Bond Ladders and Optimal Portfolios

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We analyze complex bond portfolios within the framework of a dynamic general equilibrium asset-pricing model. Equilibrium bond portfolios are nonsensical and imply a trading volume that vastly exceeds observed trading volume on financial markets. Instead, portfolios that combine bond ladders with a market portfolio of equity assets are nearly optimal investment strategies. The welfare loss of these simple investment strategies, when compared to the equilibrium portfolio, converges to zero as the length of the bond ladder increases. This article, therefore, provides a rationale for naming bond ladders as a popular bond investment strategy. (*JEL* G11)

This article provides a rationale for naming bond ladders as a popular strategy for bond portfolio management. Laddering a bond portfolio requires buying and holding equal amounts of bonds that will mature over different periods. When the shortest bond matures, an equal amount of the bond with the longest maturity in the ladder is purchased. Many bond portfolio managers argue that laddering tends to outperform other bond strategies because it reduces both market price risk and reinvestment risk for a bond portfolio in the presence of interest rate uncertainty. Despite the popularity of bond ladders as a strategy for managing investments in fixed-income securities, there is surprisingly little reference to this subject in the economics and finance literature. In this article, we analyze complex bond portfolios within the framework of a dynamic general equilibrium asset-pricing model. We

characterize the resulting optimal equilibrium stock and bond portfolios and document that optimal bond investment strategies are nonsensical as well as imply a huge trading volume that bears no relationship to observed transaction volume in bond markets. The main contribution this article makes is to show that complete ladders with all bonds in the economy, combined with a market portfolio of equity assets, are excellent investment strategies, in the sense that they are nearly optimal.

For many investors, bonds of different maturities constitute an important part of their investment portfolios. When the maturity date of a bond does not coincide with an investor's investment horizon, he faces two possible risks. If he sells a bond with time remaining until maturity, he is exposed to market price risk because changes in interest rates may strongly affect the value of the bonds in his portfolio. If bonds in the portfolio mature before the investor needs the invested funds, then he is exposed to reinvestment risk, i.e., the risk that he will not be able to reinvest the returned principal at maturity with the same interest rate as that of the initial investment. Instead, he is forced to roll over maturing bonds into new investments at uncertain interest rates. Reinvestment risk also arises if the investor receives periodic payments from a security, such as periodic coupon payments from a bond, long before its maturity date. A popular tool for lessening the impact of both market price and reinvestment risk is bond ladders.

An investor builds a bond ladder by investing an equal amount of capital into bonds that will mature on different dates. For example, an investor may want to create a ladder of bonds maturing in one, two, three, four, and five years. The strategy is then to invest one-fifth of the capital into bonds of each maturity. Once the one-year bond matures, the returned principal, and possible coupon payments from all five bonds, will be reinvested into a new five-year bond. At this point, the bond portfolio consists again of investments in bonds of each maturity. This bond portfolio strategy delivers much more stable returns over time than does investing the entire capital into bonds of identical maturity, since only a portion of the portfolio matures at any one time.

Many financial advisors advocate to investors the creation of bond ladders (see, e.g., Bohlin and Strickland 2004 or Gardner and Gardner 2004). Morgan Stanley[1] advertises laddered portfolio strategies as a way to save for retirement and college. Strickland et al. (2009) stresses that laddered bond portfolios yield consistent returns with reduced market price and reinvestment risk.

Despite the well-documented advantages and resulting popularity of bond ladders as a strategy for managing bond investments, there is, to the best of our knowledge, no thorough analysis of bond ladders in modern portfolio theory. The classical portfolio literature (see French 2008 for a history), starting with Markowitz (1952) and Tobin (1958), examines investors' portfolio decisions in

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[1] Morgan Stanley, Bond Investment Strategies for the Way You Live (accessed March 31, 2011), http://www.morganstanleyindividual.com/investmentproducts/bonds/strategies/default.asp.

one-period models. One-period models, by their very nature, do not allow the analysis of bond ladders. The first decade of the 21st century has seen a rapidly evolving literature on optimal asset allocation in stochastic environments. One string of this literature builds on the general dynamic continuous-time framework of Merton (1973) and assumes exogenously specified stochastic processes for stock returns or the interest rate. Recent examples of this literature include Brennan and Xia (2000) and Wachter (2003), among many other works. A second string of literature uses discrete-time factor models to examine optimal asset allocation (see, e.g., Campbell and Viceira 2001, 2002). Most of these papers focus on aspects of the optimal choice of the stock-bond-cash mix but do not examine the details of a stock or bond portfolio. For reasons of tractability, a particular feature of these factor models is that only very few factors are included. So, in turn, only a few assets are needed for security markets to be complete. For example, the model of Brennan and Xia (2002) can exhibit complete security markets with only four securities and only two of which are bonds. Also, Campbell and Viceira (2001) report computational results on portfolios with only three-month and ten-year bonds. Due to the small number of bonds, the described portfolios in these models do not include bond ladders. Analyzing more bonds in these models would certainly be possible, but additional bonds would be redundant securities, since markets are already complete. As a result, there would be a continuum of optimal asset allocations, so any further analysis of particular bond portfolios would depend on quite arbitrary modeling choices. In sum, neither the classical finance literature of one-period models nor the modern literature on optimal asset allocation in dynamic models can adequately analyze portfolios in the presence of large families of (nonredundant) bonds in a stochastic dynamic framework. An immediate consequence is that neither literature can examine bond ladders and their impact on investors' welfare. These observations motivate the current article.

We employ a Lucas-style (Lucas 1978) discrete-time, infinite-horizon general equilibrium model with a finite set of exogenous shocks per period for our analysis of complex bond portfolios because this model offers three advantages. First, when markets are dynamically complete, efficient equilibria are stationary. This feature allows for a simple description of equilibrium. Second, general equilibrium restrictions preclude us from making possibly inconsistent assumptions on agents' tastes and asset price processes. Instead, general equilibrium conditions enforce a perfect consistency between tastes, stock dividends, and the prices of all securities,[2] making the model an excellent expositional tool for our analysis. Third, we can include any desired number of financial securities without causing asset redundancy by choosing a sufficiently

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[2] The traditional partial-equilibrium approach simply postulates asset return processes. These processes could be inconsistent in equilibrium. A better approach in partial-equilibrium analysis is to postulate a pricing kernel and to then derive asset prices from the assumed kernel (see, e.g., Campbell and Viceira 2001).

large number of exogenous shocks. This facet of the model makes it ideally suited for the analysis of portfolios with many stocks and bonds. Into this model, we then introduce the classical assumption of equicautious HARA utility for all agents. This assumption guarantees that consumption allocations follow a linear sharing rule (see Wilson 1968; Rubinstein 1974a,b). Linear sharing rules imply that portfolios exhibit the classical property of two-fund monetary separation (Hakansson 1969; Cass and Stiglitz 1970), if agents can trade a riskless asset (see Rubinstein 1974a,b). In our infinite-horizon economy, a consol, i.e., a perpetual bond, plays the role of the riskless asset. In the presence of a consol, agents hold the market portfolio of all stocks and have a position in the consol. If the consol is replaced by a one-period bond ("cash"), then two-fund monetary separation fails to hold (generically). The agents no longer hold the market portfolio of stocks. Our analysis of economies with a single bond serves us as a helpful benchmark for our subsequent analysis of portfolios with many bonds.

