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A nonlinear certainty equivalent approximation method for dynamic stochastic problems

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This paper introduces a nonlinear certainty-equivalent approximation method for dynamic stochastic problems. We first introduce a novel, stable, and efficient method for computing the decision rules in deterministic dynamic economic problems. We use the results as nonlinear and global certainty-equivalent approximations for solutions to stochastic problems, and compare their accuracy to the common linear and local certainty-equivalent methods. Our examples demonstrate that this method can be applied to solve high-dimensional problems with up to 400 state variables with acceptable accuracy. This method can also be applied to solve problems with inequality constraints. These features make the nonlinear certainty-equivalent approximation method suitable for solving complex economic problems, where other algorithms, such as log-linearization, fail to produce a valid global approximation or are far less tractable.

Keywords. New Keynesian DSGE model, competitive equilibrium, parallel computing, sparse grid approximation, real business cycle model.

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1. Introduction

Many important problems across different fields of economics require solving dynamic stochastic general equilibrium (DSGE) or optimal decision-making problems. Numerical dynamic programming (DP) is a typical method to solve such problems by formulating them as Bellman equation (Bellman (1957)) and then solving them with value function iteration (or some accelerating methods like policy function iteration) or time iteration.[1] However, implementing numerical DP faces challenging problems such as time-consuming high-dimensional integration, keeping the shape properties of the value/policy function approximation (Cai and Judd (2013, 2015)), choosing appropriate approximation domains, avoiding possible nonconvergence because of accumulated approximation errors, and dealing with the kinks from inequality constraints that occasionally bind. Choosing a good approximation can be particularly challenging for multidimensional dynamic stochastic problems where the domain of state variables expands quickly over time, while a wider domain requires a higher degree approximation. Its implementation for high-dimensional problems is very time-consuming even if we take advantage of recent innovations, such as parallel dynamic programming methods (Cai, Judd, Thain, and Wright (2015b)) in a supercomputer or a computational grid.

Because of these challenges it is common in applied economics to rely on methods other than numerical DP and sacrifice accuracy of results for the greater ease of numerical implementation. The most common method is log-linearization; Magill (1977) introduced linearization methods for dynamic stochastic models to economics. Linear (and log-linear) approximations produce decision rules that depend only on the state of a deterministic dynamical system. They are also called certainty-equivalent approximations because they do not depend on the variance of any random variable. Because of its local nature, log-linearization often fails to give a good solution on states that are not near the steady state.[2] Moreover, perturbation methods are unsuitable for problems with inequality constraints.

This paper introduces a new method for solving dynamic problems in economics that we call the nonlinear certainty-equivalent (NLCEQ) approximation method. Application of certainty-equivalent approximations goes back to Simon (1956) and Theil (1957), who suggested solving dynamic programming problems with quadratic objectives and linear transition laws by optimizing under perfect foresight, and then using optimal deterministic forecasts for approximating unknown future values. They also demonstrated that for some stochastic control problems, the certainty-equivalent approximation is the exact solution for the optimal decision rules.[3] The NLCEQ method is a natural extension of the idea of a certainty equivalent in that it solves for a nonlin-

[1] For a detailed discussion of these methods, see Judd (1998), Bertsekas (2005, 2007), Rust (2008), and Cai and Judd (2014).

[2] For more detailed discussion of perturbation methods in economics, see Gaspar and Judd (1997), Jin and Judd (2002), Schmitt-Grohe and Uribe (2004), Fernandez-Villaverde and Rubio-Ramirez (2006), Kim, Kim, Schaumburg, and Sims (2008), and Den Haan and De Wind (2012).

[3] For a formal derivation of this result see, for example, Hansen and Sargent (2005, Section 3.2).

ear decision rule for the nonstochastic problem that is globally valid and applies this decision rule to the stochastic model.[4]

The NLCEQ method chooses a finite set of points in the state space, solves the deterministic dynamic optimization problem using each of those points as the initial condition, and then applies numerical approximation methods to those results to construct a global nonlinear approximation for the value function and decision rules. This method is simple, stable, and efficient, and it can be naturally parallelized with high efficiency for high-dimensional problems. Furthermore, it avoids the challenges faced by the numerical DP. Like log-linearization (and other certainty-equivalent approximations) it ignores the impact of uncertainty on the decision rule, but it is better than log-linearization over nontrivial neighborhoods of the deterministic steady state.

For deterministic dynamic problems (both social planner's problems and competitive equilibrium problems), NLCEQ can provide very accurate solutions. For stochastic dynamic problems, similar to other numerical approaches, it sacrifices some accuracy of the solution for the ease of numerical implementation. However, NLCEQ has a number of important advantages over those methods.

NLCEQ can exploit parallelism to solve high-dimensional problems (up to 400 state variables in our examples) in minutes with acceptable accuracy. Moreover, NLCEQ is also appropriate for solving dynamic stochastic problems with inequality constraints that occasionally bind, where perturbation is well known for its failure to get solutions with acceptable accuracy. Furthermore, NLCEQ provides a global solution that can be used for effective nonlinear impulse function analysis.

These attractive features make NLCEQ suitable for solving complex economic problems, where other algorithms fail or are too costly to get solutions with acceptable accuracy. Of course, like any numerical methods, NLCEQ has its own limitations: it may be not applicable to problems where uncertainty significantly affects optimal decision rules, such as dynamic portfolio optimization. Like other certainty-equivalent approximations, NLCEQ also has a limit on its accuracy for stochastic problems (NLCEQ can solve deterministic dynamic problems very accurately) but the accuracy of a NLCEQ solution can be checked. In fact, our results show that NLCEQ is about 100 times more accurate than log-linear or log-linear–quadratic perturbation methods for multicountry real business cycle problems.

In this paper we apply the NLCEQ method to solve two social planner's optimal decision-making problems and one competitive equilibrium problem. Our first example is a multicountry real business cycle (RBC) problem (Den Haan, Judd, and Juillard (2011)). We first show that NLCEQ achieves higher accuracy than log-linear or log-linear–quadratic perturbation methods in low-dimensional RBC problems, and then demonstrate that NLCEQ can solve up to a 200-country RBC problem (400 state variables) in minutes by parallelism with acceptable accuracy. Our second example is a dynamic stochastic model of food and clean energy (Chakravorty, Magne, and Moreaux (2008)), which has inequality constraints that occasionally bind. Moreover, the problem's initial state is far away from its steady state and even its state path cannot reach

[4]Solvability of the NLCEQ method follows directly from the global concavity of the value function by implicit differentiation; see, for example, Theorems 1–6 in Jin and Judd (2002).

its steady state in a finite time. Our results show that NLCEQ achieves acceptable accuracy in solving these problems, which are quite challenging for all other general numerical methods to the best of our knowledge. Our final example is a New Keynesian DSGE model with zero lower bound (Guerrieri and Iacoviello (2015)). Solving New Keynesian DSGE models has been studied frequently in the literature, such as Woodford (2003), Del Negro, Schorfheide, Smets, and Wouters (2007), Smets and Wouters (2007), Gali (2008), Maliar and Maliar (2015), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), and Guerrieri and Iacoviello (2015). Our results show that NLCEQ can easily solve competitive equilibrium problems with occasionally binding constraints.[5]

The paper is organized as follows. Section 2 introduces the NLCEQ method. Section 3 describes a road map to numerical illustrations of NLCEQ, which are given in Sections 4–6 for solving multicountry RBC problems, a dynamic stochastic model of food and clean energy, and a New Keynesian DSGE model with zero lower bound. Section 7 concludes. Appendixes A–D and replication files are available in supplementary files on the journal website, http://qeconomics.org/supp/533/supplement.pdf and http://qeconomics.org/supp/533/code_and_data.zip.

2. NLCEQ method

An infinite-horizon stochastic optimal decision-making problem can be expressed by the general model

$$V(\mathbf{x}_0) = \max_{\mathbf{a}_t \in \mathcal{D}(\mathbf{x}_t)} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u(\mathbf{x}_t, \mathbf{a}_t)\right\}, \quad \text{s.t.} \quad \mathbf{x}_{t+1} = g(\mathbf{x}_t, \mathbf{a}_t, \boldsymbol{\varepsilon}_t), \tag{1}$$

where $\mathbf{x}_t \in \mathbb{R}^d$ is a state vector process with an initial state x_0 (each state variable could be either continuous or discrete), $\mathbf{a}_t \in \mathbb{R}^n$ is the vector of action variables at time t, $\boldsymbol{\varepsilon}_t$ is a serially uncorrelated random vector process with identical and independent distributions across time (for simplicity, we assume that the mean or median of $\boldsymbol{\varepsilon}_t$ is zero), $u(\mathbf{x}, \mathbf{a})$ is a utility function, $g(\mathbf{x}, \mathbf{a}, \boldsymbol{\varepsilon})$ is the stochastic law of motion for the state variable vector $\mathbf{x}$, β is the discount factor ($0 < \beta < 1$), $\mathcal{D}(\mathbf{x}_t)$ is a feasible set of action $\mathbf{a}_t$, and $\mathbb{E}\{\cdot\}$ is the expectation operator. Here, g is a general transition law of the vector of state variables, but some elements of the state variable vector $\mathbf{x}$ could be exogenous or have a deterministic transition law independent of ε.

[5] In this paper, we use GAMS (McCarl et al. (2011)) code for all examples except for high-dimensional problems in Section 4.4. The NLCEQ method can also easily be implemented in other programming languages like MATLAB or Dynare (Adjemian et al. (2011)). We use CONOPT (Drud (1996)) as the optimization solver in our GAMS code, run them on one 3.5-GHz Intel processor, and get the solution in seconds/minutes for each case. For high-dimensional problems in Section 4.4, we use Fortran code and SNOPT (Gill, Murray, and Saunders (2005)) as the optimization solver, implement parallelism on a supercomputer, and then get solutions in minutes.

To solve the problem (1), value function iteration is often used by solving the Bellman equation backward:

$$\begin{aligned} V_t(\mathbf{x}_t) &= \max_{\mathbf{a}_t \in \mathcal{D}(\mathbf{x}_t)} u(\mathbf{x}_t, \mathbf{a}_t) + \beta \mathbb{E}\big\{V_{t+1}(\mathbf{x}_{t+1})\big\}, \\ \text{s.t.} \quad \mathbf{x}_{t+1} &= g(\mathbf{x}_t, \mathbf{a}_t, \boldsymbol{\epsilon}_t). \end{aligned} \tag{2}$$

Numerical implementation of value function iteration can be challenging for a number of reasons. It requires choosing an appropriate approximation domain for the state variables, which can be way wider than the one we are interested in, because of the stochasticity in the transition law of the states. In particular, when $\boldsymbol{\epsilon}_t$ has an infinite support, this may lead to an infinite support for $\mathbf{x}_{t+1}$ so that we have to use some remedies like truncation methods. Their impact on the solution is, however, hard to measure. Moreover, a wider domain requires a higher degree approximation for the value functions, and then requires more time for an optimization solver to find the optimal solution of the Bellman equation. In addition, in the presence of multiple uncertainties, the integration part of the Bellman equation can be very time-consuming, and may even become infeasible, to get good accuracy for high-dimensional integration. Finally, many problems have occasionally binding constraints that lead to kinks in value functions, a big challenge for multidimensional value function approximation.[6]

However, in many cases it is acceptable to obtain a solution to the problem (1) with less demanding accuracy. For these cases, we propose Algorithm 1—a simple and fast nonlinear certainty-equivalent (NLCEQ) approximation method—to obtain the value function V and corresponding optimal decision rules.

