect the size of the resulting error bounds. Second, there are different ways of modeling approximation errors in conditional expectations; in particular, we can represent

<div align="center">TABLE SI</div>

<div align="center">APPROXIMATION ERRORS IN THE CURRENT VARIABLES AND THE EXPECTATION FUNCTIONS IN THE
NEOCLASSICAL STOCHASTIC GROWTH MODEL[a]</div>

| Norm | $\gamma = \frac{1}{10}$ | | | $\gamma = 1$ | | | $\gamma = 10$ | | |
	δ_{c_t}	$\delta_{k_{t+1}}$	δ_{E_t}	δ_{c_t}	$\delta_{k_{t+1}}$	δ_{E_t}	δ_{c_t}	$\delta_{k_{t+1}}$	δ_{E_t}
PER1									
L_1	−5.41	−4.12	−4.60	−4.77	−4.12	−4.65	−4.10	−3.75	−5.14
L_∞	−4.26	−3.03	−3.55	−3.96	−3.04	−3.75	−3.52	−2.62	−4.00
PER2									
L_1	−6.61	−5.80	−6.28	−6.35	−5.69	−6.18	−5.31	−4.75	−6.18
L_∞	−5.44	−4.42	−5.19	−5.11	−4.43	−4.97	−4.06	−3.65	−4.21

[a]Notes: PER1 and PER2 denote the first- and second-order perturbation solutions; δ_{c_t}, $\delta_{k_{t+1}}$, and δ_{E_t} are t-period absolute value of approximation errors in consumption, capital, and conditional expectation function, respectively; L_1 and L_∞ are, respectively, the average and maximum of absolute values of the corresponding approximation errors across optimality condition and test points (in log10 units) on a stochastic simulation of 10,000 observations; and γ is the coefficient of risk aversion.

errors in Euler equation (22) as

$$u'\bigl(\widehat{c}_t(1+\delta_{c_t})\bigr) = \beta \underbrace{(1+\delta_{E_t})\widehat{E}_t}_{=E_t[u'(c_{t+1})(1-d+\exp(\rho\theta_t+\varepsilon_{t+1})Af'(k_{t+1}))]}, \tag{A.9}$$

where δ_{E_t} is an approximation error in conditional expectation function $E_t[\cdot]$. We can use a new condition (A.9) as a restriction in the least-squares problem (28), instead of (26), by changing the objective function to $\delta_{c_t}^2 + \delta_{k_{t+1}}^2 + \delta_{E_t}^2$.

In Table SI, we show the error bounds obtained from the conditions (25), (A.9) on a stochastic simulation following the same methodology as described in Section 3.3.

The advantage of this representation is that it does not require to approximate future values of the variables and hence, it does not involve additional errors from numerical integration in the construction of lower error bound. A potential shortcoming of this alternative representation is that the error in $E_t[\cdot]$ depends on the marginal utility function, so that $\delta_{c_t}^2$, $\delta_{k_{t+1}}^2$, $\delta_{E_t}^2$ are not expressed in comparable units, and introducing a trade-off between the model's variables and marginal utility in the objective function may lead to accuracy results that are more difficult to interpret. In contrast, our baseline representation (28) contains only approximation errors in the model's variables and is not subject to this shortcoming. To deal with this issue, Kubler and Schmedders (2005) measured the error in the conditional expectation function δ_{E_t} by the average adjustment of the future consumption $\delta_{c_{t+1}}$ to satisfy the Euler equation exactly; this approach can be used in our case as well.

APPENDIX B: NEW KEYNESIAN MODEL

In this section, we implement our error bound analysis for the new Keynesian model. In Appendix B.1, we present the new Keynesian model considered; in Appendix B.2, we derive the first-order conditions (FOCs) of the studied model; in Appendix B.3, we define a lower error bound; in Appendix B.4, we derive a lower error bound by using linearized model's equations; in Appendix B.5, we define residuals in the model's equations; finally, in Appendix B.6, we describe the details of our numerical analysis and report the constructed lower error bounds.

B.1. *The Model*

The economy is populated by labor packers, households, final-good firms, intermediate-good firms, monetary authority, and government; see Galí (2008, Chapter 6) for a detailed description of a new Keynesian model with sticky wages and prices.

Labor Packers

Labor inputs of heterogeneous households are packed by labor packers to be sold to firms. A labor packer buys $N_t(l)$ units of labor of a household $l \in [0, 1]$ at price $W_t(l)$ and sells $N_t(l)$ units of labor at price W_t in a perfectly competitive market. The profit-maximization problem is

$$\max_{N_t(l)} W_t N_t - \int_0^1 W_t(l) N_t(l) \, dl \tag{B.1}$$

$$\text{s.t. } N_t = \left(\int_0^1 N_t(l)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} \, di \right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}}, \tag{B.2}$$

where (B.2) is a Dixit–Stiglitz aggregator function with $\varepsilon_w \geq 1$. Problem (B.1), (B.2) implies the demand for labor of type l:

$$N_t(l) = N_t \left(\frac{W_t(l)}{W_t} \right)^{-\varepsilon_w}. \tag{B.3}$$

Households

There is a continuum of monopolistically competitive households who supply differentiated labor input to a labor packer and are indexed by $l \in [0, 1]$. Markets are complete: the households can trade state-contingent claims to insure themselves against aggregate uncertainty. As a result, in equilibrium, the households will be identical in all their choices, except of wages and hours worked (the household's index l will be suppressed elsewhere except of nominal wage $W_t(l)$ and labor $N_t(l)$).

The household of type l maximizes expected discounted lifetime-time utility subject to the capital accumulation equation, (B.5), and the period budget constraint, (B.6),

$$\max_{\{C_t, B_{t+1}, K_{t+1}, u_t, Q_{t+1}\}_{t=0,\dots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln(C_t - bC_{t-1}) - \psi \frac{N_t(l)^{1+\eta} - 1}{1 + \eta} \right] \tag{B.4}$$

$$\text{s.t. } K_{t+1} = Z_t \left(1 - \frac{\tau}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right) I_t + (1 - d) K_t, \tag{B.5}$$

$$C_t + I_t + \frac{B_{t+1}}{P_t} + T_t + q_{t+1,t} Q_{t+1} + \left(\chi_1 (u_t - 1) + \frac{\chi_2}{2} (u_t - 1)^2 \right) \frac{K_t}{Z_t}$$
$$= \frac{W_t(l)}{P_t} N_t(l) + R_t u_t K_t + (1 + i_{t-1}) \frac{B_t}{P_t} + Q_t + \frac{D_t}{P_t}, \tag{B.6}$$

$$\ln Z_t = \rho_z \ln Z_{t-1} + \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0, \sigma_z^2), \tag{B.7}$$

where E_t is the expectation conditional on the information of period t, and (B.7) is a process for investment shock Z_t to the efficiency of transforming investment into capital.

Here, C_t, $N_t(l)$, I_t, K_{t+1}, B_{t+1}, and Q_{t+1} are consumption, labor, investment, capital holdings, nominal-bond holdings, and a vector of state-contingent claims, respectively; P_t, $W_t(l)$, R_t, i_{t-1}, and $q_{t+1,t}$ are, respectively, the commodity price, nominal wage, real return on capital, (net) nominal interest rate, and a price vector of state-contingent claims (each of its elements is a price of a claim that pays one unit of good in a particular aggregate state of nature, x_t, in the subsequent period $t + 1$); T_t is lump-sum taxes; D_t is the profit (dividends) of intermediate-good firms; $\beta \in (0, 1)$ is the discount factor; $\psi > 0$ is the utility-function parameter; $\chi_1 \geq 0$ and $\chi_2 \geq 0$ are the parameters in the cost-of-utilization function which is quadratic in utilization relative to its normalized steady-state value, that is equal to 1; $\tau \geq 0$ is the parameter that governs the size of the adjustment cost of capital; ρ_z and σ_z are the autocorrelation coefficient and the standard deviation of disturbances, respectively.

Wages are subject to Calvo's (1983) pricing frictions. Each period, a fraction $1 - \phi_w$ of the households sets wages optimally, $W_t(l)$ for $l \in [0, 1]$, and the fraction ϕ_w is not allowed to change the price. When the household cannot reoptimize its posted nominal wage, it will index to lagged inflation at $\zeta_w \in (0, 1)$. Let $\Pi_{t,t+s-1} \equiv \frac{P_{t+s-1}}{P_t}$ be a cumulative gross price inflation rate between periods $t - 1$ and $t + s - 1$. A non-reoptimizing household sets a $t + s$-period nominal wage rate at

$$W_{t+s}(l) = \Pi_{t,t+s-1}^{\zeta_w} W_t(l),$$

and hence, real wage at

$$w_{t+s}(l) = w_t(l) \Pi_{t,t+s}^{-1} \Pi_{t-1,t+s-1}^{\zeta_w}, \tag{B.8}$$

where $w_{t+s}(l)$ is real wage of the household of type l in period $t + s$. Note that (B.3) and (B.8) imply

$$N_{t+s}(l) = N_{t+s} \left(\frac{w_t(l) \Pi_{t,t+s}^{-1} \Pi_{t-1,t+s-1}^{\zeta_w}}{w_{t+s}} \right)^{-\varepsilon_w}, \tag{B.9}$$

where w_{t+s} is real wage of packed labor. A reoptimizing household $l \in [0, 1]$ maximizes the current discounted lifetime utility over the time period when $w_t(l)$ remains effective, subject to the demand for labor (B.9) and budget constraint (B.6),

$$\max_{\{w_t(l)\}_{t=0,\ldots,\infty}} E_t \sum_{s=0}^{\infty} \beta^s \phi_w^s \left[\cdots - \psi \frac{N_{t+s}(l)^{1+\eta} - 1}{1 + \eta} \right]$$

s.t. (B.6), (B.9).

Final-Good Firms

Perfectly competitive final-good firms produce final goods using intermediate goods. A final-good firm buys $Y_t(i)$ of an intermediate good $i \in [0, 1]$ at price $P_t(i)$ and sells Y_t of the final good at price P_t in a perfectly competitive market. The profit-maximization problem is

$$\max_{Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) \, di \tag{B.10}$$

$$\text{s.t. } Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} \, di \right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}}, \tag{B.11}$$

where (B.11) is a Dixit–Stiglitz aggregator function with $\varepsilon_p \geq 1$. The problem (B.10), (B.11) implies the demand for an intermediate good of type i:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} Y_t. \tag{B.12}$$

Intermediate-Good Firms

Monopolistic intermediate-good firms produce intermediate goods using capital and labor and are subject to sticky prices. A firm i produces the intermediate good i. To choose capital and labor in each period t, the firm i minimizes the nominal total cost, TC, subject to the constraint that its output is sufficient to meet demand:

$$\min_{N_t(i), K_t^{\#}(i)} \mathrm{TC}\big(Y_t(i)\big) = W_t N_t(i) + R_t^n K_t^{\#}(i) \tag{B.13}$$

$$\text{s.t. } A_t K_t^{\#}(i)^{\alpha} N_t(i)^{1-\alpha} \geq Y_t \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p}, \tag{B.14}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t}, \quad \varepsilon_{a,t+1} \sim \mathcal{N}\big(0, \sigma_a^2\big), \tag{B.15}$$

where (B.15) is a process for the productivity level, A_t; $N_t(i)$ is the labor input; $K_t^{\#}(i) \equiv u_t K_t$ is capital; A_t is the productivity level; R_t^n is the nominal rental rate; ρ_a is the auto-correlation coefficient; and σ_a is the standard deviation of the disturbance.

The firm discounts profits s periods into the future by $\widetilde{M}_{t+s} \phi_p^s$, where $\widetilde{M}_{t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ is a stochastic discount factor with λ_t being the marginal value of an extra unit of income (it is equal to the Lagrange multiplier on the household's budget constraint (B.6)). The firms are subject to Calvo-type price setting, namely, a fraction $1 - \phi_p$ of the firms sets prices optimally, $P_t(i)$ for $i \in [0, 1]$, and the fraction ϕ_p is not allowed to change the price. A non-reoptimizing firm indexes its price to lagged inflation at $\zeta_p \in [0, 1]$. The price charged in period $t + s$ if it is still not revised since period t is

$$P_{t+s}(i) = \Pi_{t-1, t+s-1}^{\zeta_p} P_t(i). \tag{B.16}$$

A reoptimizing firm $i \in [0, 1]$ maximizes the current expected value of profit over the time period when $P_t(i)$ remains effective,

$$\max_{P_t(i)} \sum_{s=0}^{\infty} \beta^s \phi_p^s E_t \left\{ \frac{\lambda_{t+s}}{\lambda_t} \left[\frac{\Pi_{t-1, t+s-1}^{\zeta_p} P_t(i)}{P_{t+s}} Y_{t+s}(i) - \mathrm{mc}_{t+s} Y_{t+s}(i) \right] \right\} \tag{B.17}$$

$$\text{s.t. } Y_{t+s}(i) = \left(\frac{\Pi_{t-1, t+s-1}^{\zeta_p} P_t(i)}{P_{t+s}}\right)^{-\varepsilon_p} Y_{t+s}, \tag{B.18}$$

where (B.18) follows from (B.12) and (B.16); P_{t+s} is the price of the final good; mc_{t+s} is the real marginal cost of output at time $t + s$ (which is identical across the firms), that is, $\mathrm{mc}_{t+s} P_{t+s} \equiv \mathrm{MC}_{t+s}$.

Government

Government finances a stochastic stream of public consumption by levying lump-sum taxes and by issuing nominal debt. The government budget constraint is

$$T_t + \frac{B_{t+1}}{P_t} = \omega_t^g Y_t + (1 + i_{t-1}) \frac{B_t}{P_t}, \qquad (B.19)$$

where $\omega_t^g Y_t = G_t$ is government spending, and ω_t^g is a government-spending shock,

$$\omega_t^g = (1 - \rho_g) \omega^g + \rho_g \omega_{t-1}^g + \varepsilon_{g,t}, \quad \varepsilon_{g,t} \sim \mathcal{N}(0, \sigma_g^2), \qquad (B.20)$$

where ρ_g is the autocorrelation coefficient, and σ_g is the standard deviation of disturbance.

Monetary Authority

The monetary authority follows a Taylor rule:

$$i_t = (1 - \rho_i) i + \rho_i i_{t-1} + (1 - \rho_i) \big[\phi_\pi (\pi_t - \pi^*) + \phi_y (\ln Y_t - \ln Y_{t-1}) \big] + \varepsilon_{i,t}, \qquad (B.21)$$

where $i = 1/\beta - 1$ is the steady-state interest rate; $\phi_\pi \geq 0$ and $\phi_y \geq 0$ are the parameters; $\pi_t \equiv \frac{P_t}{P_{t-1}} - 1$ is net inflation; $\varepsilon_{i,t}$ is a monetary shock, $\varepsilon_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$.

B.2. *Deriving FOCs*

We derive the FOCs of the studied new Keynesian model below.

Labor Packers

The FOC of the labor packer's problem (B.1), (B.2) with respect to $N_t(l)$ yields the demand for the *l*th type of labor, given by (B.3),

$$N_t(l) = N_t \left(\frac{W_t(l)}{W_t} \right)^{-\varepsilon_w}. \qquad (B.22)$$

A zero-profit condition of a labor packer implies $W_t N_t = \int_0^1 W_t(l) N_t(l) \, dl$. Substituting (B.22) into the latter equation gives

$$W_t = \left(\int_0^1 W_t(l)^{1-\varepsilon_w} \, di \right)^{\frac{1}{1-\varepsilon_w}}. \qquad (B.23)$$

Households

The FOCs of the household's problem (B.4)–(B.7) with respect to C_t, B_{t+1}, K_{t+1}, I_t, u_t, Q_{t+1}, respectively, are

$$\lambda_t = \frac{1}{C_t - bC_{t-1}} - \beta b E_t \left[\frac{1}{C_{t+1} - bC_t} \right], \qquad (B.24)$$

$$\lambda_t = \beta E_t \left[\lambda_{t+1} (1 + i_t) \frac{P_t}{P_{t+1}} \right], \qquad (B.25)$$

$$\mu_t = \beta E_t \left[\lambda_{t+1} \left(R_{t+1} u_{t+1} - \frac{1}{Z_{t+1}} \left(\chi_1(u_{t+1} - 1) + \frac{\chi_2}{2}(u_{t+1} - 1)^2 \right) \right) \right.$$
$$\left. + \mu_{t+1}(1 - d) \right], \tag{B.26}$$

$$\lambda_t = \mu_t Z_t \left[1 - \frac{\tau}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 - \tau \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right]$$
$$+ \beta E_t \mu_{t+1} Z_{t+1} \tau \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2, \tag{B.27}$$

$$R_t = \frac{1}{Z_t} \left[\chi_1 + \chi_2(u_t - 1) \right], \tag{B.28}$$

$$\lambda_t q_{t+1,t}(x) = \beta \lambda_{t+1} \Pr\{x_{t+1} = x | x_t = x'\}, \tag{B.29}$$

where λ_t and μ_t are the Lagrange multipliers associated with (B.6) and (B.5); $x_t = \{Z_t, A_t, \omega_t^g, \varepsilon_{i,t}\}$ is the economy's aggregate state; $q_{t+1,t}(x)$ is the price of a state-contingent claim, bought in period t, that pays one unit of consumption in case aggregate state x in period $t + 1$.