We begin our analysis of complex bond portfolios with numerical experiments that lead to several interesting results. We first replicate these results in a properly calibrated version of the model and then make the observations that follow. First, although agents' stock portfolios deviate from the market portfolio, they rapidly converge to the market portfolio as the number of states and bonds in the economy grows. Second, as the stock portfolios converge to the market portfolio agents' bond portfolios, they effectively synthesize the consol. The agents use the available bonds with finite maturity to approximately generate the safe income stream that a consol would deliver exactly. Third, the equilibrium portfolios of the bonds with relatively short maturity approximately constitute a bond ladder, i.e., we observe an endogenous emergence of bond ladders as a substantial part of optimal portfolios. Fourth, the portfolios of bonds with the longest maturity significantly deviate from a ladder structure. Equilibrium positions are implausibly large, and the implied trading volume bears no relation to actual bond markets.

The numerical results motivate the further analysis present in the article. We establish sufficient conditions under which the observed separation between the stock and the bond market holds not only approximately but exactly. Specifically, we state conditions on the underlying stochastic structure of stock dividends that guarantee that the consol can be perfectly replicated by a few finite-maturity bonds. When this happens, the two-fund monetary separation holds in generalized form. Each investor divides her wealth between the market portfolio and the bond portfolio that is replicating the consol. Our conditions hold for natural specifications of exogenous shocks, but they are nongeneric. Small perturbations of the stochastic structure of stock dividends destroy the exact replication property.

The bond portfolios that are replicating the consol (either approximately or exactly) always exhibit the same qualitative properties once the number of bonds is sufficiently large. The portfolios of short-term bonds display

a laddered structure, while the holdings of long-term bonds exhibit large fluctuations that render them unrealistic. These results motivate the final part of our analysis. We examine how well an investor who is restricted to hold the market portfolio of stocks and a ladder of all bonds available in the economy can do. We perform a welfare comparison of equilibrium holdings and laddered portfolios for illustrative examples as well as for the calibrated model. We find that such a simple investment strategy[3] is an excellent alternative to the equilibrium portfolio. The welfare loss of the simple portfolio is very small and converges to zero as the number of bonds increases. In fact, we find an important role for redundant bonds in that they do not increase the span of the traded securities, since adding bonds with a previously unavailable long maturity improves the performance of bond ladder strategies. We also show that the optimal portfolio weights between the bond ladder and the market portfolio deviate from the allocation between a consol and the market portfolio. The reason for this deviation is that the bond ladder decreases the reinvestment risk, but it cannot decrease this risk to zero entirely.

The remainder of this article is organized as follows: Section 1 presents the basic dynamic general equilibrium asset market model. Section 2 discusses the classical two-fund separation theory for our dynamic model and examines portfolios with a consol. In Section 3, we present results from extensive numerical experiments, which motivate and guide our further analysis of optimal portfolios. In Section 4, we describe sufficient conditions for a small number of bonds with finite maturity to span the consol. Section 5 examines portfolios that consist of an investment in the market portfolio of stocks and bond ladders. We show that as the number of bonds with finite maturities increases, the welfare loss from holding such a nonequilibrium portfolio tends to zero. Section 6 concludes the article with more details on some related literature and a discussion of the results and limitations of our analysis.

1. Dynamic General Equilibrium Model with HARA Utility

In Section 1.1, we describe a Lucas-style (Lucas 1978) discrete-time, infinite-horizon general equilibrium model. Section 1.2 then provides an intuitive explanation for the simple description of equilibria in this model if markets are dynamically complete. We state agents' budget equations, which are the central object of our analysis. Finally, in Section 1.3, we introduce the classical assumption of equicautious HARA utility into our model. This assumption guarantees that consumption allocations follow a linear sharing rule in equilibrium.

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[3] Empirical evidence supports the consideration of simple investment strategies. Not only are the practical advantages of bond ladders well documented, there is also support for simple equity strategies. DeMiguel, Garlappi, and Uppal (2009) compare the out-of-sample performance of fourteen different equity portfolio strategies, some of which are constructed with sophisticated econometric techniques, to an equally weighted equity portfolio. They find that none of the sophisticated strategies consistently outperforms the simple equally weighted portfolio. They argue that errors in the estimation of means and covariances of returns offsets all the gains from optimal diversification when compared to the simple equally weighted portfolio.

1.1 The asset market economy

We examine a standard Lucas asset-pricing model (Lucas 1978) with hetero-
geneous agents (investors) and infinite discrete time, $t = 0, 1, \ldots$. Uncertainty
is represented by exogenous shocks y_t that follow a Markov chain with a finite
state space $\mathcal{Y} = \{1, 2, \ldots, Y\}, Y \geq 3$, and transition matrix $\Pi >> 0$. At time
$t = 0$, the economy is in state y_0. A date-event is a finite history of shocks,
$(y_0, y_1, \ldots, y_t)$.

We assume that there is a finite number of types $\mathcal{H} = \{1, 2, .., H\}$ of
infinitely lived agents. There is a single perishable consumption good. Each
agent h has a time-separable expected utility function

$$U_h(c) = E\left\{ \sum_{t=0}^{\infty} \beta^t u_h(c_t) \right\},$$

where c_t is consumption at time t. All agents have the same discount factor
$\beta \in (0, 1)$ and calculate expectations using the transition matrix Π. We specify
functional forms for the utility functions u_h below.

Agents have no initial endowment of the consumption good. Their initial
endowment consists solely of shares in some firms (stocks). Each period, the
firms distribute output to its owners through dividends. Investors trade shares
of firms and other securities in order to transfer wealth across time and states.
We assume that there are $J \geq 2$ stocks, $j \in \mathcal{J} \equiv \{1, 2, \ldots, J\}$, traded on
financial markets. A stock is an infinitely lived asset ("Lucas tree") that is
characterized by its state-dependent dividends. We denote the dividend of stock
$j \in \mathcal{J}$ by $d^j : \mathcal{Y} \to \mathbb{R}_+$ and assume that the dividend vectors d^j are linearly
independent. Agent h has an initial endowment $\psi_j^{h,0}$ of stock $j \in \mathcal{J}$. We
assume that all stocks are in unit net supply, i.e., $\sum_{h \in \mathcal{H}} \psi_j^{h,0} = 1$ for all $j \in \mathcal{J}$,
and so the social endowment in the economy in state y is the sum of all firms'
dividends in that state, $e_y \equiv \sum_{j \in \mathcal{J}} d_y^j$ for all $y \in \mathcal{Y}$. We assume that all
stocks have nonconstant dividends and that aggregate dividends (i.e., aggregate
endowments) are also not constant.

Our model includes the possibility of two types of bonds. One type of bond
we analyze is a consol. The consol pays one unit of the consumption good in
each period in each state, i.e., $d_y^c = 1$ for all $y \in \mathcal{Y}$. We also study finite-lived
bonds. There are $K \geq 1$ bonds of maturities $1, 2, \ldots, K$ traded on financial
markets. We assume that all finite-lived bonds are zero coupon bonds. (This
assumption does not affect any of the results that concern stock investments,
since any other bond of a similar maturity is equivalent to a sum of zero-coupon
bonds.) A bond of maturity k delivers one unit of the consumption good k
periods in the future. If at time t an agent owns a bond of maturity k and holds
this bond into the next period, it turns into a bond of maturity $k - 1$. Agents
do not have any initial endowment of the bonds. All bonds are thus in zero net
supply.

As usual, financial markets, equilibrium consists of consumption allocations for all agents and prices for stocks and bonds at each date-event such that all asset markets clear and agents maximize their utility subject to their respective budget constraint and a standard transversality condition. For the purpose of this article, we do not need a formal definition of financial markets equilibrium. We thus omit a statement of the formal definition and refer to Judd, Kubler, and Schmedders (2003).

1.2 Dynamically complete markets

For the remainder of the article, we assume that financial markets are dynamically complete. This assumption holds, e.g., if there are as many financial assets with independent dividend or payoff vectors as there are shocks, Y. Under this assumption, Judd, Kubler, and Schmedders (2003) derive two results that are important for our analysis here. First, equilibrium is Markovian: Individual consumption allocations and asset prices depend only on the current state. Second, after one initial round of trading, each agent's portfolio is constant and depends neither on time t nor on the current state.