Algorithm 1 contains three steps: (i) the transformation step, which transforms the infinite-horizon stochastic problem (1) into a finite-horizon deterministic optimal decision-making problem; (ii) the optimization step, which solves the finite-horizon deterministic decision-making problems from the transformation step; (iii) the approximation step, which collects the results from the optimization step and uses them to construct approximations of the decision rules. These steps are laid out in Algorithm 1. Sections 2.1–2.3 below discuss the steps in more detail.

NLCEQ is a natural extension of the certainty-equivalent approximation idea that the locally accurate linearization (log-linearization) method implements, but it solves the deterministic optimization problems to find approximate values of value/policy functions at prespecified state nodes and then uses global nonlinear approximation methods to get the approximate value/policy functions, so NLCEQ is a globally accurate method. Because (3) is a convex optimization problem for most of dynamic programming problems in economics, NLCEQ is stable and can work well for problems with occasionally binding constraints. Stability of the NLCEQ algorithm ensures that solution accuracy is little changed by variations in model parameter values (we illustrated this in examples of Section 4.2).[7]

[6]Cai and Judd (2012) propose a rational spline interpolation method for value function approximation so that the value function iteration is stable and accurate for the problems with kinks, but it applies to problems with only one continuous state variable.

[7]A standard way to avoid any problems with model calibration and sensitivity analysis is to choose a wide approximation domain, so that changing calibrated parameter values does not push state variables

Algorithm 1 Nonlinear Certainty-Equivalent Approximation Method for Infinite-Horizon Stochastic Dynamic Programming Problems.

Step 1. Transformation step. Transform the infinite-horizon stochastic problem into a finite-horizon deterministic optimal decision-making problem,

$$\begin{aligned}\widetilde{V}(\mathbf{x}_0) = &\max_{a_t\in\mathcal{D}(\mathbf{x}_t)} \sum_{t=0}^{T-1} \beta^t u(\mathbf{x}_t, \mathbf{a}_t) + \beta^T \widetilde{V}_T(\mathbf{x}_T),\\ &\text{s.t.} \quad \mathbf{x}_{t+1} = g(\mathbf{x}_t, \mathbf{a}_t, 0),\end{aligned} \tag{3}$$

where $\widetilde{V}_T$ is a terminal value function given by an initial guess of the value function V.

Step 2. Optimization step. Choose a set of approximation nodes, $\mathbb{X} = \{\mathbf{x}_0^j : 1 \le j \le m\} \subset \mathbb{R}^d$, and compute $v^j = \widetilde{V}(\mathbf{x}_0^j)$ and its corresponding optimal initial action $\mathbf{a}_0^j \in \mathbb{R}^n$ using an optimization solver to solve (3), for each $\mathbf{x}_0^j \in \mathbb{X}$, $1 \le j \le m$.

Step 3. Approximation step. Using an appropriate approximation method, such that $\hat{V}(\mathbf{x}_0; \mathbf{b}_v)$ approximates $\{(\mathbf{x}_0^j, v^j) : 1 \le j \le m\}$ data and a vector of functions $\hat{\mathbf{P}}(\mathbf{x}_0; \mathbf{b}_a)$ approximates $\{(\mathbf{x}_0^j, \mathbf{a}_0^j) : 1 \le j \le m\}$, that is, $v^j \approx \hat{V}(\mathbf{x}_0^j; \mathbf{b}_v)$ and $\mathbf{a}_0^j \approx \hat{\mathbf{P}}(\mathbf{x}_0^j; \mathbf{b}_a)$ for all $\mathbf{x}_0^j \in \mathbb{X}$, where $\mathbf{b}_v$ and $\mathbf{b}_a$ are vectors of parameters.

If there is no uncertainty in the underlying problem, the NLCEQ method gives us a very accurate value and policy function for large enough T. For the stochastic problems, NLCEQ can give an estimate of the value/policy functions, which can be subsequently employed in the economic analysis, such as impulse function analysis and sensitivity analysis. To obtain more accurate approximation, if necessary, we can use the solutions of NLCEQ—$\hat{V}(\mathbf{x}_0; \mathbf{b}_v)$ and $\hat{\mathbf{P}}(\mathbf{x}_0; \mathbf{b}_a)$—as the initial guess for the value/policy functions, and then apply other more accurate methods like numerical value function iteration (Cai and Judd (2014)). When there is some freedom in choosing T, $\widetilde{V}_T$, approximation nodes, and approximation methods, we make choices that imply small global errors as defined later in equation (19). We will discuss the steps in more details below.[8]

After we get the optimal policy functions $\hat{\mathbf{P}}(\mathbf{x}_0; \mathbf{b}_a)$, it is easy to do a forward simulation: with a given initial state $\mathbf{x}_0$ and one simulation path ε_t, we use $\mathbf{a}_t = \hat{\mathbf{P}}(\mathbf{x}_t; \mathbf{b}_a)$ to get $\mathbf{x}_{t+1} = g(\mathbf{x}_t, \mathbf{a}_t, \varepsilon_t)$ for any time $t = 0, 1, 2, \ldots$. That is, in the simulation process, we do not need to repeatedly apply NLCEQ or solve its optimization problem (3); instead we only need to use the solved policy functions $\hat{\mathbf{P}}(\mathbf{x}; \mathbf{b}_a)$ while making sure that $\mathbf{x}_t$ is located inside the approximation domain.[9] We can do an impulse response analysis in a similar way.

outside the approximation domain, and choose large enough T so that the terminal value functions do not have a significant effect on the solution.

[8]For a more complete and general discussion on approximation and optimization in solving dynamic stochastic problems, see Judd (1998) and Miranda and Fackler (2002).

[9]That is, we have to choose an appropriate approximation domain in the optimization step, so that it is wide enough to contain simulated future states. This can be done in an iterative way: first guess a wider approximation domain, and then use the NLCEQ solution over it to do simulation: if the simulated states

2.1 *Transformation*

In the transformation step of the NLCEQ method, it is usually straightforward to obtain deterministic transition laws for continuous state variables. For example, if an exogenous state θ_t has a transition law $\ln(\theta_{t+1}) = \rho \ln(\theta_t) + \sigma \epsilon_{t+1}$, where $\epsilon_{t+1} \sim \mathcal{N}(0,1)$ enters linearly into the law of motion of the exogenous state, then a simple transformation is to set $\ln(\theta_{t+1}) = \rho \ln(\theta_t)$. A more general choice of the transformation is $\ln \theta_{t+1} = \rho \ln \theta_t - f(\sigma)$ with function f chosen in such way that the deterministic $u(\mathbf{x}_t, \mathbf{a}_t)$ is close to the expectation of stochastic utility, that is, the deterministic $u(\mathbf{x}_t, \mathbf{a}_t)$ is nearly a certainty equivalent of its stochastic version. One example of such transformation is shown in Appendix B, where we obtain a more accurate solution by choosing $\ln(\theta_{t+1}) = \rho \ln(\theta_t) - 0.5\sigma^2$ with $\sigma = 0.05$. Using this general transformation, we can deal with problems such as stochastic volatility (see, e.g., Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012)).

A typical way to choose the terminal value function is that for each terminal state $\mathbf{x}_T$, we find a corresponding action $\mathbf{a}_T^*(\mathbf{x}_T)$ so that its next state $\mathbf{x}_{T+1} = g(\mathbf{x}_T, \mathbf{a}_T^*(\mathbf{x}_T), 0) = \mathbf{x}_T$, and then we let $\mathbf{x}_t = \mathbf{x}_T$ and $\mathbf{a}_t^*(\mathbf{x}_t) = \mathbf{a}_T^*(\mathbf{x}_t)$ for all $t > T$, implying that the terminal value function $\widetilde{V}_T(\mathbf{x}_T) = u(\mathbf{x}_T, \mathbf{a}_T^*(\mathbf{x}_T))/(1-\beta)$ is equal to the sum of discounted utilities from $t = T$ to $t = \infty$ with the policy function $\mathbf{a}_t^*(\mathbf{x}_t)$. We use this in our multicountry RBC examples. Another potential way is to use a second-order perturbation as the terminal value function, but we do not apply it in this paper because we want to focus on NLCEQ only, without hybrid algorithms. The terminal time, T, depends on the terminal value function and the discount factor. If the terminal value function is close to the true value function, then T could be small, for example, we choose $T = 20$ for some large-dimensional multicountry RBC examples in Section 4; otherwise, T could be chosen such that $\beta^T < 10^{-4}$ if it is hard to find a terminal value function close to the true value function. For example, we choose $T = 200$ in the example with $\beta = 0.95$ in Section 5 such that $\beta^T = 3.5 \times 10^{-5}$.

We can also apply NLCEQ to problems with a discrete stochastic state θ_t by replacing it by its expected value conditional on its initial value, that is, $\mathbb{E}\{\theta_t \mid \theta_0\}$. For example, let θ_t be an exogenous Markov chain with k possible values, $\{\vartheta_1, \ldots, \vartheta_k\}$, and let P be its $k \times k$ transition matrix, where its (i, j) element represents the probability of $\theta_{t+1} = \vartheta_i$ conditional on $\theta_t = \vartheta_j$. If the initial-time value of θ_t is ϑ_i (i.e., $\theta_0 = \vartheta_i$), then we know that its unconditional probability vector at time t is $p_{t,i} = P^t \mathbf{e}_i$, where $\mathbf{e}_i$ is the column vector with 1 at the ith element and 0 everywhere else. Thus, in the transformation step, we set the transformed deterministic value for θ_t as its expected value, $\sum_{j=1}^k p_{t,i,j} \vartheta_j$, conditional on its initial value $\theta_0 = \vartheta_i$, where $p_{t,i,j}$ is the jth element of the vector $p_{t,i}$. Our example in Section 5 has a discrete stochastic state and applies this method.