As for wage setting, the FOC with respect to real wage, chosen by a reoptimizing household, is

$$\varepsilon_w w_t(l)^{-\varepsilon_w(1+\eta)-1} E_t \sum_{s=0}^{\infty} \beta^s \phi_w^s \psi \Pi_{t,t+s}^{\varepsilon_w(1+\eta)} \Pi_{t-1,t+s-1}^{\zeta_w \varepsilon_w(1+\eta)} w_{t+s}^{\varepsilon_w(1+\eta)} N_{t+s}^{1+\eta}$$
$$+ (1 - \varepsilon_w) w_t(l)^{-\varepsilon_w} \sum_{s=0}^{\infty} \beta^s \phi_w^s \lambda_{t+s} \Pi_{t,t+s}^{\varepsilon_w-1} \Pi_{t-1,t+s-1}^{\zeta_w(1-\varepsilon_w)} w_{t+s}^{\varepsilon_w} N_{t+s} = 0.$$

Note that the household-specific index l enters just $w_t(l)$, so that all reoptimizers choose the same wage, that is, $w_t(l) \equiv w_t^\#$, given by

$$\left(w_t^\# \right)^{1+\varepsilon_w \eta} = \frac{\varepsilon_w}{1 - \varepsilon_w} \frac{E_t \sum_{s=0}^{\infty} \beta^s \phi_w^s \psi \Pi_{t,t+s}^{\varepsilon_w(1+\eta)} \Pi_{t-1,t+s-1}^{\zeta_w \varepsilon_w(1+\eta)} w_{t+s}^{\varepsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{\sum_{s=0}^{\infty} \beta^s \phi_w^s \lambda_{t+s} \Pi_{t,t+s}^{\varepsilon_w-1} \Pi_{t-1,t+s-1}^{\zeta_w(1-\varepsilon_w)} w_{t+s}^{\varepsilon_w} N_{t+s}}.$$

We can rewrite it recursively as

$$\left(w_t^\# \right)^{1+\varepsilon_w \eta} = \frac{\varepsilon_w}{1 - \varepsilon_w} \frac{F_{1,t}}{F_{2,t}}, \tag{B.30}$$

where

$$F_{1,t} = \psi w_t^{\varepsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w (1 + \pi_t)^{-\zeta_w \varepsilon_w(1+\eta)} \beta E_t \left[(1 + \pi_{t+1})^{\varepsilon_w(1+\eta)} F_{1,t+1} \right], \tag{B.31}$$

$$F_{2,t} = \lambda_t w_t^{\varepsilon_w} N_t + \phi_w (1 + \pi_t)^{\zeta_w(1-\varepsilon_w)} \beta E_t \left[(1 + \pi_{t+1})^{\varepsilon_w-1} F_{2,t+1} \right], \tag{B.32}$$

where $1 + \pi_t \equiv \Pi_{t-1,t}$.

A power $1 + \varepsilon_w \eta$ in equation (B.30) could take very large values for empirically plausible parameterizations of the model (e.g., we calibrate $\eta = 1$ and $\varepsilon_w = 10$), which may lead to numerical problems. To deal with this issue, first, we divide both sides of (B.30) by $(w_t^\#)^{\varepsilon_w(1+\eta)}$,

$$\left(w_t^\#\right)^{1-\varepsilon_w} = \frac{\varepsilon_w}{1 - \varepsilon_w} \frac{f_{1,t}}{F_{2,t}}, \tag{B.33}$$

where $f_{1,t} \equiv \frac{F_{1,t}}{(w_t^\#)^{\varepsilon_w(1+\eta)}}$. Then, equation (B.31) becomes

$$f_{1,t} = \psi \left(\frac{w_t}{w_t^\#}\right)^{\varepsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w(1 + \pi_t)^{-\zeta_w \varepsilon_w(1+\eta)}$$
$$\times \beta E_t \left[(1 + \pi_{t+1})^{\varepsilon_w(1+\eta)} f_{1,t+1} \left(\frac{w_{t+1}^\#}{w_t}\right)^{\varepsilon_w(1+\eta)}\right]. \tag{B.34}$$

Second, we multiply both sides of (B.33) by $(w_t^\#)^{\varepsilon_w}$,

$$w_t^\# = \frac{\varepsilon_w}{1 - \varepsilon_w} \frac{f_{1,t}}{f_{2,t}}, \tag{B.35}$$

where $f_{2,t} \equiv \frac{F_{2,t}}{(w_t^\#)^{\varepsilon_w}}$. Then, equation (B.32) becomes

$$f_{2,t} = \lambda_t \left(\frac{w_t}{w_t^\#}\right)^{\varepsilon_w} N_t + \phi_w(1 + \pi_t)^{\zeta_w(1-\varepsilon_w)}$$
$$\times \beta E_t \left[(1 + \pi_{t+1})^{\varepsilon_w-1} f_{2,t+1} \left(\frac{w_{t+1}^\#}{w_t^\#}\right)^{\varepsilon_w}\right]. \tag{B.36}$$

Final-Good Producers

The FOC of the final-good producer's problem (B.10), (B.11) with respect to $Y_t(i)$ yields the demand for the ith intermediate good

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p}. \tag{B.37}$$

A zero-profit condition of a final-good producer implies $P_t Y_t = \int_0^1 P_t(i) Y_t(i) \, di$. Substituting (B.22) into the latter equation yields

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon_p} \, di\right)^{\frac{1}{1-\varepsilon_p}}. \tag{B.38}$$

Intermediate-Good Producers

The FOCs of the cost-minimization problem (B.13)–(B.15) with respect to $N_t(i)$ and $K_t^\#(i)$ are

$$R_t^n = \Theta_t(i)\alpha A_t K_t^\#(i)^{\alpha-1} N_t(i)^{1-\alpha}, \tag{B.39}$$
$$W_t = \Theta_t(i)(1-\alpha) A_t K_t^\#(i)^{\alpha} N_t(i)^{-\alpha}, \tag{B.40}$$

where $\Theta_t(i)$ is the Lagrange multiplier associated with (B.14). Combining (B.39) and (B.40) yields

$$\frac{W_t}{R_t^n} = \frac{1-\alpha}{\alpha} \frac{K_t^\#(i)}{N_t(i)}.$$

This condition implies that all the firms will rent capital and hire labor in the same proportion. In real terms, the latter condition becomes

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha}\left(\frac{K_t^\#}{N_t}\right),$$

where $R_t \equiv \frac{R_t^n}{P_t}$. The derivative of the total cost in (B.13) is the nominal marginal cost, $\mathrm{MC}_t(i)$,

$$\mathrm{MC}_t(i) \equiv \frac{d\,\mathrm{TC}(Y_t(i))}{dY_t(i)} = \Theta_t(i). \tag{B.41}$$

The real marginal cost is the same for all firms,

$$\mathrm{mc}_t(i) = \frac{\Theta_t(i)}{P_t} = \mathrm{mc}_t. \tag{B.42}$$

This is because all the firms face the same factor prices, and they rent capital and hire labor in the same proportion. Conditions (B.39) and (B.40), together with (B.42), can be rewritten, respectively, as

$$R_t = \mathrm{mc}_t \alpha A_t \left(\frac{K_t^\#}{N_t}\right)^{\alpha-1}, \tag{B.43}$$

$$w_t = \mathrm{mc}_t(1-\alpha)A_t\left(\frac{K_t^\#}{N_t}\right)^{\alpha}. \tag{B.44}$$

The period-t real-flow profit of the ith firm is

$$\frac{D_t(i)}{P_t} = \frac{P_t(i)}{P_t}Y_t(i) - \mathrm{mc}_t(1-\alpha)A_tK_t^\#(i)^\alpha N_t(i)^{1-\alpha} - \mathrm{mc}_t\alpha A_tK_t^\#(i)^\alpha N_t(i)^{1-\alpha}$$

$$= \frac{P_t(i)}{P_t}Y_t(i) - \mathrm{mc}_t Y_t(i).$$

This result was used to derive (B.17). Substituting constraint (B.18) into the objective function yields

$$\max_{P_t(i)} \sum_{s=0}^{\infty} \beta^s \phi_p^s E_t \left\{ \frac{\lambda_{t+s}}{\lambda_t}\left(\frac{\Pi_{t-1,t+s-1}^{\zeta_p}P_t(i)}{P_{t+s}}\right)^{-\varepsilon_p} Y_{t+s}\left[\frac{\Pi_{t-1,t+s-1}^{\zeta_p}P_t(i)}{P_{t+s}} - \mathrm{mc}_{t+s}\right]\right\}.$$

This problem can be rewritten as

$$\max_{P_t(i)} \sum_{s=0}^{\infty} \beta^s \phi_p^s E_t \frac{\lambda_{t+s}}{\lambda_t}\left\{\Pi_{t-1,t+s-1}^{\zeta_p(1-\varepsilon_p)}P_t(i)^{1-\varepsilon_p}P_{t+s}^{\varepsilon_p-1}Y_{t+s} - \Pi_{t-1,t+s-1}^{-\zeta_p\varepsilon_p}P_t(i)^{-\varepsilon_p}\mathrm{mc}_{t+s}P_{t+s}^{\varepsilon_p}Y_{t+s}\right\}.$$

The FOC of the reoptimizing intermediate-good firm with respect to $P_t(i)$ is

$$
(1 - \varepsilon_p) P_t(i)^{-\varepsilon_p} E_t \sum_{s=0}^{\infty} \beta^s \phi_p^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{\zeta_p(1-\varepsilon_p)} P_{t+s}^{\varepsilon_p-1} Y_{t+s}
$$
$$
+ \varepsilon_p P_t(i)^{-\varepsilon_p-1} E_t \sum_{s=0}^{\infty} \beta^s \phi_p^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{-\zeta_p \varepsilon_p} \mathrm{mc}_{t+s} P_{t+s}^{\varepsilon_p} Y_{t+s} = 0.
\tag{B.45}
$$

Expressing $P_t(i)$, we get

$$
P_t(i) = \frac{\varepsilon_p}{1 - \varepsilon_p} \frac{E_t \sum_{s=0}^{\infty} \beta^s \phi_p^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{-\zeta_p \varepsilon_p} \mathrm{mc}_{t+s} P_{t+s}^{\varepsilon_p} Y_{t+s}}{E_t \sum_{s=0}^{\infty} \beta^s \phi_p^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{\zeta_p(1-\varepsilon_p)} P_{t+s}^{\varepsilon_p-1} Y_{t+s}}.
\tag{B.46}
$$

Since nothing on the right side depends on the firm-specific index i, we have that all reoptimizing firms set the same price at t, that is, $P_t(i) = P_t^{\#}$,

$$
P_t^{\#} = \frac{\varepsilon_p}{1 - \varepsilon_p} \frac{X_{1t}}{X_{2t}},
\tag{B.47}
$$

where

$$
X_{1t} \equiv E_t \sum_{s=0}^{\infty} \beta^s \phi_p^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{-\zeta_p \varepsilon_p} \mathrm{mc}_{t+s} P_{t+s}^{\varepsilon_p} Y_{t+s},
\tag{B.48}
$$

$$
X_{2t} \equiv E_t \sum_{s=0}^{\infty} \beta^s \phi_p^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{\zeta_p(1-\varepsilon_p)} P_{t+s}^{\varepsilon_p-1} Y_{t+s}.
\tag{B.49}
$$

For X_{1t}, a recursive formula is

$$
X_{1t} = \lambda_t \mathrm{mc}_{t+s} P_t^{\varepsilon_p} Y_t + \beta \phi_p (1 + \pi_t)^{-\zeta_p \varepsilon_p} E_t X_{1t+1},
\tag{B.50}
$$

while for X_{2t}, the corresponding recursive formula is

$$
X_{2t} = \lambda_t P_t^{\varepsilon_p-1} Y_t + \beta \phi_p (1 + \pi_t)^{\zeta_p(1-\varepsilon_p)} E_t X_{2t+1}.
\tag{B.51}
$$

Let us divide (B.50) and (B.51) by $P_t^{\varepsilon_p}$ and $P_t^{\varepsilon_p-1}$, respectively, so that they become

$$
x_{1t} = \lambda_t \mathrm{mc}_{t+s} Y_t + \beta \phi_p (1 + \pi_t)^{-\zeta_p \varepsilon_p} E_t \big[(1 + \pi_{t+1})^{\varepsilon_p} x_{1t+1} \big],
\tag{B.52}
$$

$$
x_{2t} = \lambda_t Y_t + \beta \phi_p (1 + \pi_t)^{\zeta_p(1-\varepsilon_p)} E_t \big[(1 + \pi_{t+1})^{\varepsilon_p-1} x_{2t+1} \big],
\tag{B.53}
$$

where $x_{1t} \equiv \frac{X_{1t}}{P_t^{\varepsilon_p}}$ and $x_{2t} \equiv \frac{X_{2t}}{P_t^{\varepsilon_p-1}}$. In terms of the new variables x_{1t} and x_{2t}, condition (B.47) becomes

$$
1 + \pi_t^{\#} = \frac{\varepsilon_p}{1 - \varepsilon_p} (1 + \pi_t) \frac{x_{1t}}{x_{2t}},
\tag{B.54}
$$

with $\pi_t^{\#} \equiv P_t^{\#} / P_{t-1} - 1$.

Aggregate Price Relationship

The condition (B.31) can be rewritten as

$$
\begin{aligned}
P_t &= \left(\int_0^1 P_t(i)^{1-\varepsilon_p}\, di \right)^{\frac{1}{1-\varepsilon_p}} \\
&= \left[\int_{\text{reopt.}} P_t(i)^{1-\varepsilon_p}\, di + \int_{\text{non-reopt.}} P_t(i)^{1-\varepsilon_p}\, di \right]^{\frac{1}{1-\varepsilon_p}},
\end{aligned}
\tag{B.55}
$$

where "reopt." and "non-reopt." denote, respectively, the firms that reoptimize and do not reoptimize their prices at t.