Theorem 1 (Judd, Kubler, and Schmedders 2003). In every Pareto-efficient equilibrium, consumption allocations and asset prices are functions of the exogenous state $y \in \mathcal{Y}$ only. The end-of-period security portfolio of each agent is constant across states and time.

Remark. The intuition for these equilibrium properties follows directly from market completeness and portfolio spanning. Complete-market equilibria are Pareto efficient. This property implies that, under our assumptions of time-separable utility and identical constant discount factors, consumption allocations depend only on the current exogenous shock y but do not depend on the previous history of shocks nor endogenous state variables (see also Duffie 1988). Asset prices are priced with the stochastic discount factor, which is derived from the agents' Euler equations. Since consumption allocations are time-homogeneous Markov processes, the same is true for the agents' marginal utilities of consumption, which appear in the stochastic discount factor. The dividend and payoff vectors of all assets are similarly Markovian, so the asset prices are also Markovian. Put differently, asset prices only depend on the current exogenous shock. We do not need to introduce endogenous state variables to describe the price processes in this model.

The agents finance their consumption allocations with asset payoffs from their security portfolio. In dynamically complete markets, with Y exogenous states and the same number of assets with linearly independent dividends or payoffs, the agents need to span their (Y-dimensional) consumption vector with an appropriate combination of the Y dividend and payoff vectors. Because dividends, asset prices, and consumption allocations are all Markovian and

time-homogeneous, the portfolio spanning problem does not change over time. (Put differently, the investment opportunity set is constant over time.) So, the state-contingent consumption plan equals the payoffs that are generated by some unique *fixed and constant* combination of the Y assets. If this target portfolio is not the agent's asset endowment, then he can obtain that portfolio through trading in the initial period. Therefore, any consumption plan can be implemented by some trade-once-and-hold-forever trading strategy.[4]   ∎

Theorem 1 allows us to express equilibrium in a simple manner with dynamically complete markets. We do not need to express equilibrium values of all variables in the model as functions of the date-event $(y_0, y_1, \ldots, y_t)$ or as functions of a set of sufficient state variables. Instead, we let c_y^h denote consumption of agent h in state y. In addition, q_y^k denotes the price of bond k in state y, and the price of the consol is q_y^c. Similarly, p_y^j denotes the price of stock j in state y. All portfolio variables can be expressed without reference to a state y. The holdings of household h consist of θ_k^h bonds of maturity k or θ_c^h consols, and ψ_j^h units of stock j. For ease of reference, we summarize the notation for portfolio and price variables.

p_y^j	price of stock j in state y	ψ_j^h	agent h's holding of stock j
q_y^k	price of maturity k bond in state y	θ_k^h	agent h's holding of maturity k bond
q_y^c	price of the consol in state y	θ_c^h	agent h's holding of the consol

In Appendix 6.2, we describe a generalized version of Judd et al.'s (2003) three-step algorithm, which we use to calculate the equilibrium values for all variables in the model.

The analysis of our article focuses on investors' portfolios, and for this purpose, we analyze the budget constraints in great detail. We explicitly state them here. If the economy has a consol but no short-lived bonds, then the budget constraint in state y (after time 0) is

$$c_y^h = \underbrace{\sum_{j=1}^{J} \psi_j^h \left(d_y^j + p_y^j\right) + \theta_c^h \left(1 + q_y^c\right)}_{\text{beginning-of-period wealth}} - \underbrace{\left(\sum_{j=1}^{J} \psi_j^h p_y^j + \theta_c^h q_y^c\right)}_{\text{end-of-period investment}}$$

$$= \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_c^h. \qquad (1)$$

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[4] A somewhat similar no-trade result also holds in the continuous-time version of the Lucas model. Berrada, Hugonnier, and Rindisbacher (2007) develop necessary and sufficient conditions for zero equilibrium trading volume in a continuous-time Lucas model with heterogeneous agents, multiple goods, and multiple securities. They also relate their conditions to those of Judd, Kubler, and Schmedders (2003) for the discrete-time economy.

The budget constraint greatly simplifies since portfolios are constant over time and the prices of infinitely lived assets cancel out. If all bonds are of finite maturity, then an agent's budget constraint in state y is

$$c_y^h = \underbrace{\sum_{j=1}^{J} \psi_j^h \left(d_y^j + p_y^j \right) + \theta_1^h + \sum_{k=2}^{K} \theta_k^h q_y^{k-1}}_{\text{beginning-of-period wealth}} - \underbrace{\left(\sum_{j=1}^{J} \psi_j^h p_y^j + \sum_{k=1}^{K} \theta_k^h q_y^k \right)}_{\text{end-of-period investment}}$$

$$= \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{K} \theta_k^h (q_y^{k-1} - q_y^k). \tag{2}$$

Again, the prices of stocks cancel out. Only the prices of the (finitely lived) bonds appear in the simplified budget constraints. There may be trade on financial markets even though portfolios are constant over time. From one period to the next, a k-period bond turns into a $(k-1)$-period bond and agent h needs to rebalance the portfolio whenever $\theta_{k-1}^h \neq \theta_k^h$ in order to maintain a constant portfolio over time. In addition, the agent needs to reestablish the position in the bond of longest maturity.

1.3 HARA utility and linear sharing rules
The budget Equations (1) and (2) enable us to analyze the portfolios that deliver the equilibrium consumption allocations. For this analysis, a simple description of allocations is clearly helpful. We say that consumption for household h follows a "linear sharing rule" if there exist real numbers m^h, b^h so that in each shock $y \in \mathcal{Y}$,

$$c_y^h = m^h e_y + b^h. \tag{3}$$

Linear sharing rules partition the consumption vector c^h into a "safe" portion b^h and a "risky" portion $m^h e$. This partition proves important for our analysis of equilibrium portfolios.

The connection between linear consumption sharing rules (as exposited in Wilson 1968) and (static) asset market equilibrium was made by Rubinstein (1974a). We follow Rubinstein's approach in our dynamic economy and make the same assumption on investors' utility functions in order to ensure the emergence of linear sharing rules in equilibrium. Agents need to have equicautious HARA utility functions, i.e., (per-period) utility functions $u_h, h \in \mathcal{H}$, must exhibit linear absolute risk tolerance with identical slopes. We consider two special cases of HARA utility functions: power utility and constant absolute risk aversion. We use the following notation for the utility function of

household h:

power utility functions: $\quad u_h(c) = \begin{cases} \frac{1}{1-\gamma}(c - A^h)^{1-\gamma}, & \gamma > 0, \gamma \neq 1, c > A^h \\ \ln(c - A^h), & \gamma = 1, \qquad c > A^h \end{cases}$

CARA utility functions: $u_h(c) = -\frac{1}{a^h} e^{-a^h c}$.

If investors have equicautious HARA utility, then equilibrium consumption allocations for all agents follow a linear sharing rule of the form (3) and it holds that $\sum_{h=1}^{H} m^h = 1$ and $\sum_{h=1}^{H} b^h = 0$. Using the Negishi approach (Negishi 1960) from Appendix 6.2, we can calculate the sharing rules as functions of Negishi weights (see Appendix 6.2).

The assumption of HARA utility rules out Epstein-Zin recursive preferences. These preferences have proven useful in portfolio analysis (see, e.g., Campbell and Viceira 2001) since they allow for the separation between risk aversion and elasticity of intertemporal substitution. Although the assumption of HARA utility is certainly a restriction, it is common in both theoretical and applied work. For example, Brunnermeier and Nagel (2008) use a power utility function in their model of asset allocation that underlies their empirical examination of how investors change their portfolio allocations in response to changes in their wealth level. In another recent paper, Espino (2010) examines the neoclassical growth model with heterogeneous agents that have HARA utility functions and shows that, under some assumptions on the production function, equilibrium portfolios are constant and have a simple structure.