2.2 *Optimization*

In the NLCEQ method, the optimization step will be time-consuming for high-dimensional problems, but they can be naturally parallelized across the approximation nodes,

locate in a much more narrow domain, choose the narrower domain to rerun NLCEQ algorithm for a more accurate solution.

as every node corresponds to one optimization problem, which is independent of the others. Moreover, each optimization problem has a sparsity structure: the action variables and state variables at time t are only connected with the state variables at $t-1$ and $t+1$, that is, it has the blockwise tridiagonal pattern in the constraints. We employ this sparsity in optimization solvers like the one we used in our high-dimensional multicountry RBC examples, SNOPT (Gill, Murray, and Saunders (2005)), so that each optimization problem can be solved more efficiently.[10]

2.3 *Approximation*

For low-dimensional problems, we can use a variety of approximation methods like, for example, multidimensional Chebyshev polynomial approximation (see Appendix A). However, one advantage of NLCEQ is that it can be applied to large-dimensional problems. For large-dimensional problems, we will use sparse grid approximation methods. For example, in our large-dimensional examples, we employ Smolyak grid points as the approximation nodes and Chebyshev–Smolyak polynomials as the approximation method (Smolyak (1963) and Malin, Krueger, and Kubler (2011)).[11] Moreover, we can also implement adaptive sparse grid methods (Brumm and Scheidegger (2014)) in NLCEQ.

After we get the approximated value/policy functions, $\widehat{V}(\mathbf{x}_0;\mathbf{b}_v)$ and $\widehat{\mathbf{P}}(\mathbf{x}_0;\mathbf{b}_a)$, it is essential to estimate their errors to the "true" solution so we know whether NLCEQ gives an acceptable solution. In our examples below we implement the unit-free Euler error measure. We also compute approximation errors for the approximation functions. That is, we choose a set of out-of-sample points, $\widehat{\mathbb{X}} = \{\widehat{\mathbf{x}}_0^j : 1 \le j \le \widehat{m}\} \subset \mathbb{R}^d$, and compute $\widehat{v}^j = \widetilde{V}(\widehat{\mathbf{x}}_0^j)$ and its corresponding optimal initial action $\widehat{\mathbf{a}}_0^j \in \mathbb{R}^n$ using an optimization solver to solve (3), for each $\widehat{\mathbf{x}}_0^j \in \widehat{\mathbb{X}}$, $1 \le j \le \widehat{m}$. Using these $\widehat{v}^j$, we compute the approximation errors in the $\mathcal{L}^\infty$ or $\mathcal{L}^1$ norm for the value function with the formulas

$$\widehat{E}_{\mathcal{L}^\infty} = \max_{1\le j\le \widehat{m}} \frac{\left|\widehat{v}^j - \widehat{V}\left(\widehat{\mathbf{x}}_0^j;\mathbf{b}_v\right)\right|}{1+\left|\widehat{v}^j\right|},$$

$$\widehat{E}_{\mathcal{L}^1} = \frac{1}{\widehat{m}} \sum_{1\le j\le \widehat{m}} \frac{\left|\widehat{v}^j - \widehat{V}\left(\widehat{\mathbf{x}}_0^j;\mathbf{b}_v\right)\right|}{1+\left|\widehat{v}^j\right|}.$$

[10] The optimization step of the NLCEQ algorithm can be employed with a variety of modern nonlinear optimization solvers, such as, for example, SNOPT (Gill, Murray, and Saunders (2005)), CONOPT (Drud (1996)), and KNITRO (Byrd, Nocedal, and Waltz (2006)). These solvers are also freely available at the NEOS server (Czyzyk, Mesnier, and Moré (1998); Gropp and Moré (1997)) with two popular high-level modeling languages: GAMS (McCarl et al. (2011)) and AMPL (Fourer, Gay, and Kernighan (2003)). If the code is written in MATLAB, it can call its internal optimization routine, fmincon, or an external solver such as KNITRO.

[11] Smolyak polynomials do not preserve the shape of the value functions, so using them in standard value function iteration can easily make it fail because the optimization problem in the Bellman equation becomes a nonconcave/nonconvex problem; thus it is very challenging to find the global maximizer by a standard optimization solver. See Cai and Judd (2013, 2015) for discussion about the importance of shape preservation in numerical DP. However, with the NLCEQ algorithm we do not need to use the approximate value functions in the objective of an optimization problem, so it does not face the shape-preservation challenge while the value function iteration does.

Similarly, we can compute approximation errors for the policy functions. In our examples, we let $\widehat{\mathbb{X}}$ be a set of 1000 points uniformly and randomly drawn in the approximation domain. Note that the computation of $\widehat{v}^j = \widetilde{V}(\widehat{\mathbf{x}}_0^j)$ and its corresponding $\widehat{\mathbf{a}}_0^j$ can be parallelized naturally together with the optimization step of NLCEQ.

2.4 *NLCEQ method for competitive equilibrium*

Algorithm 1 describes the NLCEQ method for social planner's decision-making problems, but it can also be modified for solving competitive equilibrium. Similar to the transformation step of Algorithm 1, we first remove the stochasticity of models by replacing those shocks by their mean or median. For the transformed deterministic problem, its equilibrium solution should satisfy a set of equations (including the deterministic version of Euler equations, deterministic transition laws of states, market clearing conditions, and other first-order conditions)

$$\mathbf{F}(\mathbf{x}_t, \mathbf{a}_t, \mathbf{x}_{t+1}, \mathbf{a}_{t+1}) = 0, \quad t = 0, 1, 2, \ldots, \tag{4}$$

where $\mathbf{x}_t$ is the state vector and $\mathbf{a}_t$ is the action vector that should satisfy constraints $\mathbf{a}_t \in \mathcal{D}(\mathbf{x}_t)$. If there are occasionally binding constraints, then the arguments of $\mathbf{F}$ should also contain corresponding Lagrange multipliers that we omit below without loss of generality. Moreover, we know that its state and control variables will converge to its steady values $(\mathbf{x}_{ss}, \mathbf{a}_{ss})$ as time goes to infinity, that is,

$$\mathbf{x}_\infty = \mathbf{x}_{ss}, \qquad \mathbf{a}_\infty = \mathbf{a}_{ss}.$$

To solve the infinite-horizon system (4), we approximate it as the minimization problem with a finite horizon and a given initial state $\mathbf{x}_0^j$,

$$\begin{aligned} &\min_{\mathbf{a}_t \in \mathcal{D}(\mathbf{x}_t)} \left\| \mathbf{x}_T^{\text{Endo}} - \mathbf{x}_{ss}^{\text{Endo}} \right\| + \|\mathbf{a}_T - \mathbf{a}_{ss}\| \\ &\quad \text{s.t.} \quad \mathbf{F}(\mathbf{x}_t, \mathbf{a}_t, \mathbf{x}_{t+1}, \mathbf{a}_{t+1}) = 0, \quad t = 0, 1, \ldots, T-1, \\ &\qquad\quad \mathbf{x}_0 = \mathbf{x}_0^j, \end{aligned} \tag{5}$$

where $\|\cdot\|$ is a norm and $\mathbf{x}^{\text{Endo}}$ represents the endogenous state variables. By sweeping over the approximation nodes of $\mathbf{x}_0^j$, we can construct the approximation of policy functions over the state space. Algorithm 2 summarizes the NLCEQ method for solving competitive equilibrium.

Algorithm 2 yields a very accurate solution of a deterministic competitive equilibrium problem for large enough T. Similar to Algorithm 1, Algorithm 2 is also stable and efficient, and can be naturally parallelized in its optimization step, so that it can solve large-dimensional problems using sparse grid approximation methods.

Algorithm 2 can also be applied to solve social planner's stochastic dynamic programming problems, but Algorithm 1 is easier to implement as it does not require formulating the first-order conditions. Therefore, in the examples of this paper, we will use Algorithm 1 for solving social planner's problems and Algorithm 2 for computing competitive equilibrium.

Algorithm 2 Nonlinear Certainty-Equivalent Approximation Method for Competitive Equilibrium.

Step 1. Transformation step. Transform the stochastic problem into a finite-horizon deterministic system (5).

Step 2. Optimization step. Choose a set of approximation nodes $\mathbb{X} = \{\mathbf{x}_0^j : 1 \leq j \leq m\} \subset \mathbb{R}^d$. For each $\mathbf{x}_0^j \in \mathbb{X}$, solve (5) and get its corresponding optimal initial action with $\mathbf{a}_0^j$.

Step 3. Approximation step. Use an appropriate approximation method, such that $\hat{\mathbf{P}}(\mathbf{x}_0; \mathbf{b}_a)$ approximates $\{(\mathbf{x}_0^j, \mathbf{a}_0^j) : 1 \leq j \leq m\}$, where $\mathbf{b}_a$ is a vector of parameters.

3. Guide for numerical illustrations

The previous section introduces NLCEQ with two algorithms: one for social planners' problems; another for competitive equilibrium problems. In the next sections, we illustrate these algorithms with three examples from macroeconomics and environmental economics.[12] The first two examples are the social planners' problems and we employ NLCEQ (Algorithm 1) to solve them. Specifically, we solve a multicountry real business cycle model in Section 4 and demonstrate that NLCEQ can be applied to solve up to 400 dimensional dynamic stochastic problems. In Section 5 we solve an optimal resource extraction problem and show that NLCEQ can be applied to solve stochastic problems with kinks (occasionally binding constraints). In the last example (Section 6), we apply NLCEQ (Algorithm 2) to solve a competitive equilibrium problem, which is the New Keynesian DSGE model with zero lower bound. All these examples show that NLCEQ is a powerful tool for solving these complex economic problems.

In our numerical illustrations, we focus on comparison between NLCEQ and (log-) linearization, and we show that NLCEQ is much more accurate for problems with wide state domains and/or kinks. One main reason to choose (log-) linearization for comparison is that it is as easy to apply as NLCEQ and it is the most known and popular tool in economic analysis.

It is important to note several other new tools for solving dynamic stochastic problems that have recently emerged in the economic literature, such as the generalized stochastic simulation method (GSSA) of Judd, Maliar, and Maliar (2011), the OccBin method (Guerrieri and Iacoviello (2015)), and the method of Maliar and Maliar (2015).

The OccBin method is a piecewise linear interpolation method, "linking the first-order approximation of the model around the same point under each regime" (Guerrieri and Iacoviello (2015)), where one region represents that the inequality constraint does not bind, and another region represents that the inequality constraint is binding. In the absence of kinks, the OccBin method becomes the order-1 perturbation method (i.e., (log-) linearization method), so our comparison with (log-) linearization in the multicountry RBC example can also be applied to the OccBin method. That is, the OccBin method will also have the same big errors from the (log-) linearization method on the wide state domain in our examples. Guerrieri and Iacoviello (2015) also admit that the OccBin method "is not able to capture precautionary behavior linked to the possibility

[12] For an entry-level illustration we also solve a simple real business cycle model in Appendix B.

that a constraint may become binding in the future, as a result of shocks yet unrealized," while the NLCEQ method has no such limit.