Note that $\int_{\text{non-reopt.}} P_t(i)^{1-\varepsilon_p}\, di = \int_0^1 (1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)} P(j)^{1-\varepsilon_p} \omega_{t-1,t}(j)\, dj$, where $\omega_{t-1,t}(j)$ is the measure of non-reoptimizers at t that had the price $P(j)$ at $t-1$. Furthermore, $\omega_{t-1,t}(j) = \phi_p \omega_{t-1}(j)$, where $\omega_{t-1}(j)$ is the measure of firms with the price $P(j)$ in $t-1$, which implies

$$
\begin{aligned}
\int_{\text{non-reopt.}} P_t(i)^{1-\varepsilon_p}\, di &= \int_0^1 \phi_p (1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)} P(j)^{1-\varepsilon_p} \omega_{t-1}(j)\, dj \\
&= \phi_p (1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)} P_{t-1}^{1-\varepsilon_p}.
\end{aligned}
\tag{B.56}
$$

Substituting (B.56) into (B.55) and using the fact that all reoptimizers set $P_t^{\#}$, we get

$$
P_t^{1-\varepsilon_p} = (1-\phi_p)\left(P_t^{\#}\right)^{1-\varepsilon_p} + \phi_p (1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)} P_{t-1}^{1-\varepsilon_p}.
\tag{B.57}
$$

We divide both sides of (B.57) by $P_{t-1}^{1-\varepsilon_p}$,

$$
(1+\pi_t)^{1-\varepsilon_p} = (1-\phi_p)\left(1+\pi_t^{\#}\right)^{1-\varepsilon_p} + \phi_p (1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)}.
\tag{B.58}
$$

Aggregate Wage Relationship

Similarly to equation (B.57), aggregate wage index can be written as

$$
W_t^{1-\varepsilon_w} = (1-\phi_w)\left(W_t^{\#}\right)^{1-\varepsilon_w} + \phi_w (1+\pi_{t-1})^{\zeta_w(1-\varepsilon_w)} W_{t-1}^{1-\varepsilon_w},
$$

where the second term on the right side corresponds to aggregate wage, set by non-reoptimizing households. Dividing both sides by $P_t^{1-\varepsilon_w}$, we get

$$
w_t^{1-\varepsilon_w} = (1-\phi_w)\left(w_t^{\#}\right)^{1-\varepsilon_w} + \phi_w (1+\pi_{t-1})^{\zeta_w(1-\varepsilon_w)} (1+\pi_t)^{\varepsilon_w-1} w_{t-1}^{1-\varepsilon_w}.
\tag{B.59}
$$

Aggregate Output

Since all the firms rent capital and hire labor in the same proportion, we get

$$
Y_t(i) = A_t K_t^{\#}(i)^{\alpha} N_t(i)^{1-\alpha} = A_t \left(\frac{K_t^{\#}}{N_t} \right)^{\alpha} N_t(i).
$$

Let us define aggregate output

$$
\begin{aligned}
\overline{Y}_t &\equiv \int_0^1 Y_t(i)\, di = \int_0^1 A_t K_t^\#(i)^\alpha N_t(i)^{1-\alpha}\, di \\
&= A_t\left(\frac{K_t^\#}{N_t}\right)^\alpha \int_0^1 N_t(i)\, di = A_t K_t^\# N_t^{1-\alpha}.
\end{aligned}
\tag{B.60}
$$

We substitute demand for $Y_t(i)$ from (B.12) into (B.60) to get

$$
\overline{Y}_t = \int_0^1 Y_t \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} di = Y_t P_t^{\varepsilon_p} \int_0^1 P_t(i)^{-\varepsilon_p}\, di.
\tag{B.61}
$$

Let us introduce a new variable $\overline{P}_t$,

$$
(\overline{P}_t)^{-\varepsilon_p} \equiv \int_0^1 P_t(i)^{-\varepsilon_p}\, di.
\tag{B.62}
$$

Substituting (B.60) and (B.62) into (B.61) gives us

$$
Y_t \equiv \overline{Y}_t \left(\frac{\overline{P}_t}{P_t}\right)^{\varepsilon_p} = \frac{A_t\left(K_t^\#\right)^\alpha N_t^{1-\alpha}}{\Delta_t^p},
\tag{B.63}
$$

where Δ_t^p is a measure of price dispersion across firms, defined by

$$
\Delta_t^p \equiv \left(\frac{\overline{P}_t}{P_t}\right)^{-\varepsilon_p}.
\tag{B.64}
$$

Note that if $P_t(i) = P_t(i')$ for all i and $i' \in [0, 1]$, then $\Delta_t^p = 1$, that is, there is no price dispersion across firms.

Law of Motion for Price Dispersion Δ_t^p

By analogy with (B.57), the variable $\overline{P}_t$, defined in (B.62), satisfies

$$
\overline{P}_t^{-\varepsilon_p} = (1 - \phi_p)\left(P_t^\#\right)^{-\varepsilon_p} + \phi_p(1 + \pi_{t-1})^{-\zeta_p \varepsilon_p} \overline{P}_{t-1}^{-\varepsilon_p}.
\tag{B.65}
$$

By using (B.65) in (B.64), we get

$$
\Delta_t^p = (1 - \phi_p)\left(\frac{P_t^\#}{P_t}\right)^{-\varepsilon_p} + \phi_p(1 + \pi_{t-1})^{-\zeta_p \varepsilon_p} \left(\frac{\overline{P}_{t-1}}{P_t}\right)^{-\varepsilon_p}.
$$

This implies

$$
\Delta_t^p = (1 - \phi_p)\left(\frac{P_t^\#}{P_t}\right)^{-\varepsilon_p}\left(\frac{P_{t-1}}{P_{t-1}}\right)^{-\varepsilon_p} + \phi_p(1 + \pi_{t-1})^{-\zeta_p \varepsilon_p}\left(\frac{\overline{P}_{t-1}}{P_t}\right)^{-\varepsilon_p}\left(\frac{P_{t-1}}{P_{t-1}}\right)^{-\varepsilon_p}.
$$

Simplifying the latter expression, we obtain the law of motion for Δ_t^p,

$$
\Delta_t^p = (1 + \pi_t)^{\varepsilon_p} \cdot \left[(1 - \phi_p)\left(1 + \pi_t^\#\right)^{-\varepsilon_p} + \phi_p(1 + \pi_{t-1})^{-\zeta_p \varepsilon_p}\left(\frac{\overline{P}_t}{\overline{P}_t}\right)^{-\varepsilon_p} \Delta_{t-1}^p\right].
\tag{B.66}
$$

Aggregate Resource Constraint

Summing up the household's budget constraint (B.6) across all agents eliminates the state-contingent claims as they are in a zero net supply. Combining the resulting household's budget constraint (B.6) with the government budget constraint (B.19), we have the aggregate resource constraint

$$C_t + I_t + \omega_t^g Y_t = \frac{W_t N_t}{P_t} + R_t u_t K_t - \left(\chi_1 (u_t - 1) + \frac{\chi_2}{2} (u_t - 1)^2 \right) \frac{K_t}{Z_t} + \frac{D_t}{P_t}, \qquad \text{(B.67)}$$

where $W_t N_t = \int_0^1 W_t(l) N_t(l)\, dl$. Note that the ith intermediate-good firm's profit at t is $D_t(i) \equiv P_t(i) Y_t(i) - W_t N_t(i) - R_t^n K_t^\#$. Consequently,

$$\begin{aligned}
D_t &= \int_0^1 D_t(i)\, di = \int_0^1 P_t(i) Y_t(i)\, di - W_t \int_0^1 N_t(i)\, di + R_t^n \int_0^1 K_t^\#(i)\, di \\
&= P_t Y_t - W_t N_t - R_t^n K_t^\#,
\end{aligned}$$

where $P_t Y_t = \int_0^1 P_t(i) Y_t(i)\, di$ follows by a zero-profit condition of the final-good firms. Hence, (B.67) can be rewritten as

$$C_t + I_t + G_t + \left(\chi_1 (u_t - 1) + \frac{\chi_2}{2} (u_t - 1)^2 \right) \frac{K_t}{Z_t} = Y_t. \qquad \text{(B.68)}$$

Full Set of Optimality Conditions

Below, we summarize the full set of the equilibrium conditions in the studied new Keynesian model (B.1)–(B.21):

$$\lambda_t = \frac{1}{C_t - b C_{t-1}} - \beta b E_t \frac{1}{C_{t+1} - b C_t}, \qquad \text{(B.69)}$$

$$R_t = \frac{1}{Z_t} \left[\chi_1 + \chi_2 (u_t - 1) \right], \qquad \text{(B.70)}$$

$$\lambda_t = \beta E_t \lambda_{t+1} (1 + i_t)(1 + \pi_{t+1})^{-1}, \qquad \text{(B.71)}$$

$$\begin{aligned}
\lambda_t &= \mu_t Z_t \left[1 - \frac{\tau}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 - \tau \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right] \\
&\quad + \beta E_t \mu_{t+1} Z_{t+1} \tau \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2,
\end{aligned} \qquad \text{(B.72)}$$

$$\begin{aligned}
\mu_t &= \beta E_t \left[\lambda_{t+1} \left(R_{t+1} u_{t+1} - \frac{1}{Z_{t+1}} \left[\chi_1 (u_{t+1} - 1) + \frac{\chi_2}{2} (u_{t+1} - 1)^2 \right] \right) \right. \\
&\quad \left. + \mu_{t+1} (1 - d) \right],
\end{aligned} \qquad \text{(B.73)}$$

$$w_t^\# = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\widehat{f}_{1,t}}{\widehat{f}_{2,t}}, \qquad \text{(B.74)}$$

$$\widehat{f}_{1,t} = \psi \left(\frac{w_t}{w_t^\#} \right)^{\varepsilon_w(1+\eta)} N_t^{1+\eta}$$
$$+ \phi_w \beta (1+\pi_t)^{-\zeta_w \varepsilon_w(1+\eta)} E_t \left[(1+\pi_{t+1})^{\varepsilon_w(1+\eta)} \left(\frac{w_{t+1}^\#}{w_t^\#} \right)^{\varepsilon_w(1+\eta)} \widehat{f}_{1,t+1} \right], \tag{B.75}$$

$$\widehat{f}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^\#} \right)^{\varepsilon_w} N_t$$
$$+ \phi_w \beta (1+\pi_t)^{\zeta_w(1-\varepsilon_w)} E_t \left[(1+\pi_{t+1})^{\varepsilon_w-1} \left(\frac{w_{t+1}^\#}{w_t^\#} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \right], \tag{B.76}$$

$$w_t^{1-\varepsilon_w} = (1-\phi_w)\left(w_t^\#\right)^{1-\varepsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\varepsilon_w)}(1+\pi_t)^{\varepsilon_w-1}\phi_w w_{t-1}^{1-\varepsilon_w}, \tag{B.77}$$

$$Y_t = \frac{A_t \left(K_t^\#\right)^\alpha N_t^{1-\alpha}}{\Delta_t^p}, \tag{B.78}$$

$$\Delta_t^p = (1+\pi_t)^{\varepsilon_p}\left[(1-\phi_w)\left(1+\pi_t^\#\right)^{-\varepsilon_p} + (1+\pi_{t-1})^{-\zeta_p \varepsilon_p}\phi_p \Delta_{t-1}^p\right], \tag{B.79}$$

$$(1+\pi_t)^{1-\varepsilon_p} = (1-\phi_p)\left(1+\pi_t^\#\right)^{1-\varepsilon_p} + \phi_p(1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)}, \tag{B.80}$$

$$1+\pi_t^\# = \frac{\varepsilon_p}{\varepsilon_p - 1}(1+\pi_t)\frac{x_{1,t}}{x_{2,t}}, \tag{B.81}$$

$$x_{1,t} = \lambda_t \mathrm{mc}_t Y_t + \phi_p \beta (1+\pi_t)^{-\zeta_p \varepsilon_p} E_t\left[(1+\pi_{t+1})^{\varepsilon_p} x_{1,t+1}\right], \tag{B.82}$$

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta (1+\pi_t)^{\zeta_p(1-\varepsilon_p)} E_t\left[(1+\pi_{t+1})^{\varepsilon_p-1} x_{2,t+1}\right], \tag{B.83}$$

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha} \cdot \frac{K_t^\#}{N_t}, \tag{B.84}$$

$$w_t = \mathrm{mc}_t(1-\alpha)A_t\left(\frac{K_t^\#}{N_t}\right)^\alpha, \tag{B.85}$$

$$i_t = (1-\rho_i)i + \rho_i i_{t-1}$$
$$+ (1-\rho_i)\left[\phi_\pi\left(\pi_t - \pi^*\right) + \phi_y(\ln Y_t - \ln Y_{t-1})\right] + \varepsilon_{i,t}, \tag{B.86}$$

$$Y_t = C_t + I_t + G_t + \left(\chi_1(u_t - 1) + \chi_2(u_t - 1)^2\right)\frac{K_t}{Z_t}, \tag{B.87}$$

$$K_{t+1} = Z_t\left[1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right]I_t + (1-d)K_t, \tag{B.88}$$

where $K_t^\# = u_t K_t$, $G_t = \omega_t^g Y_t$, and exogenous shocks A_t, Z_t, and ω_t^g follow (B.15), (B.7), (B.20), respectively; and $f_{1,t}$, $f_{2,t}$ and $x_{1,t}$, $x_{2,t}$ are supplementary variables introduced for writing the problem in a recursive form; and Δ_t^p is a measure of price dispersion across firms. In total, there are 25 equations in 25 variables:

$$\left\{\lambda_t, C_t, R_t, Z_t, u_t, i_t, \pi_t, I_t, \mu_t, w_t^\#, \widehat{f}_{1,t}, \widehat{f}_{2,t}, w_t, Y_t,\right.$$
$$\left. A_t, N_t, \Delta_t^p, \pi_t^\#, x_{1,t}, x_{2,t}, \mathrm{mc}_t, K_t, K_t^\#, G_t, \omega_t^g\right\}.$$

B.3. *Defining a Lower Error Bound*

Defining Approximation Errors in Variables

The approximation errors in the model's variables are defined by the following twenty equations that correspond to the optimality conditions (B.69)–(B.88), respectively:

$$
\widehat{\lambda}_t(1 + \delta_{\lambda_t}) = \frac{1}{\widehat{C}_t(1 + \delta_{C_t}) - bC_{t-1}}
$$

$$
- E_t\left[\frac{\beta b}{\widehat{C}_{t+1}(1 + \delta_{C_{t+1}}) - b\widehat{C}_t(1 + \delta_{C_t})}\right], \tag{B.89}
$$

$$
\widehat{R}_t(1 + \delta_{R_t}) = \frac{1}{Z_t}\left[\chi_1 + \chi_2\left(\widehat{u}_t(1 + \delta_{u_t}) - 1\right)\right], \tag{B.90}
$$

$$
\widehat{\lambda}_t(1 + \delta_{\lambda_t}) = \beta E_t\left[\frac{\widehat{\lambda}_{t+1}(1 + \delta_{\lambda_{t+1}})\left(1 + \widehat{i}_t(1 + \delta_{i_t})\right)}{1 + \widehat{\pi}_{t+1}(1 + \delta_{\pi_{t+1}})}\right], \tag{B.91}
$$

$$
\widehat{\lambda}_t(1 + \delta_{\lambda_t}) = \widehat{\mu}_t(1 + \delta_{\mu_t})
$$

$$
\times Z_t\left[1 - \frac{\tau}{2}\left(\frac{\widehat{I}_t(1 + \delta_{I_t})}{I_{t-1}} - 1\right)^2\right.
$$

$$
\left. - \tau\left(\frac{\widehat{I}_t(1 + \delta_{I_t})}{I_{t-1}} - 1\right)\frac{\widehat{I}_t(1 + \delta_{I_t})}{I_{t-1}}\right] \tag{B.92}
$$

$$
+ \beta E_t\left[\widehat{\mu}_{t+1}(1 + \delta_{\mu_{t+1}})Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}(1 + \delta_{I_{t+1}})}{I_t(1 + \delta_{I_t})} - 1\right)\right.
$$

$$
\left. \times \left(\frac{\widehat{I}_{t+1}(1 + \delta_{I_{t+1}})}{I_t(1 + \delta_{I_t})}\right)^2\right],
$$

$$
\widehat{\mu}_t(1 + \delta_{\mu_t}) = \beta E_t\left[\widehat{\lambda}_{t+1}(1 + \delta_{\lambda_{t+1}}) \times \left\{\widehat{R}_{t+1}(1 + \delta_{R_{t+1}}) \cdot \widehat{u}_{t+1}(1 + \delta_{u_{t+1}})\right.\right.
$$

$$
- \frac{1}{Z_{t+1}}\left[\chi_1\left(\widehat{u}_{t+1}(1 + \delta_{u_{t+1}}) - 1\right)\right.
$$

$$
\left.\left. + \frac{\chi_2}{2}\left(\widehat{u}_{t+1}(1 + \delta_{u_{t+1}}) - 1\right)^2\right]\right\} \tag{B.93}
$$

$$
\left. + \widehat{\mu}_{t+1}(1 + \delta_{\mu_{t+1}})(1 - d)\right],
$$

$$
\widehat{w}_t^\#(1 + \delta_{w_t^\#}) = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{\widehat{f}_{1,t}(1 + \delta_{f_{1t}})}{\widehat{f}_{2,t}(1 + \delta_{f_{2t}})}, \tag{B.94}
$$