2. Portfolios with a Consol

The assumption of equicautious HARA utility functions implies linear sharing rules for equilibrium consumption. This fact is of interest to us because Rubinstein (1974a) showed that, in the context of a static equilibrium model, linear sharing rules lead to investors' portfolios, which satisfy the classical two-fund monetary separation property, if the investors can trade a riskless asset. We now examine this connection in the context of our infinite-horizon model. In Section 2.1, we define the notion of two-fund separation for our model. Section 2.2 explains why the consol serves in our model as a riskless asset and why the one-period bond is a risky asset. Section 2.3 illustrates that the absence of a riskless asset has a nontrivial impact on equilibrium portfolios.

In the later sections of this article, equilibrium portfolios in dynamic economies with stocks and a consol (but no finite-maturity bonds) serve as a useful benchmark for our analysis of complex bond portfolios.

2.1 Two-fund monetary separation

There exists a variety of portfolio separation theorems[5] in the economic literature, but the notion that most people now have in mind when they talk about two-fund separation is what Cass and Stiglitz (1970) termed two-fund "monetary" separation. (For examples of an application of this notion, see Canner, Mankiw, and Weil 1997 and Elton and Gruber 2000.) Two-fund monetary separation states that investors who must allocate their wealth between a number of risky assets and a riskless security should all hold the same mutual fund of risky assets. An investor's risk aversion only affects the proportions of wealth that she invests in the risky mutual fund and the riskless security. But, the allocation of wealth across the different risky assets does not depend on the investor's preferences.

Hakansson (1969) and Cass and Stiglitz (1970) showed that the assumption of HARA utility is a necessary and sufficient condition on investors' utility functions for the optimal portfolio in investors' static asset demand problems to satisfy the monetary separation property. Ross (1978) presents conditions on asset return distributions under which two-fund separation holds for static demand problems. In this article we stay away from analyses that rely on distributional assumptions about asset prices, since we focus on equilibrium prices and portfolios, and there is no reason to believe that equilibrium asset prices fall into any of the special families that produce portfolio separation.

We define the notion of two-fund monetary separation for our dynamic general equilibrium model with heterogeneous agents. This form of separation requires the proportions of wealth invested in any two stocks to be the same for all agents in the economy.

Definition 2.1. Suppose an asset with a riskless payoff vector (i.e., a one-period bond or a consol) is available for trade. The remaining J assets are risky stocks in unit net supply. We say that portfolios exhibit two-fund monetary separation if

$$\frac{\psi_j^h \, p_y^j}{\psi_{j'}^h \, p_y^{j'}} = \frac{\psi_j^{h'} \, p_y^j}{\psi_{j'}^{h'} \, p_y^{j'}}$$

for all stocks j, j', and all agents $h, h' \in \mathcal{H}$ in all states $y \in \mathcal{Y}$.

Stocks are in unit net supply, so market clearing and the requirement from the definition immediately imply that all agents' portfolios exhibit two-fund separation if, and only if, each agent has a constant share of each stock in the economy, i.e., $\psi_j^h = \psi_{j'}^h$ for all stocks j, j', and all agents $h \in \mathcal{H}$. This constant share typically varies across agents. In the remainder of this article, we

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[5] The literature starts with Tobin's (1958) two-fund result in a mean-variance setting. For textbook overviews, see Ingersoll (1987) or Huang and Litzenberger (1988). The standard reference for two-fund and m-fund separation in continuous-time models is Merton (1973).

identify two-fund monetary separation with the constant-share property. The ratio of wealth invested in any two stocks j, j', equals the price ratio $p_y^j/p_y^{j'}$ and, thus, depends on the state $y \in \mathcal{Y}$.

At first this constant-share property may appear unusual, but it is solely a result of the unit net supply assumption. If the total number of shares differed across stocks, then two-fund portfolios would not exhibit constant shares of all stocks. Also, for stocks with different total quantities of shares, we could always renormalize the number of shares and adjust dividends correspondingly in order to obtain an equal number of shares for all stocks. Our results on bond portfolios below do not depend on the unit-net-supply normalization.

2.2 The consol is the riskless asset

As mentioned earlier, Rubinstein (1974a) introduced the assumption of equicautious HARA utility functions for all investors, which yields linear sharing rules of consumption. In fact, the principal motivation for having linear sharing rules is that they, in turn, result in investors' portfolios that satisfy two-fund monetary separation. We can interpret Rubinstein's results as a generalization of the conditions of Hakansson (1969) and Cass and Stiglitz (1970) for static asset demand problem to a static general equilibrium model. We obtain the same connection between linear sharing rules and two-fund monetary separation in our dynamic model if agents can trade a consol.[6] The consol serves as the riskless asset in an infinite-horizon economy (see, e.g., Rubinstein 1974b; Connor and Korajczyk 1989; Bossaerts and Green 1989). More recently, this fact has been emphasized by Campbell and Viceira (2001) and Cochrane (2008).

Theorem 2 (Two-fund Monetary Separation). Consider an economy with $J \leq Y - 1$ stocks and a consol. If all agents have equicautious HARA utilities, then their portfolios exhibit two-fund monetary separation in all efficient equilibria.

Proof. The statement of the theorem follows directly from the budget constraint (1). Agents' sharing rules are linear, $c_y^h = m^h e_y + b^h$ for all $h \in \mathcal{H}$, $y \in \mathcal{Y}$, so the budget constraints immediately yield $\theta_c^h = b^h$ and $\psi_j^h = m^h$ for all $j = 1, \ldots, J$. ∎

The consol is the riskless asset in an infinite-horizon dynamic economy. An agent establishes once and forever a position in the consol at time 0. Fluctuations in the price of the consol, therefore, do not affect the agent in the

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[6] Due in part to linear sharing rules, an asset market economy with stocks and a consol has efficient equilibria even when markets are nominally incomplete, i.e., when the number of assets $J + 1$ is smaller than the number of states, Y. For analog static and two-period versions of this result, see Rubinstein (1974a) and Detemple and Gottardi (1998), respectively. For a multi-period version, see Rubinstein (1974b). The agents' portfolios are unique as long as $J + 1 \leq Y$ since the vectors d^c and d^j, $j \in \mathcal{J}$, are linearly independent.

same way that she is unaffected by stock price fluctuations. This fact allows her to hold a portfolio that exhibits two-fund monetary separation. We can read off agents' portfolios from their linear sharing rules. The holding b^h of the consol delivers the safe portion of the consumption allocation, $m^h e + b^h$, and the holding m^h of the market portfolio of all stocks delivers the risky portion $m^h e$ of the allocation. In the special case of CRRA utility functions, $b^h = 0$ for all $h \in \mathcal{H}$, and so the agents do not trade the consol. This is a corollary to the theorem: Whenever the intercept terms of the sharing rules are zero, the agents do not trade the consol and the stock markets are dynamically complete even without a bond market. However, under the additional condition $\sum_{h \in \mathcal{H}} A^h \neq 0$, sharing rules have a nonzero intercept for a generic set of agents' initial stock portfolios. That is, with the exception of a set of initial portfolios that has measure zero and is closed, sharing rules will have nonzero intercepts (see Schmedders 2007).