The GSSA method (Judd, Maliar, and Maliar (2011)) is a stochastic simulation method, and the method of Maliar and Maliar (2015) combines simulation and projection methods. Both methods solve for the impact of uncertainty on solutions but are not as simple to implement as the NLCEQ method. They can also be unstable as their iteration cannot guarantee convergence, but the NLCEQ method is stable and robust. Moreover, NLCEQ can be naturally and massively parallelized, but efficient parallelism is challenging for both GSSA and the Maliar and Maliar method. In addition, a solution from NLCEQ can also be used as a good initial guess for GSSA or the Maliar and Maliar method that might produce a solution with higher accuracy for some problems.

4. Application to multicountry real business cycle model

We apply NLCEQ to solve a multicountry real business cycle (RBC) model introduced in Den Haan, Judd, and Juillard (2011). We assume that there are N countries with a capital stock state vector $K_t = (K_{t,1}, \ldots, K_{t,N})$ and a productivity state vector $\theta_t = (\theta_{t,1}, \ldots, \theta_{t,N})$ at the beginning of period t. For the jth country, its production at time t is given by $\theta_{t,j} f(K_{t,j}, \ell_{t,j})$, where $\ell_{t,j}$ is labor supply and f is the Cobb–Douglas production function

$$f(K_{t,j}, \ell_{t,j}) = A(K_{t,j})^{\alpha} (\ell_{t,j})^{1-\alpha}, \tag{6}$$

where α is the expenditure share of capital in the production and A is the productivity parameter. The law of motion of capital is

$$K_{t+1,j} = (1-\delta)K_{t,j} + I_{t,j}, \tag{7}$$

where $I_{t,j}$ is investment and δ is the depreciation rate of capital. The law of motion of productivity is exogenous,

$$\ln(\theta_{t+1,j}) = \rho \ln(\theta_{t,j}) + \sigma(\epsilon_{t+1,j} + \varepsilon_{t+1}), \tag{8}$$

where $\epsilon_{t,j}, \varepsilon_t \sim i.i.d.\mathcal{N}(0,1)$ are a country-specific shock and a worldwide shock, respectively.

The jth country has an instantaneous utility

$$u_j(c_{t,j}, \ell_{t,j}) = \frac{(c_{t,j})^{1-\frac{1}{\gamma_j}}}{1-\frac{1}{\gamma_j}} - B_j \frac{(\ell_{t,j})^{1+\frac{1}{\eta_j}}}{1+\frac{1}{\eta_j}}, \tag{9}$$

where $c_{t,j}$ is consumption, γ_j is the intertemporal elasticity of substitution, η_j is the Frisch elasticity of labor supply, and $B_j = (1-\alpha)A^{(\gamma_j-1)/\gamma_j}$ is the relative weight of consumption and leisure in the welfare.

We want to solve a social planner's problem with an aggregate utility $U(c_t, \ell_t)$, which is the weighted sum of the instantaneous utilities of all countries, that is,

$$U(c_t, \ell_t) = \sum_{j=1}^{N} \tau_j u_j(c_{t,j}, \ell_{t,j}),$$

where $\tau_j = A^{1/\gamma_j}$ are Negishi weights, $c_t = (c_{t,1}, \ldots, c_{t,N})$, and $\ell_t = (\ell_{t,1}, \ldots, \ell_{t,N})$. Let β be the discount factor, and let $\Gamma_{t,j}$ be an adjustment cost

$$\Gamma_{t,j} \equiv \frac{\phi}{2} K_{t,j} \left(\frac{I_{t,j}}{K_{t,j}} - \delta \right)^2, \tag{10}$$

with ϕ as the intensity of the friction. The social planner problem then becomes

$$\max_{c,\ell,I} \mathbb{E} \left(\sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t) \right) \tag{11}$$

subject to (7) and the aggregate world resource constraint

$$\sum_{j=1}^{N} (c_{t,j} + I_{t,j} - \delta K_{t,j}) = \sum_{j=1}^{N} \big(\theta_{t,j} f(K_{t,j}, \ell_{t,j}) - \Gamma_{t,j} \big). \tag{12}$$

Therefore, the problem has $2N$ state variables, K_t and θ_t, and $3N$ control variables, c_t, ℓ_t, and I_t.

In our example, we set $A = (1-\beta)/(\alpha\beta)$ and use the parameter values in Juillard and Villemot (2011) as the default. That is, we set $\beta = 0.99$, $\alpha = 0.36$, $\delta = 0.025$, $\rho = 0.95$, $\phi = 0.5$, and $\sigma = 0.01$ as the default values. Since the optimal solution depends only on the states and not the time t, we use (K, θ) to denote current states by omitting the subscript t, and use (K^+, θ^+) to denote next-period states. We are interested in the policy solutions over the domain of the state variables $(K, \theta) \in [0.7, 1.3]^{2N}$.

4.1 *Error measure*

For a given current state vector (K, θ), the first-order conditions of the RBC model (11) tell us that the optimal policy (c, ℓ, I) should satisfy

$$\frac{\partial u_j}{\partial c}(c_j, \ell_j)\tau_j = \frac{\partial u_{j'}}{\partial c}(c_{j'}, \ell_{j'})\tau_{j'}, \quad j' \neq j, \tag{13}$$

$$\frac{\partial u_j}{\partial \ell}(c_j, \ell_j) = -\frac{\partial u_j}{\partial c}(c_j, \ell_j)\theta_j \frac{\partial f}{\partial \ell}(K_j, \ell_j), \tag{14}$$

and the N Euler equations

$$\frac{\partial u_j}{\partial c}(c_j, \ell_j)\omega_j = \mathbb{E}\left\{ \beta \frac{\partial u_j}{\partial c}\big(c_j^+, \ell_j^+\big) \left[\pi_j^+ + \theta_j^+ \frac{\partial f}{\partial K}\big(K_j^+, \ell_j^+\big) \right] \right\} \tag{15}$$

for $j = 1, \ldots, N$. Here

$$\omega_j \equiv 1 + \phi\left(\frac{I_j}{K_j} - \delta\right), \tag{16}$$

$$\pi_j^+ \equiv 1 + \frac{\phi}{2}\left(\frac{I_j^+}{K_j^+} - \delta\right)\left(2 - \delta + \frac{I_j^+}{K_j^+}\right), \tag{17}$$

and (c^+, ℓ^+, I^+) is the optimal policy in the next period.

We use NLCEQ to get the estimate of the optimal policy functions of the problem (11): $C_j(K, \theta)$ for consumption, $L_j(K, \theta)$ for labor supply, $\mathcal{I}_j(k, \theta)$ for investment, and $\mathcal{K}_j^+(K, \theta) = (1 - \delta)K + \mathcal{I}_j(K, \theta)$ for the next-period capital. Thus, for any (K, θ), we can compute $c_j = C_j(K, \theta)$, $\ell_j = L_j(K, \theta)$, $I_j = \mathcal{I}_j(K, \theta)$, $K_j^+ = \mathcal{K}_j^+(K, \theta)$, $c_j^+ = C_j(K^+, \theta^+)$, $\ell_j^+ = L_j(K^+, \theta^+)$, and $I_j^+ = \mathcal{I}_j(K^+, \theta^+)$, and then compute ω_j and π_j^+ from equations (16) and (17).

Therefore, for a given (K, θ), we can compute the unit-free Euler error

$$E_1(K, \theta) = \max_{1 \le j \le N} \left|\mathbb{E}\left\{F_j\left(K, \theta, \theta^+\right)\right\} - 1\right|, \tag{18}$$

with

$$F_j\left(K, \theta, \theta^+\right) \equiv \frac{\beta \dfrac{\partial u_j}{\partial c}\left(c_j^+, \ell_j^+\right)}{\dfrac{\partial u_j}{\partial c}(c_j, \ell_j)\omega_j}\left[\pi_j^+ + \theta_j^+ \frac{\partial f}{\partial K}\left(K_j^+, \ell_j^+\right)\right].$$

Moreover, the unit-free errors for the intratemporal-choice conditions (13) and (14) are also available:

$$E_2(K, \theta) = \max_{2 \le j \le N} \left| \frac{\dfrac{\partial u_j}{\partial c}(c_j, \ell_j)\tau_j}{\dfrac{\partial u_1}{\partial c}(c_1, \ell_1)\tau_1} - 1 \right|,$$

$$E_3(K, \theta) = \max_{1 \le j \le N} \left| \frac{\dfrac{\partial u_j}{\partial c}(c_j, \ell_j)\theta_j \dfrac{\partial f}{\partial \ell}(K_j, \ell_j)}{\dfrac{\partial u_j}{\partial \ell}(c_j, \ell_j)} + 1 \right|.$$

The unit-free error for the resource constraint is given by

$$E_4(K, \theta) = \left| \frac{\displaystyle\sum_{j=1}^{N}(c_j + I_j - \delta K_j + \Gamma_j)}{\displaystyle\sum_{j=1}^{N}\left(\theta_j f(K_j, \ell_j)\right)} - 1 \right|.$$

Using the above errors for the first-order conditions and the resource constraint, we compute the global $\mathcal{L}^\infty$ error on a domain of (K, θ), denoted $\mathcal{D}$, to measure the accuracy

of our solution:

$$\mathcal{E} = \max_{(K,\theta)\in\mathcal{D}} \left\{ \max_{1\le i\le 4} E_i(K,\theta) \right\}. \tag{19}$$

Note that the estimated policy functions C_j, L_j, and $\mathcal{I}_j$ should be defined not only in the domain of $(K,\theta)\in\mathcal{D}$ (in our examples, we let $\mathcal{D}=[0.7,1.3]^{2N}$), but also in a wider domain for (K^+,θ^+). Therefore, so as to get the Euler errors E_1, we should apply NLCEQ in a wider domain than we are interested in. In our examples, we choose a domain $[0.5,1.5]^{2N}$ for approximating NLCEQ policy functions, and then estimate the global error in $[0.7,1.3]^{2N}$. In addition, we could have E_2 and E_3 (and even E_4) to be zero: for any (K,θ), we let $c_1 = C_1(K,\theta)$ and compute other c_j from equation (13) instead of letting $c_j = C_j(K,\theta)$, and then compute ℓ_j from (14) instead of letting $\ell_j = L_j(K,\theta)$. This method may obtain a smaller global error as it has smaller approximation errors from C_j and L_j. But this method may require solving a complicated system of nonlinear equations, so we do not apply it in this paper for more generality.