$$
\widehat{f}_{1,t}(1 + \delta_{f_{1t}}) = \psi\left(\frac{\widehat{w}_t(1 + \delta_{w_t})}{\widehat{w}_t^\#(1 + \delta_{w_t^\#})}\right)^{\varepsilon_w(1+\eta)}\left[\widehat{N}_t(1 + \delta_{N_t})\right]^{1+\eta}
$$

$$
+ \phi_w\beta\left(1 + \widehat{\pi}_t(1 + \delta_{\pi_t})\right)^{-\zeta_w\varepsilon_w(1+\eta)} \tag{B.95}
$$

$$\times E_t\Big[\big[1+\widehat{\pi}_{t+1}(1+\delta_{\pi_{t+1}})\big]^{\varepsilon_w(1+\eta)}$$

$$\times\left(\frac{\widehat{w}_{t+1}^{\#}(1+\delta_{w_{t+1}^{\#}})}{\widehat{w}_t^{\#}(1+\delta_{w_t^{\#}})}\right)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1}(1+\delta_{f_{1t+1}})\Big],$$

$$\widehat{f}_{2,t}(1+\delta_{f_{2t}})=\widehat{\lambda}_t\left(\frac{\widehat{w}_t(1+\delta_{w_t})}{\widehat{w}_t^{\#}(1+\delta_{w_t^{\#}})}\right)^{\varepsilon_w}\widehat{N}_t(1+\delta_{N_t})$$

$$+\phi_w\beta\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)^{\zeta_w(1-\varepsilon_w)}$$

$$\text{(B.96)}$$

$$\times E_t\Big[\big(1+\widehat{\pi}_{t+1}(1+\delta_{\pi_{t+1}})\big)^{\varepsilon_w-1}$$

$$\times\left(\frac{\widehat{w}_{t+1}(1+\delta_{w_{t+1}})}{\widehat{w}_{t+1}^{\#}(1+\delta_{w_{t+1}^{\#}})}\right)^{\varepsilon_w}\widehat{f}_{2,t+1}(1+\delta_{f_{2t+1}})\Big],$$

$$\widehat{w}_t^{1-\varepsilon_w}(1+\delta_{w_t})^{1-\varepsilon_w}=(1-\phi_w)\big(\widehat{w}_t^{\#}\big)^{1-\varepsilon_w}(1+\delta_{w_t^{\#}})^{1-\varepsilon_w}$$

$$+(1+\pi_{t-1})^{\zeta_w(1-\varepsilon_w)}\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)^{\varepsilon_w-1}\phi_w w_{t-1}^{1-\varepsilon_w},$$

$$\text{(B.97)}$$

$$\widehat{Y}_t(1+\delta_{Y_t})=A_t\big(\widehat{K}_t^{\#}\big)^{\alpha}(1+\delta_{K_t^{\#}})^{\alpha}\widehat{N}_t^{1-\alpha}(1+\delta_{N_t})^{1-\alpha}\big[\widehat{\Delta}_t^p\big]^{-1}(1+\delta_{\Delta_t^p})^{-1},\quad\text{(B.98)}$$

$$\widehat{\Delta}_t^p(1+\delta_{\Delta_t^p})=\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)^{\varepsilon_p}$$

$$\times\Big[(1-\phi_w)\big(1+\widehat{\pi}_t^{\#}(1+\delta_{\pi_t^{\#}})\big)^{-\varepsilon_p}$$

$$\text{(B.99)}$$

$$+(1+\pi_{t-1})^{-\zeta_p\varepsilon_p}\phi_p\Delta_{t-1}^p\Big],$$

$$\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)^{1-\varepsilon_p}=(1-\phi_p)\big(1+\widehat{\pi}_t^{\#}(1+\delta_{\pi_t^{\#}})\big)^{1-\varepsilon_p}+\phi_p(1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)},\quad\text{(B.100)}$$

$$1+\widehat{\pi}_t^{\#}(1+\delta_{\pi_t^{\#}})=\frac{\varepsilon_p}{\varepsilon_p-1}\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)\frac{\widehat{x}_{1,t}}{\widehat{x}_{2,t}}(1+\delta_{x_{1t}})(1+\delta_{x_{2t}})^{-1},\quad\text{(B.101)}$$

$$\widehat{x}_{1,t}(1+\delta_{x_{1t}})=\widehat{\lambda}_t(1+\delta_{\lambda_t})\widehat{\mathrm{mc}}_t(1+\delta_{\mathrm{mc}_t})\widehat{Y}_t(1+\delta_{Y_t})$$

$$+\phi_p\beta\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)^{-\zeta_p\varepsilon_p}$$

$$\text{(B.102)}$$

$$\times E_t\big[\big(1+\widehat{\pi}_{t+1}(1+\delta_{\pi_{t+1}})\big)^{\varepsilon_p}\widehat{x}_{1,t+1}(1+\delta_{x_{1t+1}})\big],$$

$$\widehat{x}_{2,t}(1+\delta_{x_{2t}})=\widehat{\lambda}_t(1+\delta_{\lambda_t})\widehat{Y}_t+\phi_p\beta\big(1+\widehat{\pi}_t(1+\delta_{\pi_t})\big)^{\zeta_p(1-\varepsilon_p)}$$

$$\times E_t\big[\big(1+\widehat{\pi}_{t+1}(1+\delta_{\pi_{t+1}})\big)^{\varepsilon_p-1}\widehat{x}_{2,t+1}(1+\delta_{x_{2t+1}})\big],\quad\text{(B.103)}$$

$$\frac{\widehat{w}_t(1+\delta_{w_t})}{\widehat{R}_t(1+\delta_{R_t})}=\frac{1-\alpha}{\alpha}\cdot\widehat{K}_t^{\#}(1+\delta_{K_t^{\#}})\widehat{N}_t^{-1}(1+\delta_{N_t})^{-1},\quad\text{(B.104)}$$

$$\widehat{w}_t(1+\delta_{w_t})=\widehat{\mathrm{mc}}_t(1+\delta_{\mathrm{mc}_t})(1-\alpha)A_t\big(\widehat{K}_t^{\#}\big)^{\alpha}(1+\delta_{K_t^{\#}})^{\alpha}N_t^{-\alpha}(1+\delta_{N_t})^{-\alpha},\quad\text{(B.105)}$$

$$\widehat{i}_t(1+\delta_{i_t})=(1-\rho_i)i+\rho_i i_{t-1}$$

$$+(1-\rho_i)\big[\phi_{\pi}\big(\widehat{\pi}_t(1+\delta_{\pi_t})-\pi^*\big)$$

$$\text{(B.106)}$$

$$+ \phi_y \big(\ln \widehat{Y}_t + \ln(1 + \delta_{Y_t}) - \ln Y_{t-1} \big) \big] + \varepsilon_{i,t},$$

$$
\begin{aligned}
\widehat{Y}_t(1 + \delta_{Y_t}) &= \widehat{C}_t(1 + \delta_{C_t}) + \widehat{I}_t(1 + \delta_{I_t}) \\
&\quad + \widehat{G}_t(1 + \delta_{G_t}) + \big(\chi_1 \big(\widehat{u}_t(1 + \delta_{u_t}) - 1 \big) \\
&\quad + \chi_2 \big(\widehat{u}_t(1 + \delta_{u_t}) - 1 \big)^2 \big) \frac{K_t}{Z_t},
\end{aligned}
\tag{B.107}
$$

$$
K_{t+1}(1 + \delta_{K_{t+1}}) = Z_t \left[1 - \frac{\tau}{2} \left(\frac{\widehat{I}_t(1 + \delta_{I_t})}{I_{t-1}} - 1 \right)^2 \right] \widehat{I}_t(1 + \delta_{I_t}) + (1 - d) K_t,
\tag{B.108}
$$

where hats on the variables denote their approximated values; $f_{1,t}$, $f_{2,t}$ and $x_{1,t}$, $x_{2,t}$ are supplementary variables; Δ_t^p is a measure of price dispersion across firms.

Setting up a Minimization Problem

To construct the lower bound on approximation errors, we minimize the least-squares criterion for each t:

$$
\begin{aligned}
\min_{x_t} \ & \delta_{\lambda_t}^2 + \delta_{C_t}^2 + \delta_{\mu_t}^2 + \delta_{R_t}^2 + \delta_{u_t}^2 + \delta_{\pi_t}^2 + \delta_{I_t}^2 + \delta_{i_t}^2 + \delta_{f_{1t}}^2 + \delta_{f_{2t}}^2 + \delta_{w_t}^2 + \delta_{w_t^\#}^2 \\
& + \delta_{N_t}^2 + \delta_{Y_t}^2 + \delta_{K_t^\#}^2 + \delta_{\Delta_t^p}^2 + \delta_{\pi_t^\#}^2 + \delta_{x_{1t}}^2 + \delta_{x_{2t}}^2 + \delta_{\mathrm{mc}_t}^2 + \delta_{K_{t+1}}^2 + \delta_{G_t}^2 \\
& + \sum_{j=1}^{J} \big[\delta_{\lambda_{t+1,j}}^2 + \delta_{C_{t+1,j}}^2 + \delta_{\mu_{t+1,j}}^2 + \delta_{\pi_{t+1,j}}^2 + \delta_{I_{t+1,j}}^2 + \delta_{R_{t+1,j}}^2 + \delta_{f_{1t+1,j}}^2 \\
& \qquad + \delta_{f_{2t+1,j}}^2 + \delta_{w_{t+1,j}^\#}^2 + \delta_{x_{1t+1,j}}^2 + \delta_{x_{2t+1,j}}^2 + \delta_{u_{t+1}}^2 \big]
\end{aligned}
\tag{B.109}
$$

s.t. (B.89)–(B.108),

where $x_t \equiv \{ \delta_{\lambda_t}, \delta_{\lambda_{t+1,j}}, \ldots \}$ is a list of all approximation errors to the corresponding model's variables $\{ \lambda_t, \lambda_{t+1,j}, \ldots \}$ that appear in the objective function (B.109). Similarly to the optimal growth model, approximation errors in the current period variables are defined in a given point of the state space, while approximation errors in future variables are defined in J integration nodes. Restrictions (B.89)–(B.108) are the optimality conditions (B.69)–(B.88) written in terms of an approximation solution and the corresponding approximation errors; they are provided in Appendix B.2. Again, using linearized optimality conditions in place of nonlinear optimality conditions leads to a linear-quadratic programming problem that is more simple to solve numerically and that produces a good initial guess for the problem with the nonlinear restrictions. A linearization of the optimality conditions (B.69)–(B.88) is shown in Appendix B.4.

B.4. *Constructing Approximation Errors Using Linearized Model's Equations*

We construct approximation errors satisfying linearized model's equations (B.89)–(B.108).

Condition (B.89)

Finding a first-order Taylor expansion of equation (B.89) and omitting second-order terms, we have

$$0 = -\delta_{\lambda_t} \cdot \widehat{\lambda}_t - \delta_{C_t} \cdot \left\{ (\widehat{C}_t - b\widehat{C}_{t-1})^{-2} \widehat{C}_t + b\widehat{C}_t \cdot \beta b E_t (\widehat{C}_{t+1} - b\widehat{C}_t)^{-2} \right\}$$

$$+ \beta b \sum_{j=1}^{J} \omega_j \left[(\widehat{C}_{t+1,j} - b\widehat{C}_t)^{-2} \widehat{C}_{t+1,j} \delta_{C_{t+1,j}} \right] + \lambda_t \mathcal{R}_t^1.$$

For convenience, we introduce the following compact notation:

$$h^1 \equiv \beta b E_t (\widehat{C}_{t+1} - b\widehat{C}_t)^{-2}.$$

Introducing compact notation, we get

$$a^{1,1} \cdot \delta_{\lambda_t} + a^{1,3} \cdot \delta_{C_t} + \sum_{j=1}^{J} \omega_j a_j^{1,4} \cdot \delta_{C_{t+1,j}} + b^1 = 0,$$

where

$$a^{1,1} \equiv -\widehat{\lambda}_t,$$
$$a^{1,3} \equiv -(\widehat{C}_t - b\widehat{C}_{t-1})^{-2} \widehat{C}_t - b\widehat{C}_t \cdot h^1,$$
$$a_j^{1,4} \equiv \beta b (\widehat{C}_{t+1,j} - b\widehat{C}_t)^{-2} \widehat{C}_{t+1,j},$$
$$b^1 \equiv \widehat{\lambda}_t \mathcal{R}_t^1,$$

with $\mathcal{R}_t^1$ being the residual of this FOC, given by (B.110).

Condition (B.90)

By finding a first-order Taylor expansion in errors of condition (B.90), we obtain

$$-\delta_{R_t} + \mathcal{R}_t^{39} + \frac{1}{\widehat{R}_t Z_t} \chi_2 \widehat{u}_t \delta_{u_t} = 0.$$

Introducing compact notation, we get

$$a^{2,7} \cdot \delta_{R_t} + a^{2,8} \cdot \delta_{u_t} + b^2 = 0,$$

where

$$a^{2,7} \equiv -1, \qquad a^{2,8} = \frac{1}{\widehat{R}_t Z_t} \chi_2 \widehat{u}_t, \qquad b^2 = \mathcal{R}_t^2,$$

where $\mathcal{R}_t^2$ is the residual in equation (B.111).

Condition (B.91)

A first-order Taylor expansion of (B.91) yields

$$\delta_{\lambda_t} = \ln(1 + \mathcal{R}_t^3) + E_t \delta_{\lambda_{t+1}} + \frac{\widehat{i}_t}{1 + \widehat{i}_t} \delta_{i_t} - E_t \frac{\widehat{\pi}_{t+1}}{1 + \widehat{\pi}_{t+1}} \delta_{\pi_{t+1}}.$$

The latter condition can be rewritten as

$$a^{3,1} \cdot \delta_{\lambda_t} + a^{3,2} \cdot \sum_{j=1}^{J} \omega_j \delta_{\lambda_{t+1,j}} + \sum_{j=1}^{J} \omega_j a_j^{3,11} \cdot \delta_{\pi_{t+1,j}} + a^{3,15} \cdot \delta_{i_t} + b^3 = 0,$$

where

$$a^{3,1} \equiv -1, \qquad a^{3,2} \equiv 1, \qquad a^{3,15} \equiv \frac{\widehat{i_t}}{1 + \widehat{i_t}},$$

$$a_j^{3,11} \equiv -\sum_{j=1}^{J} \omega_j \frac{\widehat{\pi}_{t+1,j}}{1 + \widehat{\pi}_{t+1,j}}, \qquad b^3 \equiv \ln(1 + \mathcal{R}_t^3),$$

with $\mathcal{R}_t^3$ being a residual, defined in (B.112).