A close examination of the linear sharing rules shows that the more risk averse an agent, the larger the safe portion b^h of her consumption stream will be and thus her consol holding (see, also, the example in Section 2.3). This equilibrium outcome is very much in the spirit of similar results by Campbell and Viceira (2001) and Wachter (2003). Campbell and Viceira find in their asset demand model that as risk aversion increases, an investor tends to hold a portfolio that is equivalent to a consol and provides a stable income stream. Wachter (2003) shows that as risk aversion approaches infinity, the portfolio of an investor with utility over consumption at a final time T converges to a portfolio that consists solely of a bond maturing at time T.

The fact that sharing rules have generically nonzero intercepts immediately implies that a one-period bond cannot serve as the riskless asset. Even when markets are complete, there will be no two-fund separation. The economic intuition for this fact follows directly from the budget equations in an economy with $Y = J + 1$ states, J stocks, and a one-period bond,

$$m^h \cdot e_y + b^h = \eta^h \cdot e_y + \theta_1^h (1 - q_y^1) . \tag{4}$$

In contrast to the budget equations for an economy with a consol, the bond price q_y^1 now appears. Investors have to reestablish their position in the short-lived bond in every period. As a result, they face reinvestment risk due in part to fluctuating equilibrium interest rates of the short-term bond. Fluctuations in the price of the one-period bond generically prohibit a solution to Equation (4) (see Schmedders 2007). The reinvestment risk affects agents' bond, and thus stock portfolios, and leads to a change of the portfolio weights that implement equilibrium consumption.

Obviously, the agents' portfolios do satisfy a generalized separation property. Consumption follows a linear sharing rule, so an agent's portfolio effectively consists of one fund that generates the safe payoff stream of a consol and a second fund that generates a payoff that is identical to aggregate dividends. Both funds have nonzero positions of stocks and the bond. Agent h

holds b^h units of the first fund and m^h units of the second fund. However, this generalized definition is not the notion people have in mind when they discuss two-fund separation. Instead, they think of monetary separation (see, e.g., the discussions in Canner, Mankiw, and Weil 1997 and Elton and Gruber 2000). It is exactly this notion of two-fund monetary separation that appears below in the analysis of complex bond portfolios.

2.3 Illustrative example: Consol versus short-lived bonds

We complete our discussion of equilibrium portfolios with an example that illustrates two issues. First, equilibrium portfolios in economies with a consol are very different from portfolios in economies with short-lived bonds. Second, the example shows that bonds with very long but different times to maturity are nearly perfect substitutes. This property proves to be critical for the structure of equilibrium portfolios in economies with many bonds in Sections 3 and 4.

Consider an economy with $H = 3$ agents who have CARA utility functions with coefficients of absolute risk aversion of 1, 2, and 3, respectively. The agents' discount factor is $\beta = 0.95$. There are two independent stocks with identical "high" and "low" dividends of 1.02 and 0.98, respectively. The dividends of the first stock have a persistence probability of 0.8, i.e., if the current dividend level is high (low), then the probability of having a high (low) dividend in the next period is 0.8. The corresponding probability of the second stock equals 0.6. As a result of this dividend structure, the economy has $S = 4$ exogenous states of nature. The dividend vectors[7] are

$$d^1 = (1.02, 1.02, 0.98, 0.98)^\top \text{ and } d^2 = (1.02, 0.98, 1.02, 0.98)^\top.$$

The Markov transition matrix for the exogenous dividend process is

$$\Pi = \begin{bmatrix} 0.48 & 0.32 & 0.12 & 0.08 \\ 0.32 & 0.48 & 0.08 & 0.12 \\ 0.12 & 0.08 & 0.48 & 0.32 \\ 0.08 & 0.12 & 0.32 & 0.48 \end{bmatrix}.$$

At time $t = 0$, the economy is in state $y_0 = 1$. The agents' initial holdings of the two stocks are identical, so $\psi_j^{h,0} = \frac{1}{3}$ for $h = 1, 2, 3$, $j = 1, 2$.

Applying the algorithm of Appendix 6.2, we can easily calculate consumption allocations and price any asset in this model. We do not need these values here but, for completion, state them in Appendix 6.2. If, in addition to the two stocks, a consol is available for trade, then the economy satisfies the

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[7] The only purpose of this small numerical example is to illustrate theoretical and robust qualitative results. We thus refrain from a careful calibration of dividend processes and transition probabilities in this example. We analyze a model with properly estimated parameter values in Section 3.2.

Table 1
Equilibrium portfolios with two bonds

bonds		Agent 1				Agent 2				Agent 3			
k_1	k_2	ψ_1^1	ψ_2^1	θ_1^1	$\theta_{k_2}^1$	ψ_1^2	ψ_2^2	θ_1^2	$\theta_{k_2}^2$	ψ_1^3	ψ_2^3	θ_1^3	$\theta_{k_2}^3$
1	2	0.467	0.191	−1.029	1.249	0.295	0.374	0.294	−0.357	0.238	0.435	0.735	−0.892
1	5	0.603	1.878	0.835	−45.572	0.256	−0.108	−0.239	13.020	0.141	−0.770	−0.597	32.551
1	10	0.519	0.395	−0.646	−6.828	0.280	0.316	0.185	1.951	0.201	0.289	0.462	4.877
1	25	0.518	0.381	−0.660	−13.736	0.281	0.320	0.188	3.925	0.201	0.299	0.471	9.811
1	50	0.518	0.381	−0.660	−49.517	0.281	0.320	0.188	14.148	0.201	0.299	0.471	35.369

Table 2
End-of-period investment across assets in state 1

bonds		Agent 1				Agent 2				Agent 3			
k_1	k_2	ψ_1^1	ψ_2^1	θ_1^1	$\theta_{k_2}^1$	ψ_1^2	ψ_2^2	θ_1^2	$\theta_{k_2}^2$	ψ_1^3	ψ_2^3	θ_1^3	$\theta_{k_2}^3$
1	2	9.07	3.71	−0.99	1.15	5.74	7.26	0.28	−0.33	4.62	8.44	0.71	−0.82
1	5	11.71	36.44	0.80	−36.01	4.98	−2.09	−0.23	10.29	2.74	−14.94	−0.57	25.72
1	10	10.08	7.66	−0.62	−4.18	5.45	6.13	0.18	1.19	3.90	5.62	0.44	2.98
1	25	10.07	7.40	−0.63	−3.89	5.45	6.20	0.18	1.11	3.91	5.80	0.45	2.78
1	50	10.07	7.40	−0.63	−3.89	5.45	6.20	0.18	1.11	3.91	5.80	0.45	2.78

conditions of Theorem 2 and agents' portfolios exhibit two-fund monetary separation,

$$\left(\psi_1^1, \psi_2^1, \theta_c^1\right) = \left(\frac{6}{11}, \frac{6}{11}, -0.425\right), \left(\psi_1^2, \psi_2^2, \theta_c^2\right) = \left(\frac{3}{11}, \frac{3}{11}, 0.121\right),$$

$$\left(\psi_1^3, \psi_2^3, \theta_c^3\right) = \left(\frac{2}{11}, \frac{2}{11}, 0.304\right).$$

We compare the portfolios in a consol economy with the portfolios in an economy with short-lived bonds. We need two bonds to complete the financial markets. In addition to the natural choice of having a one- and two-period bond available for trade, we also report portfolios for cases where the second bond has a longer maturity k_2. Table 1 shows portfolios for all three agents, and Table 2 reports the corresponding end-of-period investments in the four assets at asset prices in state 1, i.e., the table reports the amount of capital invested in each asset in state 1 (see Equation (2)).

Two main observations stand out. First, agents' portfolios clearly do not resemble two-fund monetary separation. The stock portfolios are not the market portfolio. Second, the equilibrium portfolios greatly depend on the set of bonds available to the investors. Any arbitrary choice of bond maturities in the model will strongly affect both the equilibrium holdings and the end-of-period wealth invested in stocks and bonds. Also, note that economies with a one-period bond and a twenty-five-period or fifty-period bond, respectively, show (almost) identical positions in stocks and the one-period bond. Moreover, the wealth levels invested in the long-maturity bonds are (almost) identical.