To compute the Euler error $E_1(K,\theta)$ for a given (K,θ), we estimate the integration in (18) using a Monte Carlo simulation method with 10,000 points randomly drawn from the distribution of θ^+ (when $N\le 4$, we can use the Gauss–Hermite quadrature rule with seven quadrature nodes in each dimension for a faster run). Since the standard deviation of $F_j(k,\theta,\theta^+)$ is around the size of σ in all of our cases, the accuracy of the numerical integration is about 10^{-4}, which is acceptable for measuring the errors of NLCEQ. In our results, the global error $\mathcal{E}$ is estimated by the maximal value of $\max_{1\le i\le 4} E_i(K,\theta)$ among 10,000 randomly and uniformly drawn points (K,θ) in the domain $[0.7,1.3]^{2N}$. This is time-consuming for high-dimensional problems, but it can also be parallelized naturally. For all of our examples, we computed the standard error of the estimated expectation, and found that the standard error is one or two orders of magnitude smaller than the Euler error, so the numerical integration error is negligible in our computation of the Euler error.

4.2 *Examples for accuracy test*

We first test NLCEQ for its accuracy for the two-country real business cycle problem (i.e., $N=2$), which has four continuous state variables: two capital stocks and two productivity levels. In the transformation step of NLCEQ (Algorithm 1), we choose $T=50$ and the problem becomes

$$\widetilde{V}(K_0,\theta_0) = \max_{c,\ell,I} \sum_{t=0}^{T-1} \beta^t U(c_t,\ell_t) + \beta^T \widetilde{V}_T(K_T,\theta_T), \tag{20}$$

subject to (7) and (12) with a deterministic process of θ_t: $\ln(\theta_{t+1,j}) = \rho\ln(\theta_{t,j})$. The terminal value function $\widetilde{V}_T(K,\theta)$ is given as $U(f(K,\ell^*),\ell^*)/(1-\beta)$ with $\ell^* = (1,\dots,1)$.

In the NLCEQ method, we first use the tensor grid of Chebyshev nodes ($D+1$ nodes in each dimension) over the domain of the state variables, $[0.5,1.5]^{2N}$, and then apply degree-D complete Chebyshev polynomials in the approximation step. Since we

TABLE 1. Global errors in the $\mathcal{L}^\infty$ norm for two-country problems.

			Global Error $\mathcal{E}$			
			Degree-D Chebyshev		Level-l Smolyak	
β	γ	η	$D=2$	$D=4$	$l=1$	$l=2$
0.99	0.25	0.1	2.4(−2)	1.7(−3)	5.3(−2)	6.7(−3)
		0.5	2.1(−2)	2.0(−3)	6.5(−2)	1.0(−2)
	0.5	0.1	2.0(−2)	1.3(−3)	6.1(−2)	5.3(−3)
		0.5	2.1(−2)	1.1(−3)	6.5(−2)	6.1(−3)
0.95	0.25	0.1	2.8(−2)	2.6(−3)	5.1(−2)	9.3(−3)
		0.5	1.8(−2)	3.7(−3)	7.0(−2)	1.3(−2)
	0.5	0.1	2.0(−2)	1.5(−3)	5.7(−2)	5.6(−3)
		0.5	1.5(−2)	1.7(−3)	6.2(−2)	8.7(−3)

Note: Note that $\zeta(-j)$ represents $\zeta \times 10^{-j}$.

will apply NLCEQ to high-dimensional problems using the level-l Smolyak points and Chebyshev–Smolyak polynomials (a subset of degree-2^l complete Chebyshev polynomials) for approximation, we also try them in the low-dimensional problems to check their accuracy.

Our starting examples have a symmetric model specification, that is, we let γ_j be a constant γ and let η_j be a constant η. Table 1 lists the global errors in the $\mathcal{L}^\infty$ norm over $[0.7, 1.3]^{2N}$ for the symmetric cases with $\beta \in \{0.99, 0.95\}$, $\gamma \in \{0.25, 0.5\}$, $\eta \in \{0.1, 0.5\}$, and $\sigma = 0.01$.[13] From Table 1, we see that degree-4 complete Chebyshev polynomials have the smallest global errors at $O(10^{-3})$.[14]

We also show that NLCEQ (Algorithm 1) can very accurately solve deterministic dynamic problems. Table 2 lists the maximal Euler errors and global errors in the $\mathcal{L}^\infty$ norm over $[0.7, 1.3]^{2N}$ for the two-country problem with $\sigma = 0$ and $\beta = 0.99$, $\gamma = 0.5$, and $\eta = 0.5$ (the results are similar for other values of (β, γ, η)). As we seek higher accuracy and the only source of errors for deterministic problems comes from truncation of the infinite horizon and the value/policy function approximation, we use a large $T = 200$ and high degree approximation. From Table 2, we see that the global errors are always larger than Euler errors so it is not good enough to use Euler errors as a criterion. More-

[13]The range for θ, $[0.7, 1.3]^N$, is narrow: from (8), if $\theta_{t,j}$ is inside the range

$$\left[\exp\left(\frac{-\sqrt{2}\sigma}{1-\rho}\right), \exp\left(\frac{\sqrt{2}\sigma}{1-\rho}\right)\right],$$

which is close to $[0.7, 1.3]$, then only when $\epsilon_{t+1,j}$ and ε_{t+1} are simulated to let $\epsilon_{t+1,j} + \varepsilon_{t+1}$ be bounded in $[-\sqrt{2}, \sqrt{2}]$, can we make sure that $\theta_{t+1,j}$ is inside the same range. That is, if $\theta_{t,j}$ is at one end of the range, then it has about 16% probability that $\theta_{t+1,j}$ is outside of the range. Kollmann, Maliar, Malin, and Pichler (2011) also checked the errors of solutions of perturbation methods on the sphere in the state space centered at the steady state with a radius 0.3.

[14]Since our solutions are independent of the value of σ but the Euler errors E_1 depend on σ, we also checked the errors with $\sigma = 0.02$. We found that the global errors of $\sigma = 0.02$ are the same as with those of $\sigma = 0.01$ although the σ-related maximal Euler errors with $\sigma = 0.02$ are a bit higher.

TABLE 2. Errors in the $\mathcal{L}^\infty$ norm for two-country problems with $\sigma = 0$.

	Degree-D Chebyshev			Level-l Smolyak		
	$D=4$	$D=6$	$D=8$	$l=2$	$l=3$	$l=4$
Euler error	4.2(−5)	2.1(−5)	2.1(−5)	2.6(−4)	2.7(−5)	2.1(−5)
Global error	1.1(−3)	5.6(−5)	3.8(−6)	5.9(−3)	6.1(−4)	7.3(−5)

Note: Note that $\zeta(-j)$ represents $\zeta \times 10^{-j}$.

over, we see that our solution can reach five-digit accuracy in $\mathcal{L}^\infty$ for the deterministic problem.

4.3 *Comparison with log-linearization*

Log-linearization is the most popular method for solving dynamic stochastic models. It is also a certainty-equivalent approximation method: it computes a log-linear approximation for the policy function of a deterministic problem and uses it in simulations of the stochastic model. While this may be acceptable for states close to the deterministic steady state, it is not likely to be a good approximation beyond a small neighborhood around the steady state. This is particularly relevant if the mean of the stochastic problem is not the deterministic steady state.

We next present one two-country example where we compare NLCEQ with log-linearization. We use an example with asymmetric model specification by assuming $\gamma_1 = 0.25$ and $\eta_1 = 0.1$ for the first country, and $\gamma_2 = 1$ and $\eta_2 = 1$ for the second country in their utility functions. The other parameters are set as their default values. We use the degree-4 complete Chebyshev polynomials for approximation in NLCEQ.

For NLCEQ, the global $\mathcal{L}^\infty$ error, $\mathcal{E}$, is 0.0014, which is similar to the examples in Table 1. Kollmann et al. (2011) report the errors of solutions from other methods for this asymmetric case (Model II with $N = 2$ in their Table 4). The perturbation methods (order 1 or order 2) have large errors on the sphere in the state space centered at the steady state with a radius 0.3 (this sphere is inside our domain $[0.7, 1.3]^{2N}$). The $\mathcal{L}^\infty$ error of the log-linear approximation (i.e., the order-1 perturbation, which is linear in $\log(K)$ and $\log(\theta)$) is 0.51. Even its extended order-2 perturbation method (with quadratic polynomials in $\log(K)$ and $\log(\theta)$) has an $\mathcal{L}^\infty$ error equal to 0.21. Thus, NLCEQ is far more accurate, up to two orders of magnitude higher, than log-linearization over the wider and more relevant domain.

4.4 *Application to high-dimensional problems*

In this subsection we use NLCEQ to solve the high-dimensional RBC problems with the default parameter values and $\gamma_j \equiv 0.25$ and $\eta_j \equiv 0.1$. We use the same transformed deterministic model (20) with $T = 20$ or 50 for more countries, and use the level-l Smolyak grid and Chebyshev–Smolyak polynomials for approximation.

Table 3 lists the Euler errors and global errors in the $\mathcal{L}^\infty$ norm over $[0.7, 1.3]^{2N}$ and running times (in wall clock time) in minutes, for cases with the number of countries

TABLE 3. Errors and running times for high-dimensional RBC problems.

N	Level l	No. of Points	No. of Cores	T	Max Euler Error	Global Error	Time (minutes)
10	1	41	41	20	3.5(−3)	2.4(−2)	0.5
				50	3.6(−3)	2.6(−2)	0.4
	2	841	288	20	3.2(−3)	4.8(−3)	1.5
				50	7.1(−4)	4.3(−3)	1.4
20	1	81	81	20	2.6(−3)	1.9(−2)	0.2
				50	2.0(−3)	1.9(−2)	1.3
	2	3281	352	20	2.1(−3)	3.3(−3)	1.7
			352	50	5.8(−4)	3.1(−3)	13.5
			3281	50	5.8(−4)	3.1(−3)	1.6
50	1	201	201	20	2.3(−3)	1.8(−2)	0.8
				50	1.9(−3)	1.8(−2)	5.7
	2	20,201	2048	20	1.5(−3)	2.7(−3)	8.3
			2048	50	3.5(−4)	2.6(−3)	58.1
			20,201	50	3.5(−4)	2.6(−3)	8.6
100	1	401	401	20	1.9(−3)	1.8(−2)	2.2
200	1	801	801	20	1.6(−3)	1.8(−2)	8.0

Note: Note that $\zeta(-j)$ means $\zeta \times 10^{-j}$.

$N = 10, 20, 50, 100, 200$ (the dimension of continuous state variables is $2N$). For example, for the case with $N = 200$ countries and $T = 20$, its maximal Euler error is 0.0016 and its global error is 0.018, and it is solved in only 8 minutes.