Condition (B.92)

A first-order Taylor expansion of (B.92) yields

$$
\begin{aligned}
0 = {} & -\widehat{\lambda}_t + \widehat{\mu}_t Z_t + -\frac{3}{2}\widehat{\mu}_t Z_t \tau \left(\frac{\widehat{I}_t}{I_{t-1}}\right)^2 + 2\widehat{\mu}_t Z_t \tau \frac{\widehat{I}_t}{I_{t-1}} + \frac{1}{2}\widehat{\mu}_t Z_t \tau + h_{1t}^4 - h_{2t}^4 \\
& - \widehat{\lambda}_t \delta_{\lambda_t} \\
& + \widehat{\mu}_t Z_t \left[2\tau \frac{\widehat{I}_t}{I_{t-1}} - \frac{3}{2}\tau\left(\frac{\widehat{I}_t}{I_{t-1}}\right)^2 + \frac{1}{2}\tau + 1\right]\delta_{\mu_t} \\
& + \beta E_t\left[\left(\widehat{\mu}_{t+1} Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{I_t}\right)^3 - \delta_{\mu_{t+1}}\widehat{\mu}_{t+1}Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{I_t}\right)^2\right) \cdot \delta_{\mu_{t+1}}\right] \\
& + \widehat{\mu}_t Z_t \left[2\tau \frac{\widehat{I}_t}{I_{t-1}} - 3\tau\left(\frac{\widehat{I}_t}{I_{t-1}}\right)^2 + 2h_{1t}^4 - 3h_{2t}^4\right]\delta_{I_t} \\
& + \beta E_t\left[3\widehat{\mu}_{t+1}Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{I_t}\right)^3 - 2\widehat{\mu}_{t+1}Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{I_t}\right)^2\delta_{I_{t+1}}\right],
\end{aligned}
$$

where the following compact notation is used:

$$h_1^4 \equiv \beta\widehat{\mu}_{t+1}Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{I_t}\right)^3,$$

$$h_2^4 \equiv \beta\widehat{\mu}_{t+1}Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{I_t}\right)^2.$$

Introducing compact notation, we have

$$a^{4,1} \cdot \delta_{\lambda_t} + a^{4,5} \cdot \delta_{\mu_t} + \sum_{j=1}^{J} \omega_j a_j^{4,6} \cdot \delta_{\mu_{t+1,j}} + a^{4,12} \cdot \delta_{I_t} + \sum_{j=1}^{J} \omega_j a_j^{4,13} \cdot \delta_{I_{t+1,j}} + b^4 = 0,$$

where

$$a^{4,1} \equiv -\widehat{\lambda}_t,$$

$$a^{4,5} \equiv \widehat{\mu}_t Z_t \left[2\tau \frac{\widehat{I}_t}{I_{t-1}} - \frac{3}{2}\tau \left(\frac{\widehat{I}_t}{I_{t-1}} \right)^2 + \frac{1}{2}\tau + 1 \right],$$

$$a_j^{4,6} \equiv m_{1,j}^4 - m_{2,j}^4,$$

$$a^{4,12} \equiv \widehat{\mu}_t Z_t \left[2\tau \frac{\widehat{I}_t}{I_{t-1}} - 3\tau \left(\frac{\widehat{I}_t}{I_{t-1}} \right)^2 \right] - 3h_1^4 + 2h_2^4,$$

$$a_j^{4,13} \equiv 3m_{1,j}^4 - 2m_{2,j}^4,$$

$$b^4 \equiv -\widehat{\lambda}_t + \widehat{\mu}_t Z_t + -\frac{3}{2}\widehat{\mu}_t Z_t \tau \left(\frac{\widehat{I}_t}{I_{t-1}} \right)^2 + 2\widehat{\mu}_t Z_t \tau \frac{\widehat{I}_t}{I_{t-1}} + \frac{1}{2}\widehat{\mu}_t Z_t \tau + h_1^4 - h_2^4,$$

with $m_{1,j}^4 \equiv \beta \widehat{\mu}_{t+1,j} Z_{t+1,j} \tau (\frac{\widehat{I}_{t+1,j}}{I_t})^3$ and $m_{2,j}^4 \equiv \beta \widehat{\mu}_{t+1,j} Z_{t+1,j} \tau (\frac{\widehat{I}_{t+1,j}}{I_t})^2$.

Condition (B.93)

A first-order Taylor expansion of (B.93) implies

$$
\begin{aligned}
0 = \mathcal{R}_t^5 \\
&- \sum_{j=1}^J \omega_j m_{1,j}^5 \widehat{\lambda}_{t+1,j} \left[\left(-\frac{1}{Z_{t+1,j}} \right) \chi_2 (\widehat{u}_{t+1,j} - 1) - \widehat{R}_{t+1,j} \widehat{u}_{t+1,j} \right] \cdot \delta_{\lambda_{t+1,j}} \\
&- \delta_{\mu_t} \\
&+ \sum_{j=1}^J \omega_j \left[\widehat{\mu}_{t+1} (1-d) \cdot \delta_{\mu_{t+1}} \right] \\
&+ \sum_{j=1}^J \omega_j m_{1,j}^5 \widehat{\lambda}_{t+1,j} \widehat{R}_{t+1,j} \widehat{u}_{t+1,j} \cdot \delta_{R_{t+1,j}} \\
&- \sum_{j=1}^J \omega_j m_{1,j}^5 \widehat{\lambda}_{t+1,j} \left(\left(-\frac{1}{Z_{t+1,j}} \right) [\chi_1 (2\widehat{u}_{t+1} - 1) + \chi_2 \widehat{u}_{t+1,j} (\widehat{u}_{t+1,j} - 1)] \right. \\
&\left. - \widehat{R}_{t+1,j} \widehat{u}_{t+1,j} \right) \cdot \delta_{u_{t+1,j}},
\end{aligned}
$$

where $\mathcal{R}_t^5$ is a residual defined in (B.114), and

$$m_{1,j}^5 \equiv \left[\widehat{\lambda}_{t+1,j} \left(\widehat{R}_{t+1,j} \widehat{u}_{t+1,j} - \frac{1}{Z_{t+1}, j} \left[\chi_1 (\widehat{u}_{t+1,j} - 1) + \frac{\chi_2}{2} (\widehat{u}_{t+1,j} - 1)^2 \right] \right) + \widehat{\mu}_{t+1,j} (1-d) \right]^{-1}.$$

Introducing further more compact notation, we have

$$\sum_{j=1}^J a_j^{5,2} \cdot \delta_{\lambda_{t+1,j}} + a^{5,5} \cdot \delta_{\mu_t} + \sum_{j=1}^J a_j^{5,6} \cdot \delta_{\mu_{t+1,j}} + \sum_{j=1}^J a_j^{5,14} \cdot \delta_{R_{t+1,j}} + \sum_{j=1}^J a_j^{5,33} \cdot \delta_{u_{t+1,j}} + b^5 = 0,$$

where

$$a_j^{5,2} \equiv -\omega_j m_{1,j}^5 \widehat{\lambda}_{t+1,j} \left[\left(-\frac{1}{Z_{t+1,j}} \right) \chi_2(\widehat{u}_{t+1,j} - 1) - \widehat{R}_{t+1,j} \widehat{u}_{t+1,j} \right],$$

$$a^{5,5} \equiv -1,$$

$$a_j^{5,6} \equiv \omega_j \widehat{\mu}_{t+1,j}(1-d),$$

$$a_j^{5,14} \equiv m_{1,j}^5 \widehat{\lambda}_{t+1,j} \widehat{R}_{t+1,j} \widehat{u}_{t+1,j},$$

$$a_j^{5,33} \equiv -\omega_j m_{1,j}^5 \widehat{\lambda}_{t+1,j} \left(\left(-\frac{1}{Z_{t+1,j}} \right) \left[\chi_1(2\widehat{u}_{t+1} - 1) + \chi_2 \widehat{u}_{t+1,j}(\widehat{u}_{t+1,j} - 1) \right] - \widehat{R}_{t+1,j} \widehat{u}_{t+1,j} \right),$$

$$b^5 \equiv \mathcal{R}_t^5.$$

Condition (B.94)

A first-order Taylor expansion of (B.94) leads us to

$$\delta_{w_t^\#} = \mathcal{R}_t^6 + \delta_{f_{1t}} - \delta_{f_{2t}}.$$

Introducing compact notation, we get

$$a^{6,16} \cdot \delta_{f_{1t}} + a^{6,18} \cdot \delta_{f_{2t}} + a^{6,21} \cdot \delta_{w_t^\#} + b^6 = 0,$$

where

$$a^{6,16} \equiv 1, \qquad a^{6,18} \equiv -1, \qquad a^{6,21} \equiv -1, \qquad b_t^6 \equiv \mathcal{R}_t^6.$$

Condition (B.95)

A first-order Taylor expansion of (B.95) implies

$$\begin{aligned}
0 = &\ \widehat{f}_{1,t} \mathcal{R}_t^7 \\
&+ h_{1t}^7 \delta_{\pi_t} \\
&+ \phi_w \beta (1+\widehat{\pi}_t)^{-\zeta_w \varepsilon_w (1+\eta)} \cdot \varepsilon_w (1+\eta) \\
&\cdot E_t \left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)-1} \left(\frac{\widehat{w}_{t+1}^\#}{\widehat{w}_t^\#} \right)^{\varepsilon_w(1+\eta)} \widehat{f}_{1,t+1} \cdot \widehat{\pi}_{t+1} \cdot \delta_{\pi_{t+1}} \right] \\
&- \widehat{f}_{1,t} \delta_{f_{1t}} \\
&+ \phi_w \beta (1+\widehat{\pi}_t)^{-\zeta_w \varepsilon_w (1+\eta)} E_t \left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)} \left(\frac{\widehat{w}_{t+1}^\#}{\widehat{w}_t^\#} \right)^{\varepsilon_w(1+\eta)} \widehat{f}_{1,t+1} \cdot \delta_{f_{1t+1}} \right] \\
&+ \psi \left[\frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t \right]^{(1+\eta)} (1+\eta) \varepsilon_w \delta_{w_t} \\
&- \left[\psi \left(\frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t \right)^{(1+\eta)} (1+\eta) \varepsilon_w + h_{2t}^7 \right] \delta_{w_t^\#} \\
&+ \varepsilon_w (1+\eta) \phi_w \beta (1+\widehat{\pi}_t)^{-\zeta_w \varepsilon_w (1+\eta)}
\end{aligned}$$

$$\times E_t\Big[[1 + \widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)}\big(\widehat{w}_{t+1}^{\#}\big)^{\varepsilon_w(1+\eta)}\big(\widehat{w}_t^{\#}\big)^{-\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1} \cdot \delta_{w_{t+1}^{\#}}\Big]$$

$$+ \psi\Bigg[\frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}}\widehat{N}_t\Bigg]^{(1+\eta)}(1+\eta)\delta_{N_t},$$

where $\mathcal{R}_t^7$ denotes a residual (B.116), and where

$$h_{1t}^7 \equiv -\zeta_w\varepsilon_w(1+\eta)\widehat{\pi}_t \cdot \phi_w\beta(1+\widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)-1} \cdot E_t\Bigg[[1 + \widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)}\bigg(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}}\bigg)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1}\Bigg],$$

$$h_{2t}^7 \equiv -\varepsilon_w(1+\eta)\phi_w\beta(1+\widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)}E_t\Bigg[[1 + \widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)}\bigg(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}}\bigg)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1}\Bigg].$$

Using compact notation, we get

$$a^{7,10} \cdot \delta_{\pi_t} + \sum_{j=1}^{J} a_j^{7,11} \cdot \delta_{\pi_{t+1,j}} + a^{7,16} \cdot \delta_{f_{1t}} + \sum_{j=1}^{J} a_j^{7,17} \cdot \delta_{f_{1t+1}} + a^{7,20} \cdot \delta_{w_t}$$

$$+ a^{7,21} \cdot \delta_{w_t^{\#}} + \sum_{j=1}^{J} a_j^{7,22} \cdot \delta_{w_{t+1,j}^{\#}} + a^{7,23} \cdot \delta_{N_t} + b^7 = 0,$$

where

$$a^{7,10} \equiv -\zeta_w\varepsilon_w(1+\eta)\widehat{\pi}_t \cdot \phi_w(1+\widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)-1}$$
$$\cdot \beta E_t\Bigg[[1 + \widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)}\bigg(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}}\bigg)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1}\Bigg],$$

$$a_j^{7,11} \equiv \phi_w\beta(1+\widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)} \cdot \varepsilon_w(1+\eta)$$
$$\cdot \omega_j\Bigg[[1 + \widehat{\pi}_{t+1,j}]^{\varepsilon_w(1+\eta)-1}\bigg(\frac{\widehat{w}_{t+1,j}^{\#}}{\widehat{w}_t^{\#}}\bigg)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1,j} \cdot \widehat{\pi}_{t+1,j}\Bigg],$$

$$a^{7,16} \equiv -\widehat{f}_{1,t},$$

$$m_{1,j}^7 \equiv \omega_j\phi_w(1+\widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)}\beta\Bigg[[1 + \widehat{\pi}_{t+1,j}]^{\varepsilon_w(1+\eta)}\bigg(\frac{\widehat{w}_{t+1,j}^{\#}}{\widehat{w}_t^{\#}}\bigg)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1,j}\Bigg],$$

$$a_j^{7,17} \equiv m_{1,j}^7, \qquad h_1^7 \equiv \psi\Bigg[\frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}}\widehat{N}_t\Bigg]^{(1+\eta)}(1+\eta), \qquad a^{7,20} \equiv h_1^7 \cdot \varepsilon_w,$$

$$h_2^7 \equiv \phi_w(1+\widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)}\beta E_t\Bigg[[1 + \widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)}\bigg(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}}\bigg)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1}\Bigg],$$

$$a^{7,21} \equiv -h_1^7 \cdot \varepsilon_w - h_2^7 \cdot \varepsilon_w(1+\eta), \qquad a_j^{7,22} \equiv \omega_j m_{1,j}^7 \cdot \varepsilon_w(1+\eta),$$

$$a^{7,23} \equiv h_1^7, \qquad b^7 \equiv \widehat{f}_{1,t}\mathcal{R}_t^{65}.$$

Condition (B.96)

A first-order Taylor expansion of (B.96) is

$$
\begin{aligned}
0 = {} & \widehat{f}_{2,t} \mathcal{R}_t^8 \\
& + \widehat{\lambda}_t \frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t \cdot \delta_{\lambda_t} \\
& + \zeta_w(1-\varepsilon_w)\widehat{\pi}_t \cdot \phi_w \beta (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)-1} \cdot E_t\left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w-1} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \right] \cdot \delta_{\pi_t} \\
& + \phi_w \beta (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} \cdot (\varepsilon_w - 1) \cdot E_t\left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w-2} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \cdot \widehat{\pi}_{t+1} \cdot \delta_{\pi_{t+1}} \right] \\
& - \widehat{f}_{2,t} \delta_{f_{2t}} \\
& + \phi_w \beta (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} E_t\left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w-1} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \cdot \delta_{f_{2t+1}} \right] \\
& + \widehat{\lambda}_t \frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t \varepsilon_w \cdot \delta_{w_t} \\
& - \left[\varepsilon_w \widehat{\lambda}_t \frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t + \varepsilon_w \phi_w \beta (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} E_t\left([1+\widehat{\pi}_{t+1}]^{\varepsilon_w-1} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \right) \right] \cdot \delta_{w_t^{\#}} \\
& + \varepsilon_w \phi_w \beta (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} E_t\left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w-1} \left(\widehat{w}_{t+1}^{\#} \right)^{\varepsilon_w} \left(\widehat{w}_t^{\#} \right)^{-\varepsilon_w} \widehat{f}_{2,t+1} \cdot \delta_{w_{t+1}^{\#}} \right] \\
& + \widehat{\lambda}_t \frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t \cdot \delta_{N_t},
\end{aligned}
$$

where $\mathcal{R}_t^8$ denotes a residual (B.117). Introducing new notation, we can rewrite the last equations as

$$
\begin{aligned}
& a^{8,1} \cdot \delta_{\lambda_t} + a^{8,10} \cdot \delta_{\pi_t} + \sum_{j=1}^{J} a_j^{8,11} \cdot \delta_{\pi_{t+1,j}} + a^{8,18} \cdot \delta_{f_{2t}} + \sum_{j=1}^{J} a_j^{8,19} \cdot \delta_{f_{2t+1,j}} + a^{8,20} \cdot \delta_{w_t} \\
& \quad + a^{8,21} \cdot \delta_{w_t^{\#}} + \sum_{j=1}^{J} a_j^{8,22} \cdot \delta_{w_{t+1,j}^{\#}} + a^{8,23} \cdot \delta_{N_t} + b^8 = 0,
\end{aligned}
$$

where

$$
h_1^8 \equiv \widehat{\lambda}_t \frac{\widehat{w}_t^{\varepsilon_w}}{\widehat{w}_t^{\#\varepsilon_w}} \widehat{N}_t, \qquad a^{8,1} = h_1^8, \qquad a^{8,18} = -\widehat{f}_{2,t}, \qquad a^{8,20} = h_1^8 \cdot \varepsilon_w,
$$

$$
a^{8,10} \equiv \zeta_w(1-\varepsilon_w)\widehat{\pi}_t \cdot \phi_w (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)-1} \cdot \beta E_t\left[[1+\widehat{\pi}_{t+1}]^{\varepsilon_w-1} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \right],
$$

$$
a_j^{8,11} \equiv \omega_j \phi_w (1+\widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} \cdot (\varepsilon_w - 1) \cdot \beta E_t\left[[1+\widehat{\pi}_{t+1,j}]^{\varepsilon_w-2} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1,j} \cdot \widehat{\pi}_{t+1,j} \right],
$$

$$m_j^8 \equiv \omega_j \phi_w (1 + \widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} \beta \left[[1 + \widehat{\pi}_{t+1,j}]^{\varepsilon_w-1} \left(\frac{\widehat{w}_{t+1,j}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1,j} \right],$$

$$a^{8,21} \equiv -h_{1t}^8 \cdot \varepsilon_w - h_2^8 \cdot \varepsilon_w,$$

$$h_2^8 \equiv \phi_w (1 + \widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)} \cdot \beta E_t \left([1 + \widehat{\pi}_{t+1}]^{\varepsilon_w-1} \left(\frac{\widehat{w}_{t+1}^{\#}}{\widehat{w}_t^{\#}} \right)^{\varepsilon_w} \widehat{f}_{2,t+1} \right),$$

$$a_j^{8,22} \equiv \omega_j m_j^8 \cdot \varepsilon_w, \qquad a^{8,23} = h_1^8, \qquad b^8 = \widehat{f}_{2,t} \mathcal{R}_t^8.$$