Remark. The intuition for this observed similarity is simple yet important. Over a horizon of twenty-five or fifty periods, the distribution of the exogenous state of the economy, at the time of maturity of the bonds, is very close to the stationary distribution of the underlying Markov chain of exogenous states. Therefore, these bonds are nearly perfect substitutes, i.e., the fifty-period bond is approximately a twenty-five-period bond with additional twenty-five periods of discounting. In fact, a closer analysis of the price vectors (listed in Appendix 6.2) of these bonds reveals that the price vectors approximately satisfy the relationship $q^{50} \approx \beta^{25} q^{25}$. This relationship implies that, in turn, their respective returns as well as the respective coefficients of variation are almost the same. This substitutability of long-maturity bonds turns out to be significant in our below analysis.

The described property of near-perfect substitutability of long-term bonds is not an artifact of our model but is very much present in actual bond markets. Krishnamurthy (2002) examines portfolios that consists of a short position in newly auctioned thirty-year Treasury ("on-the-run") bonds and a long position in the just previously auctioned Treasury ("off-the-run") bonds. Such a portfolio is the basis for the so-called convergence trade that tries to exploit the fact that off-the-run Treasuries are typically slightly cheaper than on-the-run bonds, presumably due to liquidity effects. But, as Krishnamurthy (2002) states: "There is little economic reason for these bonds to have different yields." The two bonds are nearly perfect substitutes. ∎

We believe that the most natural choice of bonds in our model are those bonds with consecutive maturities, but often in the literature, other combinations are chosen. In our model, with non-consecutive bond maturities, an agent would be artificially forced to sell bonds whenever a bond changed its remaining maturity to a level that is not permitted by the model. For example, after one period, a five-period bond turns into a four-period bond. The agent would then be forced to sell this bond and thus would face considerable market price risk for this transaction. Clearly, this risk would influence the optimal portfolio decisions. To avoid such unnatural restrictions on agents' portfolio choices, we only consider economies with the property that if a bond of maturity k is present, then also bonds of maturity $k-1, k-2, \ldots, 1$ are also available to investors.

3. Many Finite-maturity Bonds

We have seen that, instead of the short-lived bond, the consol is the ideal asset for generating a riskless consumption stream in our dynamic economy. But real-world investors do not have access to a consol[8] and, instead, can only trade

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[8] With the exception of some perpetual bonds issued by the British treasury in the 19th century, infinite-horizon bonds do not exist and are no longer issued. See Calvo and Guidotti (1992) for a theory of government debt structure that explains why modern governments do not issue consols.

finite-maturity bonds. While modern markets offer investors the opportunity to trade bonds with many different finite maturities, these bonds expose investors to reinvestment risk if their investment horizon exceeds the available maturity levels. As a result, investors who demand that a portion of their consumption stream be safe face the problem of generating such a constant stream without the help of a riskless asset. We now examine this problem in our dynamic framework. In our model, we can include any number of independent bonds by choosing a sufficiently large number of states. In Section 3.1, we begin our analysis of complex bond portfolios with some numerical experiments. The purpose of these experiments is to learn details about the structure of equilibrium portfolios. Section 3.2 shows that the observed properties also hold in a calibrated model. We summarize our observations in Section 3.3 and explain how they guide our subsequent analysis.

3.1 "Wild" portfolios with many finite-maturity bonds

We consider economies with $H = 2$ agents with power utility functions. Setting $A^1 = -A^2 = b$ results in the linear sharing rules[9] $c^1 = m^1 \cdot e + b \cdot 1_Y$ and $c^2 = (1 - m^1) \cdot e - b \cdot 1_Y$, where 1_Y denotes the vector of all ones. We set $m^1 = \frac{1}{2} - b$ so that both agents consume on average half of the endowment. For the subsequent examples, we use $b = 0.2$, $\gamma = 5$, and $\beta = 0.95$. The agents' sharing rules are then

$$c^1 = 0.3 \cdot e + 0.2 \cdot 1_Y \ \ \text{and} \ \ c^2 = 0.7 \cdot e - 0.2 \cdot 1_Y.$$

We consider economies with $J \in \{3, 4, 5, 6, 7\}$ independent stocks. Each stock $j \in \mathcal{J}$ in the economy has only two dividend states, a "high" and a "low" state, that result in a total of 2^J possible states in the economy. We define the persistence parameters ξ^j for each stock j and denote the dividend's 2×2 transition matrix by

$$\varXi = \begin{bmatrix} \frac{1}{2}(1 + \xi^j) & \frac{1}{2}(1 - \xi^j) \\ \frac{1}{2}(1 - \xi^j) & \frac{1}{2}(1 + \xi^j) \end{bmatrix}$$

with $\xi^j \in (0, 1)$. The Markov transition matrix $\Pi = \bigotimes_{j=1}^{J} \varXi$ for the entire economy is a Kronecker product of the individual transition matrices (see Appendix 6.2). Table 3 reports the parameter values for our examples.

These parameter values cover a reasonable range of persistence and variance in stock dividends. The varying dividend values and persistence probabilities are chosen so that the examples display generic behavior. (We calculated hundreds of examples showing qualitatively similar behavior.) To keep the expected social endowment at one, we always normalize the dividend vectors.

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[9] To simplify the analysis, we do not compute linear sharing rules for some given initial portfolios but instead take sharing rules as given and assume that the initial endowment is consistent with the sharing rules. There is a many-to-one relationship between endowments and consumption allocations, and it is more convenient to fix consumption rules.

Table 3
Stock characteristics

stock	1	2	3	4	5	6	7
high d	1.02	1.23	1.05	1.2	1.09	1.14	1.1
low d	0.98	0.77	0.95	0.8	0.91	0.86	0.9
ξ	0.08	0.62	0.22	0.48	0.32	0.4	0.36
$\frac{1}{2}(1+\xi)$	0.54	0.81	0.61	0.74	0.66	0.7	0.68

For this reason, we multiply the dividend vectors by $1/J$ for the economy with J stocks. (This normalization is unnecessary and barely affects portfolio holdings.)

The economy has J stocks, $Y = 2^J$ states of nature, and $K = 2^J - J$ bonds. For example, for $J = 5$ stocks and $Y = 32$ states, our model has twenty-seven bonds of maturities $1, 2, \ldots, 27$. All thirty-two assets are independent, and thus markets are dynamically complete. Table 4 reports the stock portfolio for Agent 1. Table 5 reports the agent's entire bond portfolio for $J \in \{3, 4\}$ and positions of selected bonds for $J \in \{5, 6, 7\}$. (There are too many bonds to report complete portfolios.)