We employ parallelism in a supercomputer. Table 3 lists the numbers of approximation points (level-l Smolyak grid) and compute cores of the supercomputer for all cases. For the level-1 Smolyak grid, the number of cores is chosen to be same as the number of points, so each core runs one approximation node corresponding to one optimization problem of the deterministic model (20). For the level-2 Smolyak grid, we see that it will be faster if we use more cores.[15]

From Table 3, we see that the level-2 Chebyshev–Smolyak polynomial approximation obtains about one more digit accuracy than the level-1 Chebyshev–Smolyak polynomial approximation for every case. Moreover, $T = 50$ does not improve much accuracy in global errors than $T = 20$ although it decreases the maximal Euler errors, while $T = 50$ is far more time-consuming. With the parallelism, although our examples have far higher numbers of countries than those in Kollmann et al. (2011), we still get the optimal solutions with acceptable accuracy in minutes, much faster than the other methods listed in Table 3 of Kollmann et al. (2011), except the perturbation methods, which will have large errors in the wide domain $[0.7, 1.3]^{2N}$.

[15]If we can employ a parallel optimization solver for one approximation point so that we can use more cores, then it could be even faster.

5. Application to a dynamic stochastic model of food and clean energy

While high dimensionality is an important computational challenge to solving dynamic stochastic problems, low-dimensional problems may have other challenges of their own. For example, if there are occasionally binding constraints for state and/or control variables, then the kinks in the value/policy functions often make the problem difficult to solve. One advantage of NLCEQ is that it can solve the problems with inequality constraints that occasionally bind. Here we illustrate this by applying NLCEQ to solve a dynamic stochastic model of food and clean energy with a constraint on pollution. In Appendix B, we also illustrate this for a RBC model with a constraint on investment.

In Appendix B we solve problems with occasionally binding constraints and their steady state in the center of their state space. However, in some problems like the one shown below, the steady state is on the boundary of the feasible space of states and is approached from only one side. Moreover, we know that usually the solution at the initial states is the most important, but the initial states could be far away from the steady state. For these problems, log-linearization is not reliable because it can only give good solutions around the steady state. This section applies NLCEQ to solve such a low-dimensional problem, where there are no reachable steady states, its initial states stay in a corner region, and there are inequality constraints that occasionally bind.

5.1 *Model setup*

In this example we apply NLCEQ method to solve a stochastic version of a dynamic model of food and clean energy introduced by Chakravorty, Magne, and Moreaux (2008). This stylized model serves as a vehicle for developing and solving more complex models aimed at understanding complicated real world economic problems related to biofuels and global land use.[16]

We assume a single-country economy with two primary factors, land and fossil fuels (e.g., oil). The economy has a fixed endowment of land, L, which can be used to produce food or biofuels. Let $L_{t,f}$ and $L_{t,b}$ be the amounts of land dedicated to produce food and biofuels crops at time t, respectively. The residual land, $L - L_{t,f} - L_{t,b}$, is unused. The total land constraint is

$$L_{t,f} + L_{t,b} \leq L. \tag{21}$$

The economy has also some stock of extractable fossil fuel resource (e.g., oil), S, with initial stock S_0. At period t the economy extracts s_t units of fossil fuel, so we have

$$S_{t+1} = S_t - s_t. \tag{22}$$

Production of food employs only the land resource. The production function for food crops is linear in the amount of land used. There is one stochastic tipping event: once

[16]For direct model extension, see Chakravorty, Hubert, Moreaux, and Nostbakken (2012). Steinbuks and Hertel (2014) present a closely related computable partial equilibrium model of land use at the global scale, which incorporates additional sectors and nonhomothetic preferences. While all these works assume perfect foresight, Cai, Steinbuks, Elliott, and Hertel (2014) incorporate uncertainty in global land use decision models.

it happens, it adversely affects the production of food crops at a level $\overline{J} < 1$, and this damage is irreversible for any later periods. Let the food production per unit of land be $\theta_{t,f}$ before the tipping event happens (we assume $\theta_{t,f} = 1$ for simplicity). Thus, the production function for food crops is

$$y_{t,f} \equiv (1 - J_t)\theta_{t,f} L_{t,f}, \tag{23}$$

where J_t denotes the stochastic damage level: $J_t = 0$ if the tipping event has not happened before time t; otherwise $J_t = \overline{J}$. We assume that the stochastic process J_t is a Markov chain with the transition probability matrix

$$P = \begin{pmatrix} 1 - p_{21} & 0 \\ p_{21} & 1 \end{pmatrix}, \tag{24}$$

where p_{21} is the probability that the tipping event happens in 1 year, and the $(2, 2)$ element of P is 1 because of the irreversibility of the tipping damage.[17]

Production of energy employs both fossil fuels and biofuels, and it is a constant elasticity of substitution (CES) function,[18]

$$y_{t,e} \equiv A\big[\alpha(\theta_{t,b} L_{t,b})^{\lambda} + (1 - \alpha)(s_t)^{\lambda}\big]^{\frac{1}{\lambda}}, \tag{25}$$

where A is the technology parameter of energy production, $\theta_{t,b}$ is the return of biofuel crops per unit of land (we assume $\theta_{t,b} = 1$ for simplicity), α is the cost share of biofuel feedstocks, and λ is the CES function parameter proportional to the elasticity of substitution of oil for biofuels.

Let M be the mass of pollution (e.g., carbon concentration), with the initial stock M_0. The law of accumulation of pollution is

$$M_{t+1} = \mu s_t + (1 - \delta)M_t, \tag{26}$$

where μ is the amount of pollution produced from combustion of 1 unit of fossil fuel (relative to biofuels), and δ is the natural rate of pollution absorption by the earth atmosphere and oceans.

The nonland production costs of food and biofuels are linear,

$$c_{t,j} \equiv \psi_j L_{t,j}, \tag{27}$$

where ψ_j is the food or biofuels cost per unit of land, for $j \in \{f, b\}$. The fossil fuel extraction cost is

$$c_{t,e} \equiv \psi_{1e} s_t (S_t)^{-\psi_{2e}}, \tag{28}$$

[17]Cai, Judd, and Lontzek (2015a) assume a more general tipping process with irreversible damage on production. We use this simpler version for illustrating the application of NLCEQ without loss of generality.

[18]Chakravorty, Magne, and Moreaux (2008) assume that fossil fuels and biofuels are perfect substitutes. Our modification of the original model makes it more realistic (as biofuels substitute imperfectly for petroleum in final liquid fuel demand), and avoids numerical problems caused by the bang–bang solutions of Chakravorty, Magne, and Moreaux (2008).

where ψ_{1e} and ψ_{2e} are two positive parameters, so that the oil extraction cost increases with depletion of the oil stock S_t.[19] Following Cai et al. (2014) we assume that at each time t there is an exogenous endowment of other primary resources (e.g., labor, physical and human capital, and materials), Π_t. A part of this endowment is used in food and energy sectors for oil extraction and refining, and production of food and biofuels. The remaining amount of other primary resources is converted to other goods, which are consumed in final demand. The production of other goods is linear in the remaining amount of other primary resources with transformation coefficient $\theta_{t,o}$ (for simplicity we assume $\theta_{t,o} = 1$):

$$y_{t,o} \equiv \theta_{t,o}(\Pi_t - c_{t,f} - c_{t,b} - c_{t,e}). \tag{29}$$

We assume a utility function, which is additively separable in food, energy, and other goods (positively), as well as pollution stock (negatively),

$$u(y_{t,f}, y_{t,e}, y_{t,o}, M_t) = \frac{(y_{t,f})^{1-\frac{1}{\gamma_f}}}{1-\frac{1}{\gamma_f}} + B_e \frac{(y_{t,e})^{1-\frac{1}{\gamma_e}}}{1-\frac{1}{\gamma_e}} + B_o \frac{(y_{t,o})^{1-\frac{1}{\gamma_o}}}{1-\frac{1}{\gamma_o}} - B_M M_t^{\eta}, \tag{30}$$

where γ_f, γ_e, γ_o, B_e, B_o, B_M, and η are positive parameters.[20] In addition, following Chakravorty, Magne, and Moreaux (2008) we assume that pollution stock is capped at a certain threshold by an international agreement, which is not necessarily consistent with an unconstrained country pollution optimum. That is, we assume that

$$M_t \leq \overline{M}$$

for all t with a given upper bound $\overline{M}$.

The objective of the social planner is to maximize the expected sum of the discounted utility with a discount factor β. That is, the social planner's problem is

$$V(S_0, M_0, J_0) = \max_{L_{t,f}, L_{t,b}, s_t \geq 0} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t \left[u(y_{t,f}, y_{t,e}, y_{t,o}, M_t)\right]\right\} \tag{31}$$

subject to (21), (22), (26), and $M_t \leq \overline{M}$, with three nonnegative control variables at each time t: land dedicated to food crops $L_{t,f}$, land dedicated to biofuels $L_{t,b}$, and extracted fossil fuels s_t.

There are two endogenous state variables: fossil fuel stock, S_t, governed by equation (22), and accumulated pollution, M_t, governed by equation (26), and one exogenous discrete state variable: damage level, J_t. Note that the steady endogenous states imply

[19] This is another modification of Chakravorty, Magne, and Moreaux (2008), who assume linear extraction costs for simplicity. Our cost specification is more commonly used in the environmental economics literature (see, e.g., Nordhaus and Boyer (2003)).

[20] This formulation is different from Chakravorty, Magne, and Moreaux (2008), who do not incorporate disutility from pollution in the utility function, but it is broadly consistent with the environmental economics and growth literature; see, for example, Andreoni and Levinson (2001) and Xepapadeas (2005).

zero extraction of fossil fuels and zero pollution from (22) and (26), that is, $s_t = 0$ and $M_t = 0$. This means that the steady state will not be reached in the optimal path in a finite horizon as M_t is always greater than 0 in our model (although M_t converges to 0 as t goes to infinity).

We set the total amount of land, L, equal to 1. We also assume that the tipping event has not happened at the initial time (i.e., $J_0 = 0$). In our example, we let $\overline{J} = 0.1$, $p_{21} = 0.0034$, $\beta = 0.95$, $\alpha = 0.5$, $\lambda = 0.5$, $\delta = 0.001$, $\mu = 0.25$, $\gamma_f = \gamma_e = \gamma_o = 0.5$, $B_e = B_o = 0.5$, $B_M = 1$, $\eta = 4$, $\Pi_t \equiv 1$, $A = 1$, $\psi_{1e} = 0.4$, $\psi_{2e} = 1$, $\psi_f = 0.3$, $\psi_b = 0.5$, and $\overline{M} = 1.06$.

5.2 *Numerical results*

Since S_t is always nonincreasing over time and M_t has an upper bound $\overline{M} = 1.06$, we set the approximation domain for the value/policy functions as $S_0 \in [0.01, 1]$ and $M_0 \in [1, 1.06]$, for each $J_0 \in \{0, \overline{J}\}$. We set the length of time path equal to $T = 200$ periods for the dynamic model of food and clean energy in the deterministic model transformed from (31) in the transformation step of NLCEQ (Algorithm 1). In the transformation step, if $J_0 = 0$, then we change J_t to its unconditional expectation at time t, $p_{t,1,2}\overline{J}$, where $p_{t,1,2} = 1 - (1 - p_{21})^t$ (the second element of the vector $P^t(1, 0)^\top$ with P given by (24)) is the probability that the tipping event happens at a time not later than t; if $J_0 = \overline{J}$, then it has been a deterministic model as J_t will always be $\overline{J}$ because of the irreversibility of the tipping damage. We assume the terminal value function to be $u(y_{T,f}, y_{T,e}, y_{T,o}, M_T)/(1-\beta)$, where $(y_{T,f}, y_{T,e}, y_{T,o})$ are given by (23), (25), and (29) with terminal extraction $s_T = 0.01X_T$.