Condition (B.97)

A first-order Taylor expansion of (B.97) leads to

$$\widehat{w}_t^{1-\varepsilon_w} \mathcal{R}_t^9 + (1 - \varepsilon_w) \widehat{w}_t^{1-\varepsilon_w} \cdot \delta_{w_t} = (1 - \phi_w)(1 - \varepsilon_w) \left(\widehat{w}_t^{\#} \right)^{1-\varepsilon_w} \delta_{w_t^{\#}}$$
$$+ (1 + \pi_{t-1})^{\zeta_w(1-\varepsilon_w)} \phi_w w_{t-1}^{1-\varepsilon_w} (\varepsilon_w - 1) \widehat{\pi}_t (1 + \widehat{\pi}_t)^{\varepsilon_w-2} \cdot \delta_{\pi_t},$$

where $\mathcal{R}_t^9$ is a residual of this equation, defined in (B.118). After introducing more compact notation, we obtain

$$a^{9,10} \cdot \delta_{\pi_t} + a^{9,20} \cdot \delta_{w_t} + a^{9,21} \cdot \delta_{w_t^{\#}} + b^9 = 0,$$

where

$$a^{9,20} \equiv -(1 - \varepsilon_w) \widehat{w}_t^{1-\varepsilon_w},$$
$$a^{9,20} \equiv -(1 - \varepsilon_w) \widehat{w}_t^{1-\varepsilon_w},$$
$$a^{9,21} \equiv (1 - \phi_w)(1 - \varepsilon_w) \left(\widehat{w}_t^{\#} \right)^{1-\varepsilon_w},$$
$$a^{9,10} \equiv (1 + \pi_{t-1})^{\zeta_w(1-\varepsilon_w)} \phi_w w_{t-1}^{1-\varepsilon_w} (\varepsilon_w - 1) \widehat{\pi}_t (1 + \widehat{\pi}_t)^{\varepsilon_w-2},$$
$$b^9 \equiv \widehat{w}_t^{1-\varepsilon_w} \mathcal{R}_t^9.$$

Condition (B.98)

A first-order Taylor expansion of (B.98) is

$$(1 - \alpha) \delta_{N_t} - \delta_{Y_t} + \alpha \delta_{K_t^{\#}} - \delta_{\Delta_t^p} + \mathcal{R}_t^{10} = 0,$$

where the residual of this equation, $\mathcal{R}_t^{10}$, is defined in (B.119). We rewrite it as

$$a^{10,23} \cdot \delta_{N_t} + a^{10,24} \cdot \delta_{Y_t} + a^{10,25} \cdot \delta_{K_t^{\#}} + a^{10,26} \cdot \delta_{\Delta_t^p} + b^{10} = 0,$$

where

$$a^{10,24} \equiv -1, \qquad a^{10,25} \equiv \alpha, \qquad a^{10,23} \equiv 1 - \alpha, \qquad a^{10,26} \equiv -1, \qquad b^{10} \equiv \ln(1 + \mathcal{R}_t^{10}).$$

Condition (B.99)

A Taylor expansion of equation (B.99) is

$$\varepsilon_p \frac{1}{1 + \widehat{\pi}_t} \widehat{\pi}_t \cdot \delta_{\pi_t} - \delta_{\Delta_t^p} - \left[(1 - \phi_w) \left(1 + \widehat{\pi}_t^{\#} \right)^{-\varepsilon_p} + (1 + \pi_{t-1})^{-\zeta_p \varepsilon_p} \phi_p \Delta_{t-1}^p \right]^{-1}$$
$$\times \varepsilon_p (1 - \phi_w) \left(1 + \widehat{\pi}_t^{\#} \right)^{-\varepsilon_p-1} \widehat{\pi}_t^{\#} \cdot \delta_{\pi_t^{\#}} + \mathcal{R}_t^{11} = 0,$$

where $\mathcal{R}_t^{11}$ is the residual (B.120) of this equation. In terms of new notation, this becomes

$$a^{11,10} \cdot \delta_{\pi_t} + a^{11,26} \cdot \delta_{\Delta_t^p} + a^{11,27} \cdot \delta_{\pi_t^\#} + b^{11} = 0,$$

where

$$a^{11,10} \equiv \varepsilon_p \frac{1}{1 + \widehat{\pi}_t} \widehat{\pi}_t, \qquad a^{11,26} = -1,$$

$$a^{11,27} \equiv -\left[(1 - \phi_w)\left(1 + \widehat{\pi}_t^\#\right)^{-\varepsilon_p} + (1 + \pi_{t-1})^{-\zeta_p \varepsilon_p} \phi_p \Delta_{t-1}^p \right]^{-1} \varepsilon_p (1 - \phi_w)\left(1 + \widehat{\pi}_t^\#\right)^{-\varepsilon_p - 1} \widehat{\pi}_t^\#,$$

$$b^{11} \equiv \mathcal{R}_t^{11}.$$

Condition (B.100)

An expansion of (B.100) is

$$-(1 - \varepsilon_p)(1 + \widehat{\pi}_t)^{-\varepsilon_p} \widehat{\pi}_t \cdot \delta_{\pi_t} + (1 + \widehat{\pi}_t)^{1 - \varepsilon_p} \mathcal{R}_t^{12} + (1 - \phi_p)(1 - \varepsilon_p)\left(1 + \widehat{\pi}_t^\#\right)^{-\varepsilon_p} \widehat{\pi}_t^\# \cdot \delta_{\pi_t^\#} = 0,$$

with $\mathcal{R}_t^{12}$ being a residual (B.121). We rewrite this equation as follows:

$$a^{12,10} \cdot \delta_{\pi_t} + a^{12,27} \cdot \delta_{\pi_t^\#} + b^{12} = 0,$$

where

$$a^{12,10} \equiv -(1 - \varepsilon_p)(1 + \widehat{\pi}_t)^{-\varepsilon_p} \widehat{\pi}_t,$$

$$a^{12,27} \equiv (1 - \phi_p)(1 - \varepsilon_p)\left(1 + \widehat{\pi}_t^\#\right)^{-\varepsilon_p} \widehat{\pi}_t^\#,$$

$$b^{12} \equiv (1 + \widehat{\pi}_t)^{1 - \varepsilon_p} \mathcal{R}_t^{12}.$$

Condition (B.101)

A first-order Taylor expansion of (B.101) implies

$$\frac{\widehat{\pi}_t}{1 + \widehat{\pi}_t} \delta_{\pi_t} - \frac{\widehat{\pi}_t^\#}{1 + \widehat{\pi}_t^\#} \delta_{\pi_t^\#} + \delta_{x_{1t}} - \delta_{x_{2t}} + \mathcal{R}_t^{13} = 0,$$

with $\mathcal{R}_t^{13}$ being this equation's residual that is defined in (B.122); this yields

$$a^{13,10} \cdot \delta_{\pi_t} + a^{13,27} \cdot \delta_{\pi_t^\#} + a^{13,28} \cdot \delta_{x_{1t}} + a^{13,29} \cdot \delta_{x_{2t}} + b^{13} = 0,$$

where

$$a^{13,10} \equiv \frac{\widehat{\pi}_t}{1 + \widehat{\pi}_t}, \qquad a^{13,27} \equiv -\frac{\widehat{\pi}_t^\#}{1 + \widehat{\pi}_t^\#}, \qquad a^{13,28} \equiv 1,$$

$$a^{13,29} \equiv -1, \qquad b^{13} \equiv \ln\left(1 + \mathcal{R}_t^{13}\right).$$

Condition (B.102)

A Taylor expansion of (B.102) is

$$\widehat{x}_{1,t} \delta_{x_{1t}} = \mathcal{R}_t^{51} \widehat{x}_{1,t} + \widehat{\lambda}_t \widehat{\mathrm{mc}}_t \widehat{Y}_t [\delta_{\lambda_t} + \delta_{\mathrm{mc}_t} + \delta_{Y_t}]$$

$$- \zeta_p \varepsilon_p \phi_p \beta (1 + \widehat{\pi}_t)^{-\zeta_p \varepsilon_p - 1} \widehat{\pi}_t E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p} \widehat{x}_{1,t+1} \big] \cdot \delta_{\pi_t}$$

$$+ \phi_p \beta (1 + \widehat{\pi}_t)^{-\zeta_p \varepsilon_p} \varepsilon_p E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p - 1} \widehat{\pi}_{t+1} \widehat{x}_{1,t+1} \cdot \delta_{\pi_{t+1}} \big]$$

$$+ \phi_p \beta (1 + \widehat{\pi}_t)^{-\zeta_p \varepsilon_p} E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p} \widehat{x}_{1,t+1} \cdot \delta_{x_{1t+1}} \big],$$

with $\mathcal{R}_t^{14}$ being a residual (B.123). In compact notation, it becomes

$$a^{14,1} \cdot \delta_{\lambda_t} + a^{14,10} \cdot \delta_{\pi_t} + \sum_{j=1}^{J} a_j^{14,11} \cdot \delta_{\pi_{t+1,j}} + a^{14,24} \cdot \delta_{Y_t} + a^{14,28} \cdot \delta_{x_{1t}}$$

$$+ \sum_{j=1}^{J} a_j^{14,30} \cdot \delta_{x_{1t+1,j}} + a^{14,34} \cdot \delta_{\mathrm{mc}_t} + b^{14} = 0,$$

where

$$h^{14} \equiv \widehat{\lambda}_t \widehat{\mathrm{mc}}_t \widehat{Y}_t, \qquad a^{14,1} \equiv h^{14},$$

$$a^{14,10} \equiv -\zeta_p \varepsilon_p \phi_p (1 + \widehat{\pi}_t)^{-\zeta_p \varepsilon_p - 1} \widehat{\pi}_t \beta E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p} \widehat{x}_{1,t+1} \big],$$

$$a_j^{14,11} \equiv \omega_j \phi_p (1 + \widehat{\pi}_t)^{-\zeta_p \varepsilon_p} \varepsilon_p \beta \big[(1 + \widehat{\pi}_{t+1,j})^{\varepsilon_p - 1} \widehat{\pi}_{t+1,j} \widehat{x}_{1,t+1,j} \big],$$

$$a^{14,24} \equiv h^{14}, \qquad a^{14,34} = h^{14}, \qquad a^{14,28} = -\widehat{x}_{1,t},$$

$$a_j^{14,30} \equiv \omega_j \phi_p (1 + \widehat{\pi}_t)^{-\zeta_p \varepsilon_p} \beta \big[(1 + \widehat{\pi}_{t+1,j})^{\varepsilon_p} \widehat{x}_{1,t+1,j} \big],$$

$$b^{14} \equiv \mathcal{R}_t^{14} \widehat{x}_{1,t}.$$

Condition (B.103)

A first-order Taylor expansion of (B.103) implies

$$\widehat{x}_{2,t} \delta_{x_{2t}} = \mathcal{R}_t^{15} \widehat{x}_{2,t} + \widehat{\lambda}_t \widehat{Y}_t \delta_{\lambda_t} + \widehat{\lambda}_t \widehat{Y}_t \delta_{Y_t}$$

$$+ \zeta_p (1 - \varepsilon_p) \phi_p \beta (1 + \widehat{\pi}_t)^{\zeta_p (1 - \varepsilon_p) - 1} \widehat{\pi}_t E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p - 1} \widehat{x}_{2,t+1} \big] \cdot \delta_{\pi_t}$$

$$+ \phi_p \beta (1 + \widehat{\pi}_t)^{\zeta_p (1 - \varepsilon_p)} (\varepsilon_p - 1) E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p - 2} \widehat{\pi}_{t+1} \widehat{x}_{2,t+1} \cdot \delta_{\pi_{t+1}} \big]$$

$$+ \phi_p \beta (1 + \widehat{\pi}_t)^{\zeta_p (1 - \varepsilon_p)} E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p - 1} \widehat{x}_{2,t+1} \cdot \delta_{x_{2t+1}} \big],$$

where $\mathcal{R}_t^{15}$ is the residual introduced in (B.124). Rearranging the terms and using new notation, we have

$$a^{15,1} \cdot \delta_{\lambda_t} + a^{15,10} \cdot \delta_{\pi_t} + \sum_{j=1}^{J} a_j^{15,11} \cdot \delta_{\pi_{t+1,j}} + a^{15,29} \cdot \delta_{x_{2t}} + \sum_{j=1}^{J} a_j^{15,31} \cdot \delta_{x_{2t+1,j}} + b^{15} = 0,$$

where

$$a^{15,1} \equiv \widehat{\lambda}_t \widehat{Y}_t,$$

$$a^{15,10} \equiv \zeta_p (1 - \varepsilon_p) \phi_p (1 + \widehat{\pi}_t)^{\zeta_p (1 - \varepsilon_p) - 1} \widehat{\pi}_t \beta E_t \big[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p - 1} \widehat{x}_{2,t+1} \big],$$

$$a_j^{15,11} \equiv \omega_j \phi_p (1 + \widehat{\pi}_t)^{\zeta_p (1 - \varepsilon_p)} (\varepsilon_p - 1) \beta \big[(1 + \widehat{\pi}_{t+1,j})^{\varepsilon_p - 2} \widehat{\pi}_{t+1,j} \widehat{x}_{2,t+1,j} \big],$$

$$a^{15,24} \equiv \widehat{\lambda}_t \widehat{Y}_t, \qquad a^{15,29} = -\widehat{x}_{2,t},$$

$$a_j^{15,31} \equiv \omega_j \phi_p (1 + \widehat{\pi}_t)^{\zeta_p(1-\varepsilon_p)} \beta \big[(1 + \widehat{\pi}_{t+1,j})^{\varepsilon_p - 1} \widehat{x}_{2,t+1,j} \big],$$

$$b^{15} \equiv \mathcal{R}_t^{15} \widehat{x}_{2,t}.$$

Condition (B.104)

A first-order Taylor expansion of (B.104) leads to

$$\delta_{R_t} - \delta_{w_t} - \delta_{N_t} + \delta_{K_t^\#} + \mathcal{R}_t^{16} = 0,$$

where $\mathcal{R}_t^{16}$ is the residual of the equation; see (B.125). In terms of coefficients, we get

$$a^{16,7} \cdot \delta_{R_t} + a^{16,20} \cdot \delta_{w_t} + a^{16,23} \cdot \delta_{N_t} + a^{16,25} \cdot \delta_{K_t^\#} + b^{16} = 0,$$

where

$$a^{16,7} = 1, \qquad a^{16,20} = -1, \qquad a^{16,23} = -1, \qquad a^{16,25} = 1, \qquad b^{16} = \mathcal{R}_t^{16}.$$

Condition (B.105)

A first-order Taylor expansion of (B.105) is

$$a^{17,20} \cdot \delta_{w_t} + a^{17,23} \delta_{\widehat{N}_t} + a^{17,25} \cdot \delta_{K_t^\#} + a^{17,34} \cdot \delta_{\mathrm{mc}_t} + b^{17} = 0,$$

where $\mathcal{R}_t^{17}$ is the residual in (B.126), and

$$a^{17,20} = -1, \qquad a^{17,23} = -\alpha, \qquad a^{17,25} = \alpha, \qquad a^{17,34} = 1, \qquad b^{17} = \ln\big(1 + \mathcal{R}_t^{17}\big).$$