We make several observations about the agents' portfolios. First, consider the stock portfolios of Agent 1 in Table 4. For $J = 3$ stocks and $Y = 8$ states, the stock portfolio deviates significantly from the market portfolio. But, for $J \in \{4, 5, 6, 7\}$, Agent 1's stock holdings are very close to the slope $m^1 = 0.3$ of the linear sharing rule. In fact, the stock positions match at least the first seven digits (to keep the table small, we report fewer than seven digits) of m^1. In other words, the agent's stock portfolios come extremely close to being the market portfolio. Second, consider the bond portfolios in Table 5. For $J = 3$, the positions in the five bonds exhibit no meaningful structure. For $J = 4$, Agent 1's holdings of the bonds of maturity $1, 2, \ldots, 5$ match the intercept term b^1 of the linear sharing rule for the first 2 digits. For $J = 5$, there is already a corresponding match for more than the first fifteen bonds. As J, Y, and K further increase, the pattern of an increasing number of bond positions approximately matching b^1 continues. So, the agent's bond positions for relatively short-term bonds come extremely close to a bond ladder in which the holding of each bond (approximately) matches the level b^1 of the riskless consumption stream. This pattern falls apart for bonds with long maturity. The longer the maturity of the bonds, the greater the deviations of holdings from b^h (with the exception of just the holdings of bonds with longest maturities). In addition, once holdings deviate significantly from b^h, they alternate in sign.[10]

To underscore our observations, we next report the deviations of the stock holdings from the slope of the linear sharing rule and the deviations of the

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[10] The nature of these bond holdings, namely the very large positions of alternating signs, may remind some readers of the unrelated literature on the optimal maturity structure of noncontingent government debt (see, e.g., Angeletos 2002; Buera and Nicolini 2004). Buera and Nicolini report very high debt positions from numerical calculations of their model with four bonds. The reason for their highly sensitive large debt positions is the close correlation of bond prices.

Table 4
Stock portfolio of agent 1

$(J, K) =$	(3,5)	(4,12)	(5,27)	(6,58)	(7,121)
ψ_1^1	0.431	0.300	0.300	0.300	0.300
ψ_2^1	0.351	0.300	0.300	0.300	0.300
ψ_3^1	0.387	0.300	0.300	0.300	0.300
ψ_4^1		0.300	0.300	0.300	0.300
ψ_5^1			0.300	0.300	0.300
ψ_6^1				0.300	0.300
ψ_7^1					0.300

Table 5
Bond portfolio of agent 1

$(J, K) =$	(3,5)	(4,12)		(5,27)		(6,58)		(7,121)	
k	θ_k^1	θ_k^1	k	θ_k^1	k	θ_k^1	k	θ_k^1
1	0.152	0.20	1	0.20	1	0.20	1	0.20
2	−0.184	0.20	2	0.20	5	0.20	10	0.20
3	2.337	0.20	8	0.20	10	0.20	30	0.20
4	−7.498	0.20	9	0.20	15	0.20	50	0.20
5	8.074	0.20	10	0.20	20	0.20	58	0.20
6		0.27	11	0.20	25	0.20	95	0.20
7		−0.66	12	0.20	27	0.20	100	0.18
8		6.33	15	0.20	40	0.20	110	−29675
9		−26.23	20	−5.2	50	1179	115	504548
10		66.16	25	556	56	10178	119	157907
11		−86.59	26	−423	57	−4627	120	−50013
12		46.58	27	146	58	998	121	7670

bond holdings from the intercept term of the sharing rule. As a measure for these deviations, define

$$\Delta^S \equiv \max_{j=1,2,\dots,J} |\psi_j^1 - m^1|$$

to be the maximal deviation of agents' equilibrium stock holdings from the appropriate holding of the market portfolio, where we maximize the difference across all stocks. (Due in part to market clearing, it suffices to calculate the difference for the first agent.) Similarly, we define

$$\Delta^k \equiv |\theta_k^1 - b^1|$$

to be the maximal difference between agents' holdings in bond k and the intercept of their linear sharing rules. Table 6 reports deviations in stock holdings and the first 5 bonds, and Table 7 reports deviations in bond positions for some selected longer maturity bonds. Throughout the article, deviations are rounded up. (To save space, we use the abbreviation 4.5 (−9) for 4.5×10^{-9}, etc.)

The results are clear. First, Table 6 shows how close equilibrium stock portfolios are to the fraction m^h of the market portfolio. Stock positions are practically identical to m^h, when there are many states and bonds. Second,

Table 6
Deviations of stock and bond holdings from m^h and b^h, resp.

J	K	Δ^S	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
4	12	4.5 (−9)	1.3 (−9)	3.5 (−8)	2.0 (−6)	1.1 (−4)	3.7 (−3)
5	27	3.5 (−33)	6.3 (−34)	8.3 (−31)	8.3 (−28)	4.6 (−25)	1.6 (−22)
6	58	9.6 (−88)	4.2 (−85)	3.1 (−81)	1.1 (−77)	2.1 (−74)	3.0 (−71)
7	121	2.0 (−222)	4.9 (−214)	1.8 (−209)	3.0 (−205)	3.2 (−201)	2.4 (−197)

The table reports deviations of stock holdings and the first 5 bond holdings from m^h and b^h, resp. We abbreviate figures such as 4.5×10^{-9} by 4.5 (−9).

Table 7
Deviations of bond holdings from b^h

k	$J = 5, K = 27$	$J = 6, K = 58$	$J = 7, K = 121$
6	3.5 (−20)	3.0 (−68)	1.4 (−193)
7	5.3 (−18)	2.4 (−65)	6.3 (−190)
10	3.0 (−12)	2.9 (−57)	2.0 (−179)
11	1.5 (−10)	9.9 (−55)	4.5 (−176)
12	5.4 (−9)	2.9 (−52)	8.9 (−173)
20	5.37	7.5 (−35)	3.5 (−148)
25	555.6	1.1 (−25)	3.9 (−134)
26	423.4	5.3 (−24)	2.0 (−131)
27	145.8	2.4 (−22)	9.1 (−129)
40	–	3.7 (−5)	1.0 (−96)
50	–	1179.3	4.3 (−75)
56	–	10178	3.0 (−63)
57	–	4627.2	2.3 (−61)
58	–	998.2	1.7 (−59)

The table reports deviations of bond positions for some selected longer maturity bonds. We abbreviate figures such as 3.5×10^{-20} by 3.5 (−20).

both Tables 6 and 7 highlight that the deviations in the bond holdings are also negligible for short maturity bonds relative to the maximally available maturity K. For example, in the model with $Y = 64$ states and $K = 58$ bonds, the holdings for the first forty bonds are close to the intercept b^h of the linear sharing rules. The agent's portfolio of these 40 bonds is practically a bond ladder. The deviations from the constant b^h increase in the maturity k of the bonds and eventually get huge. They peak for maturity levels just short of the longest maturity K. Moreover, the deviations explode as the number of states and bonds increases (see Tables 5 and 7).

Remark. At first, the results in Tables 5–7 may seem suspicious.[11] While the portfolio weights of the short-term bonds appear to be reasonable, the weights of the long-term bonds imply huge, alternating long and short positions of those bonds. Naturally, the question arises whether these weights are only an

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[11] We thank students taking the course "Computational Economics and Finance" at the University of Zurich in 2009–2011, particularly Simon Scheuring and Ole Wilms, for carefully checking many of our numerical results.

indication of numerical errors. They are not. We now describe how both the bond prices and the portfolio weights are calculated with great precision and why the portfolio weights make intuitive sense.

The bond prices in our model are calculated using the stochastic discount factor that is derived from agents' Euler equations (see Step 2 in Appendix 6.2). The calculations require only basic arithmetic operations, so we can use Mathematica to calculate all prices with extended precision. In fact, if linear sharing rules are given, as in this section, then we can calculate all prices exactly without numerical errors. The question arises to why the weights for the short- and long-term bonds can be so different.

Observe that the (common) beliefs in all computed examples are not i.i.d. Therefore, the k-step transition probabilities from a state y to a state z vary considerably for small k. Of course, as k grows, the transition probabilities converge to the stationary distribution of the Markov chain. Recall from the pricing equations for the bonds that the price vector of the k-period bond depends critically on the k-step transition probabilities (see Equations (A8) in Appendix 6.2). So, when these probabilities are different, then the same will be true for the price vectors. However, as k grows large, the k-step probabilities are nearly identical, and the bond price vectors are almost collinear, $q^{k+1} \approx \beta q^k$. In sum, the price vectors of the long-term bonds suffer from multicollinearity, while the price vector of each short-term bond is sufficiently different from all other price vectors. Therefore, the solution to the linear spanning problem of solving agents' budget equations leads to well-behaved weights for the short-term bonds and rather "crazy" weights[12] for the long-term bonds.