For this specific problem, we can compute the true value/policy functions of the model (31) so as to measure the accuracy of solutions from NLCEQ, so we do not need to test the accuracy using its Euler equations errors like we did for the RBC model (11). When $J_0 = 0$, the problem can be solved as an optimal control problem by a large-scale optimization solver in the form

$$V_{J_0=0}(S_0, M_0) = \max_{L_{t,f}, L_{t,b}, s_t \geq 0} \left\{ \sum_{t=0}^{T} \beta^t \left[\sum_{j=1}^{2} p_{t,1,j} u(y_{t,f,j}, y_{t,e}, y_{t,o}, M_t) \right] \right\} \tag{32}$$

subject to (21), (22), (26), and $M_t \leq \overline{M}$, where $p_{t,1,1} = 1 - p_{t,1,2}$, $y_{t,f,1} = \theta_{t,f} L_{t,f}$, and $y_{t,f,2} = (1 - \overline{J})\theta_{t,f} L_{t,f}$. When $J_0 = \overline{J}$, the problem is deterministic:

$$V_{J_0=\overline{J}}(S_0, M_0) = \max_{L_{t,f}, L_{t,b}, s_t \geq 0} \left\{ \sum_{t=0}^{T} \beta^t \big[u(y_{t,f,2}, y_{t,e}, y_{t,o}, M_t) \big] \right\} \tag{33}$$

subject to (21), (22), (26), and $M_t \leq \overline{M}$.

We use the initial-time solutions for the control variables at approximation nodes of S_0 and M_0 to construct the optimal policy functions for each $J_0 \in \{0, \overline{J}\}$. In the NLCEQ method, we use the tensor grid of Chebyshev nodes ($D + 1$ nodes in each dimension) over the domain of the continuous state variables, $[0.01, 1] \times [1, 1.06]$, in the optimization

TABLE 4. Errors of policy functions from NLCEQ for the model of food and clean energy.

	Error for L_f		Error for s	
D	$\mathcal{L}^\infty$	$\mathcal{L}^1$	$\mathcal{L}^\infty$	$\mathcal{L}^1$
4	9.4(−3)	1.2(−3)	1.8(−3)	2.8(−4)
6	3.7(−3)	5.8(−4)	1.1(−3)	1.6(−4)
8	2.2(−3)	3.0(−4)	7.8(−4)	9.5(−5)
10	2.1(−3)	1.8(−4)	6.0(−4)	6.1(−5)
20	8.4(−4)	4.8(−5)	2.6(−4)	1.9(−5)

Note: Note that $\zeta(-j)$ means $\zeta \times 10^{-j}$.

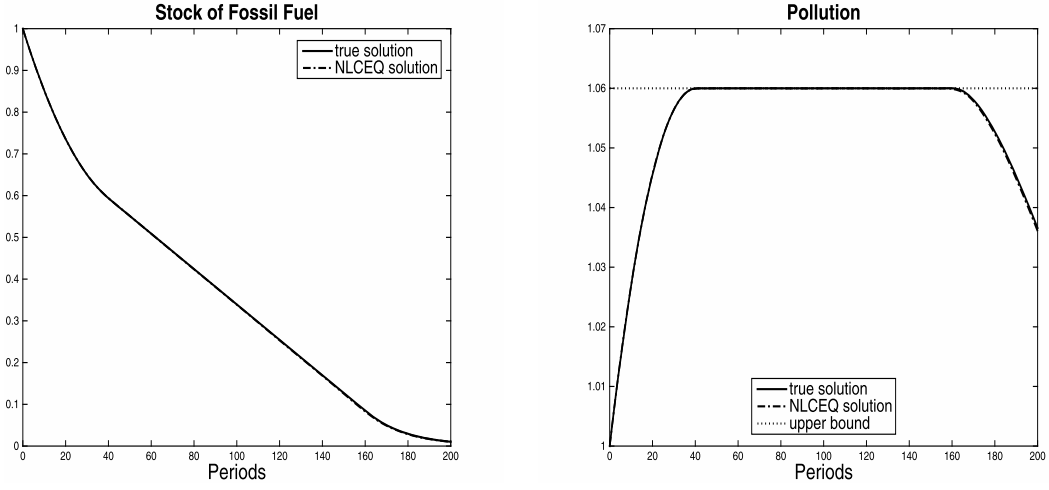

FIGURE 1. Pre-tipping state paths for the model of food and clean energy.

step. We apply the degree-D complete Chebyshev polynomials in the approximation step, for each $J_0 \in \{0, \overline{J}\}$.

Table 4 reports absolute errors[21] of the policy functions computed by NLCEQ over the approximation domain and all discrete state values for various degrees D. We see that NLCEQ gives $O(10^{-4})$ accuracy for the policy functions, after we use the degree-20 polynomial approximation. Figure 1 shows the pre-tipping paths of the stock of fossil fuel X_t and pollution M_t, and we see that our NLCEQ solutions are very close to true solutions, and that M_t hits its upper bound after 40 periods and sticks on the bound for more than 100 periods.

Note that $M_{t+1} \leq \overline{M}$ implies that

$$s_t \leq \frac{1}{\mu}\left(\overline{M} - (1-\delta)M_t\right)$$

[21]We do not use relative errors because the solution of the fossil fuel extraction s could be nearly 0.

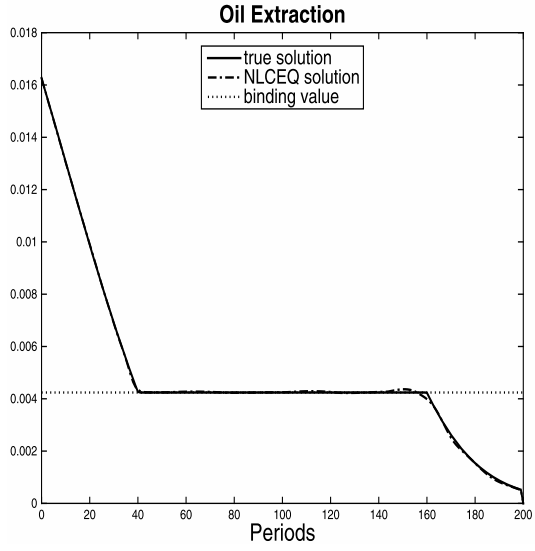

FIGURE 2. Pre-tipping path for the optimal fossil fuel extraction.

from the law of transition for M_t (i.e., equation (26)). Thus, if M_t reaches its upper bound, then $s_t \leq \delta\overline{M}/\mu = 0.00424$ will also be binding. Figure 2 shows this with the pre-tipping optimal fossil fuel extraction path, and it also shows that our degree-20 complete Chebyshev polynomial from NLCEQ can approximate the true policy function very well, although it has kinks.

6. Application to a new Keynesian model with zero lower bound

In this section, we apply NLCEQ for competitive equilibrium (Algorithm 2) to solving a New Keynesian model with zero lower bound (ZLB). We use the New Keynesian model in Guerrieri and Iacoviello (2015), a variant of the new Keynesian model with ZLB that is used in Fernández-Villaverde et al. (2015) and Maliar and Maliar (2015). The values of parameters are also chosen from Guerrieri and Iacoviello (2015).

6.1 *Model overview*

The model consists of a representative household, a government, a final-good firm, and intermediate firms. At each time t the government issues bonds that expire at $t+1$ and the nominal interest rate for the bonds is r_t (the time unit is a quarter). A representative household consumes c_t with a price p_t from the final-good firm, buys newly issued bonds with a total face value b_t from the government, sells the expired bonds b_{t-1}, earns wages from labor supply ℓ_t with a wage rate w_t, and receives a lump-sum transfer T_t from the government and profit Π_t from all firms. The budget constraint is

$$p_t c_t + \frac{b_t}{1+r_t} = w_t \ell_t + b_{t-1} + T_t + \Pi_t. \tag{34}$$

The representative household chooses consumption c_t, labor supply ℓ_t, and government bonds b_t to maximize

$$\max_{c_t,\ell_t,b_t} \mathbb{E}\left\{\sum_{t=0}^{\infty}\left(\prod_{i=0}^{t}\beta_i\right)U(c_t,\ell_t)\right\} \tag{35}$$

subject to the budget constraint (34), where

$$U(c,\ell)=\ln(c)-\frac{\ell^{1+\eta}}{1+\eta}$$

with $\eta=1$. The discount factor β_t is a stochastic process following

$$\ln(\beta_{t+1})=(1-\rho)\ln\left(\beta^*\right)+\rho\ln(\beta_t)+\sigma\epsilon_{t+1}, \tag{36}$$

where $\epsilon_t \sim i.i.d.\mathcal{N}(0,1)$, $\beta^*=0.994$, $\rho=0.8$, and $\sigma=0.005$. The first-order conditions of the household problem imply

$$1=\mathbb{E}_t\left\{\beta_{t+1}\frac{1+r_t}{\pi_{t+1}}\frac{c_t}{c_{t+1}}\right\} \tag{37}$$

and

$$w_t=p_t c_t \ell_t^{\eta}, \tag{38}$$

where $\pi_t \equiv p_t/p_{t-1}$ is the gross inflation rate.

The final-good firm purchases intermediate goods from intermediate firms to produce a final good y_t and sell it at a price p_t. The intermediate firms are assumed to have Calvo-type prices: a fraction $1-\theta$ of the firms have optimal prices and the remaining fraction θ of the firms keep the same price as in the previous period. Here the Calvo parameter θ is set as 0.9. In Appendix C we describe the detailed model specification for the final- and intermediate- goods firms and derive the equilibrium conditions

$$1=\frac{1}{\chi_{t,1}}\left(y_t\ell_t^{\eta}+\theta\mathbb{E}_t\left\{\beta_{t+1}\pi_{t+1}^{\alpha}\chi_{t+1,1}\right\}\right), \tag{39}$$

$$1=\frac{1}{\chi_{t,2}}\left(\frac{y_t}{c_t}+\theta\mathbb{E}_t\left\{\beta_{t+1}\pi_{t+1}^{\alpha-1}\chi_{t+1,2}\right\}\right), \tag{40}$$

$$q_t=\frac{\alpha\chi_{t,1}}{(\alpha-1)\chi_{t,2}}=\left(\frac{1-\theta\pi_t^{\alpha-1}}{1-\theta}\right)^{\frac{1}{1-\alpha}}, \tag{41}$$

$$v_{t+1}=\frac{\ell_t}{y_t}=(1-\theta)q_t^{-\alpha}+\theta\pi_t^{\alpha}v_t, \tag{42}$$

where $\alpha=6$, for any time $t\geq 0$.