Condition (B.106)

A Taylor expansion of (B.106) is

$$\widehat{i}_t \delta_{i_t} = (1 - \rho_i) \phi_\pi \widehat{\pi}_t \cdot \delta_{\pi_t} + (1 - \rho_i) \phi_y \cdot \delta_{Y_t} + \mathcal{R}_t^{18} \widehat{i}_t,$$

where $\mathcal{R}_t^{18}$ is the residual in (B.127). In compact notation, we get

$$a^{18,10} \cdot \delta_{\pi_t} + a^{18,15} \cdot \delta_{i_t} + a^{18,24} \cdot \delta_{Y_t} + b^{18} = 0,$$

where

$$a^{18,10} \equiv (1 - \rho_i) \phi_\pi \widehat{\pi}_t,$$

$$a^{18,15} \equiv -\widehat{i}_t,$$

$$a^{18,24} \equiv (1 - \rho_i) \phi_y,$$

$$b^{18} \equiv \mathcal{R}_t^{18} \widehat{i}_t.$$

Condition (B.107)

A first-order Taylor expansion of (B.107) leads to

$$\widehat{Y}_t \delta_{Y_t} = \mathcal{R}_t^{56} \widehat{Y}_t + \widehat{C}_t \delta_{C_t} + \widehat{I}_t \delta_{I_t} + \widehat{G}_t \delta_{G_t} + \chi_1 \widehat{u}_t \delta_{u_t} \frac{K_t}{Z_t} + 2\chi_2(\widehat{u}_t - 1) \frac{K_t}{Z_t} \widehat{u}_t \cdot \delta_{u_t},$$

Introducing compact notation, we get

$$a^{19,3} \cdot \delta_{C_t} + a^{19,8} \cdot \delta_{u_t} + a^{19,12} \cdot \delta_{I_t} + a^{19,24} \cdot \delta_{Y_t} + a^{19,36} \delta_{G_t} + b^{19} = 0,$$

where

$$a^{19,3} = \widehat{C}_t,$$

$$a^{19,8} = \chi_1 \widehat{u}_t \frac{K_t}{Z_t} + 2\chi_2(\widehat{u}_t - 1) \frac{K_t}{Z_t} \widehat{u}_t,$$

$$a^{19,12} = \widehat{I}_t, \qquad a^{19,24} = -\widehat{Y}_t, \qquad a^{19,36} = \widehat{G}_t, \qquad b^{19} = \mathcal{R}_t^{19} \widehat{Y}_t,$$

with $\mathcal{R}_t^{19}$ being the residual defined in (B.128).

Condition (B.108)

An expansion of (B.108) is

$$\delta_{K_{t+1}} \widehat{K}_{t+1} = \mathcal{R}_t^{20} \widehat{K}_{t+1}$$
$$+ Z_t \left\{ \left[-\tau\left(\frac{\widehat{I}_t}{I_{t-1}} - 1\right) \frac{\widehat{I}_t}{I_{t-1}} \right] \widehat{I}_t + \left[1 - \frac{\tau}{2}\left(\frac{\widehat{I}_t}{I_{t-1}} - 1\right)^2 \right] \widehat{I}_t \right\} \cdot \delta_{I_t},$$

where $\mathcal{R}_t^{20}$ is the residual (B.129); introducing compact notation, we get

$$a^{20,12} \cdot \delta_{I_t} + a^{20,35} \cdot \delta_{K_{t+1}} + b^{20} = 0,$$

where

$$a^{20,12} = Z_t \left\{ \left[-\tau\left(\frac{\widehat{I}_t}{I_{t-1}} - 1\right) \frac{\widehat{I}_t}{I_{t-1}} \right] \widehat{I}_t + \left[1 - \frac{\tau}{2}\left(\frac{\widehat{I}_t}{I_{t-1}} - 1\right)^2 \right] \widehat{I}_t \right\},$$

$$a^{20,35} \equiv -\widehat{K}_{t+1}, \qquad b^{20} \equiv \mathcal{R}_t^{20} \widehat{K}_{t+1}.$$

B.5. *Defining Residuals in Equations*

The unit-free residuals are defined by the following twenty equations that correspond to the optimality conditions (B.69)–(B.88), respectively:

$$\mathcal{R}_t^1 = \frac{1}{\widehat{\lambda}_t} \left[\frac{1}{\widehat{C}_t - b\widehat{C}_{t-1}} - \beta b E_t \left(\frac{1}{\widehat{C}_{t+1} - b\widehat{C}_t} \right) \right] - 1, \tag{B.110}$$

$$\mathcal{R}_t^2 = \frac{1}{\widehat{R}_t Z_t} [\chi_1 + \chi_2(\widehat{u}_t - 1) - 1] - 1, \tag{B.111}$$

$$\mathcal{R}_t^3 = \frac{1}{\widehat{\lambda}_t} \beta E_t [\widehat{\lambda}_{t+1} \cdot (1 + \widehat{i}_t) \cdot (1 + \widehat{\pi}_{t+1})^{-1}] - 1, \tag{B.112}$$

$$\mathcal{R}_t^4 = \frac{1}{\widehat{\lambda}_t}\left\{\widehat{\mu}_t Z_t\left[1 - \frac{\tau}{2}\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}} - 1\right)^2 - \tau\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}} - 1\right)\frac{\widehat{I}_t}{\widehat{I}_{t-1}}\right]\right.$$
$$\left. + \beta E_t\left[\widehat{\mu}_{t+1}Z_{t+1}\tau\left(\frac{\widehat{I}_{t+1}}{\widehat{I}_t} - 1\right)\left(\frac{\widehat{I}_{t+1}}{\widehat{I}_t}\right)^2\right]\right\} - 1, \tag{B.113}$$

$$\mathcal{R}_t^5 = \frac{1}{\widehat{\mu}_t}\beta E_t\left[\widehat{\lambda}_{t+1}\left(\widehat{R}_{t+1}\widehat{u}_{t+1} - \frac{1}{Z_{t+1}}\left[\chi_1(\widehat{u}_{t+1} - 1) + \frac{\chi_2}{2}(\widehat{u}_{t+1} - 1)^2\right]\right)\right.$$
$$\left. + \widehat{\mu}_{t+1}(1 - d)\right] - 1, \tag{B.114}$$

$$\mathcal{R}_t^6 = \frac{1}{\widehat{w}_t^\#}\frac{\varepsilon_w}{\varepsilon_w - 1}\frac{\widehat{f}_{1,t}}{\widehat{f}_{2,t}} - 1, \tag{B.115}$$

$$\mathcal{R}_t^7 = \frac{1}{\widehat{f}_{1,t}}\left\{\psi\left(\frac{\widehat{w}_t}{\widehat{w}_t^\#}\right)^{\varepsilon_w(1+\eta)}\widehat{N}_t^{1+\eta}\right.$$
$$\left. + \phi_w\beta(1 + \widehat{\pi}_t)^{-\zeta_w\varepsilon_w(1+\eta)}E_t\left[[1 + \widehat{\pi}_{t+1}]^{\varepsilon_w(1+\eta)}\left(\frac{\widehat{w}_{t+1}^\#}{\widehat{w}_t^\#}\right)^{\varepsilon_w(1+\eta)}\widehat{f}_{1,t+1}\right]\right\} - 1, \tag{B.116}$$

$$\mathcal{R}_t^8 = \frac{1}{\widehat{f}_{2,t}}\left\{\widehat{\lambda}_t\left(\frac{\widehat{w}_t}{\widehat{w}_t^\#}\right)^{\varepsilon_w}\widehat{N}_t\right.$$
$$\left. + \phi_w\beta(1 + \widehat{\pi}_t)^{\zeta_w(1-\varepsilon_w)}E_t\left[(1 + \widehat{\pi}_{t+1})^{\varepsilon_w-1}\left(\frac{\widehat{w}_{t+1}}{\widehat{w}_{t+1}^\#}\right)^{\varepsilon_w}\widehat{f}_{2,t+1}\right]\right\} - 1, \tag{B.117}$$

$$\mathcal{R}_t^9 = \frac{1}{\widehat{w}_t^{1-\varepsilon_w}}\left\{(1 - \phi_w)\left(\widehat{w}_t^\#\right)^{1-\varepsilon_w}\right.$$
$$\left. + (1 + \pi_{t-1})^{\zeta_w(1-\varepsilon_w)}(1 + \widehat{\pi}_t)^{\varepsilon_w-1}\phi_w w_{t-1}^{1-\varepsilon_w}\right\} - 1, \tag{B.118}$$

$$\mathcal{R}_t^{10} = \frac{1}{\widehat{Y}_t}A_t(\widehat{K}_t^\#)^\alpha\widehat{N}_t^{1-\alpha}\left[\widehat{\Delta}_t^p\right]^{-1} - 1, \tag{B.119}$$

$$\mathcal{R}_t^{11} = \frac{1}{\widehat{\Delta}_t^p}(1 + \widehat{\pi}_t)^{\varepsilon_p}\left[(1 - \phi_w)\left(1 + \widehat{\pi}_t^\#\right)^{-\varepsilon_p} + (1 + \pi_{t-1})^{-\zeta_p\varepsilon_p}\phi_p\Delta_{t-1}^p\right] - 1, \tag{B.120}$$

$$\mathcal{R}_t^{12} = \frac{1}{(1 + \widehat{\pi}_t)^{1-\varepsilon_p}}\left[(1 - \phi_p)\left(1 + \widehat{\pi}_t^\#\right)^{1-\varepsilon_p} + \phi_p(1 + \pi_{t-1})^{\zeta_p(1-\varepsilon_p)}\right] - 1, \tag{B.121}$$

$$\mathcal{R}_t^{13} = \frac{1}{1 + \widehat{\pi}_t^\#}\frac{\varepsilon_p}{\varepsilon_p - 1}(1 + \widehat{\pi}_t)\frac{\widehat{x}_{1,t}}{\widehat{x}_{2,t}} - 1, \tag{B.122}$$

$$\mathcal{R}_t^{14} = \frac{1}{\widehat{x}_{1,t}}\left\{\widehat{\lambda}_t\widehat{mc}_t\widehat{Y}_t + \phi_p\beta(1 + \widehat{\pi}_t)^{-\zeta_p\varepsilon_p}E_t\left[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p}\widehat{x}_{1,t+1}\right]\right\} - 1, \tag{B.123}$$

$$\mathcal{R}_t^{15} = \frac{1}{\widehat{x}_{2,t}}\left\{\widehat{\lambda}_t\widehat{Y}_t + \phi_p\beta(1 + \widehat{\pi}_t)^{\zeta_p(1-\varepsilon_p)}E_t\left[(1 + \widehat{\pi}_{t+1})^{\varepsilon_p-1}\widehat{x}_{2,t+1}\right]\right\} - 1, \tag{B.124}$$

$$\mathcal{R}_t^{16} = \frac{\widehat{R}_t}{\widehat{w}_t}\frac{1-\alpha}{\alpha}\cdot\widehat{K}_t^\#\widehat{N}_t^{-1} - 1, \tag{B.125}$$

$$\mathcal{R}_t^{17} = \frac{1}{\widehat{w}_t}\widehat{mc}_t(1-\alpha)A_t\big(\widehat{K}_t^{\#}\big)^{\alpha}N_t^{-\alpha} - 1, \tag{B.126}$$

$$\mathcal{R}_t^{18} = \frac{1}{\widehat{i}_t}\Big\{(1-\rho_i)i + \rho_i i_{t-1}$$
$$+ (1-\rho_i)\big[\phi_\pi\big(\widehat{\pi}_t - \pi^*\big) + \phi_y(\ln\widehat{Y}_t - \ln Y_{t-1})\big] + \varepsilon_{i,t}\Big\} - 1, \tag{B.127}$$

$$\mathcal{R}_t^{19} = \frac{1}{\widehat{Y}_t}\bigg[\widehat{C}_t + \widehat{I}_t + \widehat{G}_t + \big(\chi_1(\widehat{u}_t - 1) + \chi_2(\widehat{u}_t - 1)^2\big)\frac{K_t}{Z_t}\bigg] - 1, \tag{B.128}$$

$$\mathcal{R}_t^{20} = \frac{1}{\widehat{K}_{t+1}}\bigg\{Z_t\bigg[1 - \frac{\tau}{2}\bigg(\frac{\widehat{I}_t}{I_{t-1}} - 1\bigg)^2\bigg]\widehat{I}_t + (1-d)K_t\bigg\} - 1. \tag{B.129}$$

B.6. *Details of Numerical Analysis*

We describe the calibration and solution procedures, and we outline the numerical results.

B.6.1. *Calibration and Solution Procedures*

We split the parameters of the model into two sets: we calibrate the parameters

$$\big\{\varepsilon_w, \varepsilon_p, \omega^g, \pi^*, \alpha, \chi_2, \psi, \beta, d\big\}$$

to the standard values in the literature, and we fix the remaining parameters

$$\{\rho_i, \rho_a, \rho_z, \rho_g, \sigma_{\varepsilon_a}, \sigma_{\varepsilon_i}, \sigma_{\varepsilon_z}, \sigma_{\varepsilon_g}, b, \phi_w, \phi_p, \zeta_w, \zeta_p, \eta, \tau, \phi_\pi, \phi_y\}$$

in line with the estimates obtained in Sims (2014) for the U.S. economy. Finally, parameter χ_1 is calculated as $1/\beta - (1-\delta)$ (which is obtained under a normalization of u_t to unity in the steady state). Table SII summarizes our benchmark parameter choice.

TABLE SII

BENCHMARK PARAMETERIZATION OF THE NEW KEYNESIAN MODEL

	Parameters in the Processes for Shocks Estimated From the U.S. Economy Data								
	ρ_a	ρ_i	ρ_z	ρ_g	σ_a	σ_i	σ_z	σ_g	ω^g
Values	0.99	0.79	0.90	0.96	0.0074	0.0013	0.0091	0.0038	0.2
	Other Parameters Estimated From the U.S. Economy Data								
	ϕ_π	ϕ_y	ϕ_w	ϕ_p	ζ_w	ζ_p	η	b	τ
Values	1.35	0.32	0.43	0.71	0.38	0.03	1.23	0.72	1.87
	Parameters Calibrated to the U.S. Economy Data								
	ε_w	ε_p	π^*	α	χ_1	χ_2	ψ	β	d
Values	10	10	0	1/3	0.0351	0.01	2	0.99	0.025

We evaluate the accuracy of perturbation solutions on a stochastic simulation of 10,200 observations (the first 200 observations were discarded to eliminate the effect of the initial conditions). The Dynare's representation of the state space includes the current endogenous state variables $\{\pi_{t-1}, w_{t-1}, C_{t-1}, I_{t-1}, N_{t-1}, Y_{t-1}, \Delta_{t-1}^p, i_{t-1}, K_t\}$, the past exogenous state variables $\{A_{t-1}, Z_{t-1}, \omega_{t-1}^g\}$, and the current disturbances $\{\varepsilon_{a,t}, \varepsilon_{i,t}, \varepsilon_{z,t}, \varepsilon_{g,t}\}$. We use a Dynare's option of pruning for simulating a second-order perturbation solution. To compute conditional expectations, we use a monomial integration rule with $J = 2N$ nodes, where $N = 4$ is the number of exogenous shocks. This rule delivers very accurate approximation to expectation functions (up to six accuracy digits) in the context of real business-cycle models (see Judd, Maliar, and Maliar (2011), for a detailed description of this rule).

B.6.2. *Numerical Results on the Lower Error Bound*

We report the size of approximation errors in Table SIII. For a future variable $x_{t+1,j} \in \{\delta_{\lambda_{t+1,j}}, \delta_{C_{t+1,j}}, \ldots\}$, statistics reported in columns L_1 and L_∞ are the mean and maximum of t-period absolute values of approximation errors in that variable across $J = 8$ integration nodes, that is, $\frac{1}{J}\sum_{j=1}^{J} \delta_{x_{t+1,j}}$ and $\max_{j \in J} \delta_{x_{t+1,j}}$, respectively. We consider three alternative parameterizations. The first parameterization corresponds to the benchmark values of the parameters in Table SII. The second parameterization considers the benchmark values for all the parameters, except of π^*, which is set to 0.02. In the final parameterization, we decrease the values of ε_w and ε_p relative to the benchmark parameterization; namely, ε_w and ε_p are set to 5.