Let π^*, r^*, and y^* be the steady-state gross level of inflation, the steady-state nominal interest rate, and the steady-state output, respectively. Let the government spending g_t

be always equal to $s_g y_t$ with $s_g = 0.2$. From the market clearing condition $y_t = c_t + g_t$, we have

$$c_t = (1 - s_g) y_t. \tag{43}$$

Following the Taylor rule (Taylor (1993)), we have the nominal interest rate as

$$r_t = \max(z_t, 0) \tag{44}$$

with

$$z_t = \left(1 + r^*\right)\left(\frac{\pi_t}{\pi^*}\right)^{\phi_\pi}\left(\frac{y_t}{y^*}\right)^{\phi_y} - 1, \tag{45}$$

where we choose $\phi_\pi = 2.5$, $\phi_y = 0.25$, and $\pi^* = 1.005$. We have $r^* = \pi^*/\beta^* - 1$ from (37), and the formula for y^* is given in Appendix D. Equation (44) implies that the actual policy rate r_t must be nonnegative, and this zero lower bound will be binding when the notional policy rate z_t is smaller than 0.

We now have one endogenous state variable v_t and one exogenous state variable β_t, the system of equilibrium equations (37) and (39)–(45), and the exogenous process (36). We apply NLCEQ (Algorithm 2) to compute the policy functions for $(c_t, \chi_{t,1}, \chi_{t,2}, \pi_t, q_t, v_t, \ell_t, y_t, r_t, z_t)$, and with the NLCEQ solution it follows that the consumption price is computed by $p_t = \pi_t p_{t-1}$ and then the wage is computed by (38).

6.2 *Numerical results*

In NLCEQ (Algorithm 2), we transform the stochastic process (36) to be deterministic as $\ln(\beta_{t+1}) = \rho \ln(\beta^*) + (1 - \rho)\ln(\beta_t)$, and then transform the system of equilibrium equations (37) and (39)–(45) to be deterministic by canceling their corresponding expectation operator, and choose $T = 200$ in the transformed system (5).

In the approximation step of Algorithm 2, since the control variables $(c_t, \ell_t, r_t, q_t, z_t, \pi_t)$ can be simply substituted, we only need to approximate three control variables $(\chi_{t,1}, \chi_{t,2}, y_t)$ over two state variables (v_t, β_t). We use the relative $\mathcal{L}^1$ norm (relative to the steady-state values given in Appendix D, $(v^*, \chi_1^*, \chi_2^*, y^*)$, respectively) in the objective of (5). We use the tensor grid of Chebyshev nodes ($D + 1$ nodes in each dimension) and degree-D complete Chebyshev polynomials to approximate the policy functions. We want to get a solution over the state space $[1, 1.04] \times [0.96, 1.03]$, a slightly wider domain than that used in Guerrieri and Iacoviello (2015), so it will have a higher chance of a binding ZLB. Thus we choose the approximation domain of (v, β) as $[1, 1.045] \times [0.936, 1.056]$ so that the next simulated states transited from current states in $[1, 1.04] \times [0.96, 1.03]$ using (42) and (36) will be inside the approximation domain.

Table 5 reports unit-free errors of the NLCEQ solution for various degrees D. The errors are computed on the domain $[1, 1.04] \times [0.96, 1.03]$. The global errors are defined in a similar way in Section 4.1, while we need to estimate the unit-free errors for the equations (37), (39), and (40), where we use the 15-point Gauss–Hermite quadrature rule

TABLE 5. Errors of the NLCEQ solution with degree-D complete Chebyshev polynomials for the New Keynesian DSGE model with ZLB.

	$D=4$	$D=6$	$D=8$	$D=10$
$\mathcal{L}^\infty$ global error	4.4(−3)	3.1(−3)	2.2(−3)	1.8(−3)
$\mathcal{L}^1$ global error	8.1(−4)	6.0(−4)	5.6(−4)	4.6(−4)

Note: Note that $\zeta(-j)$ means $\zeta \times 10^{-j}$.

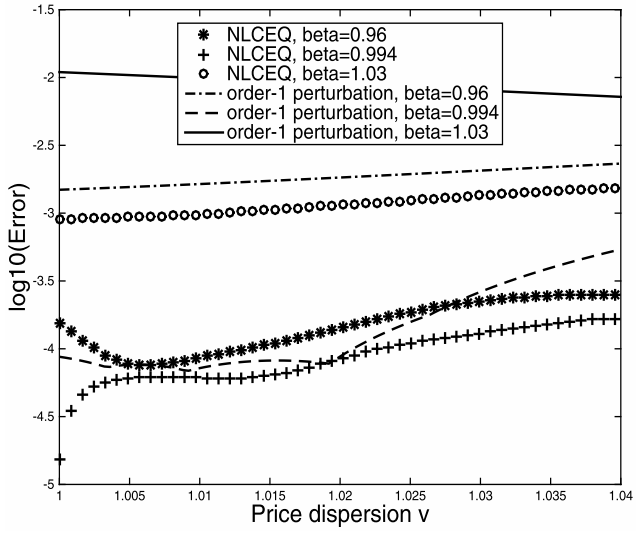

FIGURE 3. Errors of the NLCEQ solution for the New Keynesian DSGE model with ZLB.

to estimate the integrations. We see that they achieve $O(10^{-3})$ errors in $\mathcal{L}^\infty$ or $O(10^{-4})$ errors in $\mathcal{L}^1$, and a higher degree approximation improves the accuracy.

We also solve the model with the order-1 (linearization) and order-2 perturbation methods. The order-1 perturbation gives an error 0.011 in $\mathcal{L}^\infty$ and 0.0014 in $\mathcal{L}^1$. The order-2 perturbation does not improve the accuracy: its error is 0.012 in $\mathcal{L}^\infty$ and 0.0012 in $\mathcal{L}^1$. NLCEQ is almost 1 digit more accurate than the perturbation methods.

The comparison between NLCEQ and the linearization method is also shown in Figure 3, which shows the global errors of NLCEQ with degree-10 complete Chebyshev polynomials and of the order-1 perturbation method when $\beta = 0.96, 0.994, 1.03$. We see that NLCEQ is always more accurate than the linearization method. The errors are smaller when β is closer to its steady state $\beta^* = 0.994$. When β is the largest (i.e., $\beta = 1.03$), the errors are the largest because a higher β implies a higher chance of a binding ZLB (about 24% state points in $[1, 1.04] \times [0.96, 1.03]$ have a binding ZLB).

Figure 4 shows impulse responses of interest rate r_t, inflation π_t (the figure shows the net inflation rate in percent, i.e., $100(\pi_t - 1)\%$), and output y_t (the figure shows deviation of output from the steady state in percent, i.e., $100(y_t/y_{ss} - 1)\%$) to a shock of discount factor β_1 (with $v_1 = v_{ss}$). The left panel of the figure shows responses to a shock that brings β_1 up to 1.03, and the right panel shows responses to a shock that brings β_1 down

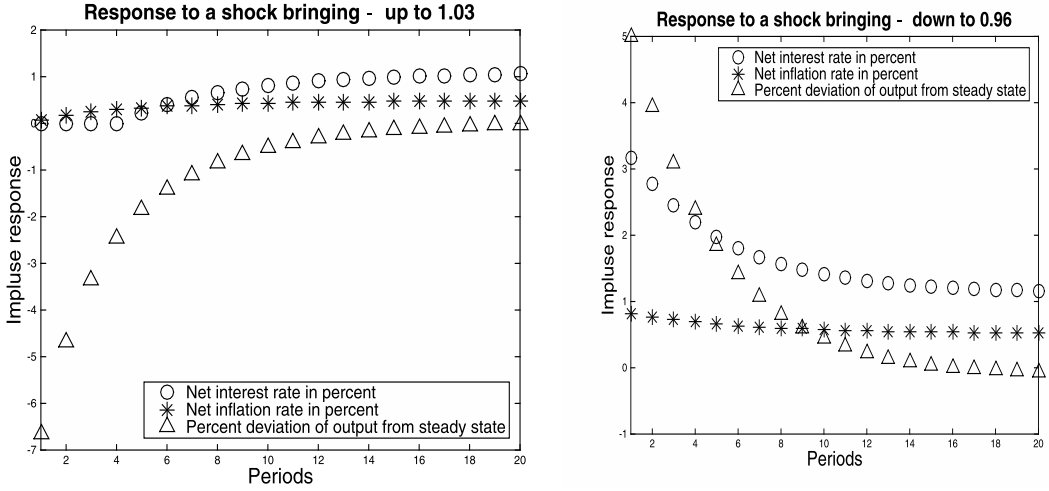

FIGURE 4. Impulse responses to a shock of discount factor.

TABLE 6. Errors of the NLCEQ solution with degree-D complete Chebyshev polynomials for the New Keynesian DSGE model with ZLB and $\sigma = 0$.

	$D = 10$	$D = 20$	$D = 50$	$D = 100$
$\mathcal{L}^\infty$ global error	1.3(−3)	7.3(−4)	5.2(−4)	1.7(−4)
$\mathcal{L}^1$ global error	2.3(−4)	8.5(−5)	4.6(−5)	1.3(−5)

Note: Note that $\zeta(-j)$ means $\zeta \times 10^{-j}$.

to 0.96. We see that the interest rate hits the ZLB in the first four periods in the left panel, and all three responses (r_t, π_t, y_t) are decreasing functions of β (β_t decreases along time t in the left panel, and β_t increases along time t in the right panel), and they are almost steady after 20 periods (i.e., 5 years).

We also show that NLCEQ (Algorithm 2) can solve deterministic competitive equilibrium problems very accurately. Table 6 lists global errors in the $\mathcal{L}^\infty$ norm over $[1, 1.04] \times [0.96, 1.03]$ for the New Keynesian DSGE problem with ZLB and $\sigma = 0$. We choose a large $T = 300$ and a higher degree approximation so as to get higher accuracy. We see that NLCEQ reaches about 4-digit accuracy in $\mathcal{L}^\infty$ for the optimal policy functions to the deterministic competitive equilibrium problem.

7. CONCLUSION

We have shown that NLCEQ can be applied to solve dynamic stochastic problems with acceptable accuracy when we combine modern approximation optimization methods with parallel computing architectures. Examples include high-dimensional optimal stochastic growth problems with up to 400 state variables and three problems with occasionally binding constraints, including a dynamic stochastic model of food and clean energy, and a New Keynesian DSGE model with zero lower bound. This approach greatly

expands the range of problems that can be solved well globally, and clearly dominates any form of linearization.

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