Under Parameterization 1, we got a lower bound on approximation errors of order $10^{0.11} \approx 129\%$, which corresponds to an approximation error in π_t. Parameterization 2 produces a similar size of approximation errors (but the biggest approximation error is obtained in variable $\pi_t^{\#}$). Under Parameterization 3, the lower error bound for PER2 reaches $10^{-0.43} \approx 37\%$, which corresponds to an approximation error in N_t. Overall, as it follows from Table SII, for the studied new Keynesian model, such variables are inflation variables π_t and $\pi_t^{\#}$, investment variables I_t, i_t, and u_t, as well as price dispersion Δ_t^p and labor variable N_t.

B.6.3. *Analysis of Residuals in the New Keynesian Model*

In Appendix B.3, we listed twenty equations (B.110)–(B.129) that define unit-free residuals $\mathcal{R}_t^1, \ldots, \mathcal{R}_t^{20}$ corresponding to the twenty FOCs (B.69)–(B.88) of the new Keynesian model. We evaluate the accuracy of perturbation solutions on the same set of simulated points as the one used for constructing the approximation errors.

We report the residuals in Table SIV. If we exclude from consideration residuals $\mathcal{R}_t^{11}$, $\mathcal{R}_t^{12}$, $\mathcal{R}_t^{13}$, and $\mathcal{R}_t^{18}$ in equations (B.79)–(B.81) and (B.86), the remaining residuals are quite low; for example, under Parameterization 1, the maximum residuals would be $10^{-4.55} \approx 0.0028\%$ for a PER2 solution. However, the residuals are enormous if we take into account these four residuals $\mathcal{R}_t^{11}$, $\mathcal{R}_t^{12}$, $\mathcal{R}_t^{13}$, and $\mathcal{R}_t^{18}$, namely, the maximum residual is $10^{-0.27} \approx 54\%$.

The analysis of residuals also provides us with some insight into which variables are approximated inaccurately. For example, equation (B.80) contains only current and past inflation measures and definition (B.120) of residual $\mathcal{R}_t^{12}$ indicates that the inflation variables $\widehat{\pi}_t$ and $\widehat{\pi}_t^{\#}$ are approximated poorly (either one or the other or both):

$$\mathcal{R}_t^{12} = \frac{1}{(1 + \widehat{\pi}_t)^{1-\varepsilon_p}}\big[(1 - \phi_p)\big(1 + \widehat{\pi}_t^{\#}\big)^{1-\varepsilon_p} + \phi_p(1 + \pi_{t-1})^{\zeta_p(1-\varepsilon_p)}\big] - 1. \qquad \text{(B.130)}$$

TABLE SIII

APPROXIMATION ERRORS IN THE CURRENT AND FUTURE VARIABLES IN THE NEW KEYNESIAN MODEL[a]

	Parameterization 1				Parameterization 2				Parameterization 3			
	PER1		PER2		PER1		PER2		PER1		PER2	
	L_1	L_∞	L_1	L_∞	L_1	L_∞	L_1	L_∞	L_1	L_∞	L_1	L_∞
δ_{λ_t}	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00	−3.00
$\delta_{\lambda_{t+1,j}}$	−3.15	−2.79	−3.19	−2.81	−2.31	−2.09	−2.31	−2.09	−3.21	−2.79	−3.25	−2.81
δ_{C_t}	−2.25	−1.99	−2.25	−1.98	−3.32	−2.63	−3.25	−2.58	−2.13	−1.84	−2.13	−1.83
$\delta_{C_{t+1,j}}$	−1.85	−1.56	−1.84	−1.56	−2.95	−2.24	−2.89	−2.20	−1.78	−1.47	1.78	−1.47
δ_{μ_t}	−1.47	−1.41	−1.47	−1.41	−1.50	−1.47	−1.49	−1.46	−1.38	−1.31	−1.37	−1.31
$\delta_{\mu_{t+1,j}}$	−1.38	−1.32	−1.38	−1.32	−1.44	−1.41	−1.44	−1.40	−1.35	−1.28	−1.35	−1.28
δ_{R_t}	−1.33	−1.09	−1.33	−1.09	−2.16	−1.69	−2.12	−1.67	−1.40	−1.13	−1.40	−1.13
δ_{u_t}	−0.79	−0.55	−0.78	−0.54	−1.62	−1.14	−1.57	−1.12	−0.85	−0.59	−0.85	−0.58
δ_{π_t}	0.07	0.30	−0.05	0.11	−1.87	−1.87	−1.86	−1.86	−0.04	−0.03	−0.04	−0.04
$\delta_{\pi_{t+1,j}}$	−5.07	−4.04	−5.12	−4.04	−3.33	−3.05	−3.40	−3.10	−5.41	−4.37	−5.40	−4.38
δ_{I_t}	−0.34	−0.33	−0.34	−0.33	−0.33	−0.33	−0.33	−0.33	−0.34	−0.34	−0.34	−0.34
$\delta_{I_{t+1,j}}$	−1.45	−1.35	−1.45	−1.35	−1.54	−1.47	−1.54	−1.47	−1.47	−1.36	−1.47	−1.36
$\delta_{R_{t+1,j}}$	−3.66	−3.59	−3.66	−3.59	−3.75	−3.71	−3.76	−3.72	−3.69	−3.62	−3.69	−3.62
δ_{i_t}	−0.78	−0.59	−0.78	−0.59	−0.70	−0.51	−0.69	−0.50	−0.80	−0.59	−0.80	−0.59
$\delta_{f_{1t}}$	−1.58	−1.50	−1.58	−1.49	−1.36	−1.31	−1.36	−1.32	−1.32	−1.23	−1.32	−1.23
$\delta_{f_{1t+1,j}}$	−2.91	−2.78	−2.91	−2.78	−2.44	−2.32	−2.45	−2.32	−2.77	−2.65	−2.77	−2.65
$\delta_{f_{2t}}$	−2.04	−1.95	−2.03	−1.94	−1.51	−1.47	−1.53	−1.48	−1.75	−1.65	−1.75	−1.65
$\delta_{f_{2t+1,j}}$	−2.51	−2.40	−2.51	−2.40	−2.13	−2.05	−2.14	−2.05	−2.28	−2.18	−2.28	−2.18
δ_{w_t}	−2.97	−2.88	−2.96	−2.87	−3.05	−3.01	−3.04	−3.00	−2.37	−2.28	−2.36	−2.28
$\delta_{w_t^\#}$	−1.77	−1.69	−1.77	−1.69	−1.88	−1.83	−1.87	−1.82	−1.52	−1.43	−1.52	−1.43
$\delta_{w_{t+1,j}^\#}$	−2.42	−2.20	−2.42	−2.20	−2.17	−1.77	−2.17	−1.75	−2.15	−2.03	−2.15	−2.03
δ_{N_t}	−0.48	−0.40	−0.48	−0.40	−0.59	−0.54	−0.59	−0.54	−0.52	−0.43	−0.52	−0.43
δ_{Y_t}	−1.59	−1.05	−1.59	−1.05	−1.04	−0.93	−1.04	−0.93	−1.61	−1.06	−1.61	−1.06
$\delta_{K_t^\#}$	−0.54	−0.49	−0.54	−0.49	−0.60	−0.57	−0.60	−0.57	−0.57	−0.52	−0.57	−0.52
$\delta_{\Delta_t^p}$	−0.47	−0.47	−0.47	−0.47	−0.46	−0.46	−0.46	−0.46	−0.50	−0.50	−0.50	−0.50
$\delta_{\pi_t^\#}$	−0.66	−0.21	−0.31	−0.14	0.10	0.10	0.10	0.10	−0.33	−0.32	−0.32	−0.31
$\delta_{x_{1t}}$	−2.29	−1.60	−2.33	−1.61	−1.82	−1.72	−1.91	−1.78	−2.33	−1.62	−2.34	−1.62
$\delta_{x_{2t}}$	−3.02	−3.01	−3.57	−3.55	−2.73	−2.72	−3.84	−3.79	−3.38	−3.37	−3.63	−3.61
$\delta_{x_{1t+1,j}}$	−3.34	−2.63	−3.38	−2.65	−2.88	−2.73	−2.96	−2.79	−3.39	−2.66	−3.40	−2.67
$\delta_{x_{2t+1,j}}$	−4.12	−4.09	−4.67	−4.63	−3.80	−3.76	−4.91	−4.82	−4.53	−4.51	−4.78	−4.75
$\delta_{u_{t+1,j}}$	−5.64	−5.27	−5.63	−5.17	−5.62	−5.36	−5.50	−5.13	−5.69	−5.32	−5.68	−5.23
δ_{mc_t}	−1.79	−1.56	−1.79	−1.55	−2.54	−2.12	−2.50	−2.11	−1.79	−1.55	−1.79	−1.55
$\delta_{K_{t+1}}$	−1.94	−1.93	−1.94	−1.93	−1.94	−1.93	−1.94	−1.93	−1.95	−1.94	−1.95	−1.94
δ_{G_t}	−0.99	−0.72	−0.98	−0.71	−2.12	−1.43	−2.07	−1.39	−1.00	−0.70	−1.00	−0.70

[a]Notes: Parameterization 1 corresponds to our benchmark parameter choice summarized in Table SII. Parameterization 2 changes the inflation target parameter π^* to 0.02. Parameterization 3 assumes that ε_w, ε_p are equal to 5. PER1 and PER2 denote the first- and second-order perturbation solutions. L_1 and L_∞ are, respectively, the average and maximum of absolute values of the corresponding approximation errors across test points (in log10 units) on a stochastic simulation of 10,000 observations. For a future variable $x_{t+1,j}$, statistics reported in columns L_1 and L_∞ are the mean and maximum of t-period absolute values of approximation errors in that variable across $J = 8$ integration nodes, that is, $\frac{1}{J}\sum_{j=1}^{J}\delta_{x_{t+1,j}}$ and $\max_{j\in J}\delta_{x_{t+1,j}}$, respectively.

However, in general, the model's equations are complex and it is difficult to see which variables are approximated poorly by looking at the size of residuals. In this respect, our lower-bound error analysis has more sharp implications.

Finally, our analysis of residuals shows that for more nonlinear models, like our new Keynesian model, a specific way of constructing the residuals might be critical for the results. For example, consider the residual $\mathcal{R}_t^{12}$ given in (B.130); the mean of $\mathcal{R}_t^{12}$ for

TABLE SIV

RESIDUALS IN THE EQUILIBRIUM CONDITIONS OF THE NEW KEYNESIAN MODEL[a]

	Parameterization 1				Parameterization 2				Parameterization 3			
	PER1		PER2		PER1		PER2		PER1		PER2	
	L_1	L_∞	L_1	L_∞	L_1	L_∞	L_1	L_∞	L_1	L_∞	L_1	L_∞
R^1	−3.86	−3.86	−5.58	−5.43	−3.88	−3.88	−4.49	−4.45	−3.88	−3.88	−5.75	−5.58
R^2	−7.53	−6.37	−6.11	−5.96	−7.61	−6.41	−5.10	−5.05	−7.56	−6.44	−6.20	−6.06
R^3	−4.53	−4.53	−7.12	−6.87	−4.54	−4.54	−5.30	−5.26	−4.54	−4.54	−7.34	−7.07
R^4	−3.29	−3.29	−6.26	−5.59	−3.49	−3.49	−5.16	−5.01	−3.35	−3.35	−6.43	−5.76
R^5	−4.60	−4.60	−6.67	−6.58	−4.64	−4.64	−5.02	−4.97	−4.59	−4.59	−6.77	−6.65
R^6	−16.04	−15.35	−15.98	−15.35	−15.97	−15.35	−15.90	−15.18	−15.88	−15.35	−15.90	−15.18
R^7	−2.90	−2.90	−4.67	−4.55	−2.33	−2.33	−4.03	−3.98	−3.18	−3.18	−5.37	−5.18
R^8	−3.65	−3.65	−5.36	−5.28	−3.17	−3.17	−4.27	−4.23	−3.94	−3.94	−6.03	−5.81
R^9	−7.57	−6.37	−6.37	−5.76	−7.36	−6.12	−4.68	−4.50	−8.24	−7.03	−6.27	−5.96
R^{10}	−15.76	−15.05	−15.78	−15.05	−15.54	−15.05	−15.70	−15.05	−15.86	−15.18	−15.79	−15.05
R^{11}	−0.54	−0.54	−0.54	−0.54	−0.53	−0.53	−0.53	−0.53	−0.55	−0.55	−0.55	−0.55
R^{12}	−0.54	−0.54	−0.54	−0.54	−0.81	−0.81	−0.80	−0.80	−0.66	−0.66	−0.66	−0.66
R^{13}	−0.30	−0.30	−0.30	−0.30	−0.32	−0.32	−0.32	−0.32	−0.32	−0.32	−0.32	−0.32
R^{14}	−3.01	−3.01	−5.63	−5.20	−2.59	−2.59	−5.11	−4.65	−3.38	−3.38	−5.86	−5.55
R^{15}	−3.13	−3.13	−5.78	−5.34	−2.74	−2.73	−5.60	−4.95	−3.60	−3.60	−5.98	−5.70
R^{16}	−15.61	−14.95	−15.41	−14.78	−15.64	−15.00	−15.42	−14.75	−15.59	−14.91	−15.46	−14.81
R^{17}	−15.86	−15.18	−15.80	−15.05	−15.67	−15.05	−15.78	−15.05	−15.85	−15.18	−15.05	−15.05
R^{18}	−1.00	−0.28	−0.99	−0.27	−1.48	−0.75	−1.47	−0.75	−1.48	−0.75	−0.75	−0.75
R^{19}	−6.83	−6.12	−6.30	−5.96	−7.34	−6.45	−5.75	−5.50	−5.66	−5.56	−5.51	−5.51
R^{20}	−8.04	−6.88	−8.18	−7.51	−8.31	−7.12	−8.62	−7.92	−8.06	−6.91	−7.38	−7.38

[a]Notes: Parameterization 1 corresponds to our benchmark parameter choice summarized in Table SII. Parameterization 2 changes the inflation target parameter π^* to 0.02. Parameterization 3 assumes that ε_w, ε_p are equal to 5. PER1 and PER2 denote the first- and second-order perturbation solutions; L_1 and L_∞ are, respectively, the average and maximum of absolute values of residuals in the model's equations across optimality condition and test points (in log10 units) on a stochastic simulation of 10,000 observations.

the PER1 method is equal to -0.5385; see Table SIV. Consider another expression for a unit-free residual in the same equation (B.80):

$$\overline{\mathcal{R}}_t^{12} = \frac{\left[(1-\phi_p)\left(1+\widehat{\pi}_t^{\#}\right)^{1-\varepsilon_p} + \phi_p(1+\pi_{t-1})^{\zeta_p(1-\varepsilon_p)}\right]^{1/(1-\varepsilon_p)} - 1}{\widehat{\pi}_t} - 1. \qquad \text{(B.131)}$$

The mean residual $\overline{\mathcal{R}}_t^{12}$ of PER1 is now equal to 4.145, which is huge (and the maximum residual is even larger)! It is easy to see why our benchmark representation $\mathcal{R}_t^{12}$ leads to much smaller residuals than the alternative representation $\overline{\mathcal{R}}_t^{12}$: in the former case, the residual is evaluated relative to the denominator $(1+\widehat{\pi}_t)^{1-\varepsilon_p} \approx 1$, while in the latter case, it is evaluated relative to $\widehat{\pi}_t \approx 0$. We find that the residuals $\mathcal{R}_t^{11}$ and $\mathcal{R}_t^{13}$ in equations (B.79) and (B.81), respectively, also significantly depend on a specific way in which they are represented. Hence, to make meaningful qualitative inferences about accuracy from the analysis of residuals, it is important to take into account the size of variables with respect to which residuals are evaluated. In turn, our lower error bounds are not subject to this shortcoming: they are independent of the way in which the model's equations are written.

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