These mathematical properties of our model have a natural economic intuition. Agents' beliefs for the near future vary a lot with time; their expectations for the (dividend) state of the economy in the next period are quite different from their expectations for the state in two or three time periods. In equilibrium, these different expectations lead to different price vectors for the bonds. Agents' beliefs for the distant future are nearly (but not quite) stationary, so they put approximately identical weights on a state y that occurs in k or $k + 1$ periods, once k is rather large. Again, in equilibrium, these expectations affect the bond prices and lead to approximately collinear bond price vectors.

The "wild" portfolios may remind readers of the optimal portfolios that are generated by the classical mean-variance optimization approach (Markowitz 1952). Such portfolios often also display unreasonably large long and short

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[12] Computing the results in Tables 4, 5, 6, and 7 requires us to solve the agents' budget Equation (2). Although these equations are linear, solving them numerically is very difficult because of the near-perfect correlation of the prices of long-term bonds which lead to a very ill-conditioned linear system of equations. This difficulty is present whether we calculate stock and bond prices (in Mathematica) with extended precision or exactly. We cannot solve the system of equations using a regular linear equation solver on a computer using sixteen decimal digits of precision. Again, we must use the extended precision offered by Mathematica and solve these equations with up to 1024 decimal digits of precision.

positions and are very sensitive to assumptions on expected returns as well as very small estimation errors (see Best and Grauer 1991; Green and Hollifed 1992). In fact, our results very much echo the theorems by Green and Hollifed (1992), who show that conditions under which mean-variance portfolios are well diversified require bounds on the average covariability between asset returns. Put differently, if returns are highly correlated, then portfolio weights that are derived from mean-variance optimization will display massive long and short positions, just as in our model. These difficulties with the mean-variance approach served as key motivation for the Black-Litterman model (see Black and Litterman 1990). The "wild" portfolios are the motivation for our analysis of bond ladders in Section 5. ■

3.2 Bond portfolios in a calibrated economy

The computed equilibria in Section 3.1 illustrate what portfolios in our general equilibrium model (can) look like. The purpose of this section is to show that the same qualitative results hold in a properly calibrated version of the model. In Appendix 6.2, we describe how we derive input parameters for our model by using some recent empirical work of Dave and Pohl (2010).

The most important input parameters for our model are the exogenous stock dividend vectors and the accompanying Markov transition matrix. We consider an economy with two such dividend vectors. The dividends of the first stock are a five-point approximation of the (detrended) dividends paid by all S&P 500 companies. The payoffs of the second "stock" are a five-point approximation for claims to the (detrended) U.S. nondividend endowments. A 25×25 transition matrix approximates the joint movements of the dividend and endowment time series. The resulting stylized general equilibrium model has, therefore, two stocks and twenty-five states. Thus, we need bonds of twenty-three different maturities to complete markets.

Based on estimates by Dave and Pohl (2010), we report results for a parameterization with $\beta = 0.97$ and $\gamma = 3$. Our results are robust to changes in these two parameters.

As in Section 3.1, we consider economies with $H = 2$ agents with power utility functions with $A^1 = -A^2 = 0.2$ and sharing rules

$$c^1 = 0.3 \cdot e + 0.2 \cdot 1_Y \text{ and } c^2 = 0.7 \cdot e - 0.2 \cdot 1_Y.$$

We note that the assumed utility functions deviate from the pure CRRA utility functions that Dave and Pohl (2010) use for their calibrations. Therefore, the resulting price process in our model deviates from that of an economy with CRRA utilities. We also computed the equilibrium holdings for the above sharing rules with a price process derived from CRRA utilities, and we not only obtained the same qualitative results, but in fact the wild holdings for the long-term bonds are similar for both price processes.

The stock portfolio of Agent 1 is $\psi_1^1 = \psi_2^1 = 0.300$ and thus, once again, is very close to the slope 0.3 of the linear sharing rule. (The stock holding 0.300

Table 8
Bond portfolio of agent 1 in the calibrated economy

k	1	2	3	7	8	9	10	12	20	21	22	23
θ_k^1	0.200	0.200	0.200	0.200	0.200	0.205	0.115	-12.1	-120941	91146	-41214	8430

matches the slope for the first 19 digits.) Table 8 shows the bond portfolio of Agent 1. The holdings of the first 8 bonds match the first 3 digits of the intercept term 0.2. The holdings of the short-term bonds come extremely close to a bond ladder. The positions of the long-term bonds swing wildly and even more so than in our previous examples. In sum, the observed qualitative results in the illustrative examples also hold in the calibrated economy.

3.3 Discussion of numerical results
Recall that the agents' portfolios are the solutions to their budget constraints,

$$c^h = m^h e + b^h 1_Y = \sum_{j=1}^{J} \psi_j^h d^j + \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k),$$

where 1_Y again denotes the vector of all ones. The computed portfolios show that with many states and bonds in equilibrium it holds that

$$m^h e \approx \sum_{j=1}^{J} \psi_j^h d^j \quad \text{with} \quad \psi_j^h \approx m^h, \ \forall \ j \in \mathcal{J},$$

$$\text{and} \ \ b^h 1_Y \approx \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k).$$

We observe that a natural generalization of two-fund monetary separation emerges. Stock holdings are approximately the market portfolio of stocks. The purpose of the bond portfolios is to synthesize the consol and to generate the safe portion of the consumption stream.

Furthermore the emerging bond ladder of bonds up to some maturity $B < K$, i.e., $\theta_k^h \approx b^h$ for $k = 1, 2, \ldots, B$ implies that

$$0 \approx b^h 1_Y - \left(\theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k) \right) \approx \sum_{k=B}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K.$$

This expression restates that the bond price vectors of the long-maturity bonds $B, B+1, \ldots, K$ are nearly dependent, as we remarked in Sections 2.3 and 3.1.

We tried many different examples and always observed the same results as those reported here for the illustrative examples and the calibrated economy. Also, the results are surprisingly invariant to the size of the stock dividends

and the utility parameter γ. We now summarize the most important robust observations.

SUMMARY OF NUMERICAL RESULTS. Equilibrium portfolios in models with many states and bonds have the following properties.

1. The portfolios approximately exhibit generalized two-fund monetary separation.

    (a) Stock portfolios are approximately the market portfolio of all stocks. Each agent h holds approximately a constant amount m^h of each stock.
    (b) Bond portfolios approximately yield the same payoff as does a consol holding.
    (c) Stock portfolios generate almost exactly the risky portion $m^h e$ of the consumption allocation. Bond portfolios generate almost exactly the safe portion $b^h 1_Y$ of the allocation.

2. The holdings of bonds of short maturity (relative to the longest available maturity K) approximately constitute a bond ladder.

3. Holdings of long bonds are highly volatile, which implies that investors are making dramatically large trades in long bonds in each period.

These results raise two sets of questions. First, the finite-maturity bonds approximately span the consol. Do there exist specifications of our dynamic economy in which finite-maturity bonds can span the consol exactly? And, if so, what do portfolios look like in such economies? Second, bond ladders emerge as the holdings of short-lived bonds, but holdings of long-lived bonds look rather unnatural. How well can investors do if they are restricted to hold only bond ladders of all bonds available for trade? And what do optimal portfolios look like under this restriction? Section 4 provides answers to the first set of questions, and Section 5 addresses the second set of questions.

4. Sufficient Conditions for Spanning the Consol

For an agent's stock holdings to be the market portfolio, there must exist stock weights $\psi_j^h \equiv \eta^h$ for all $j \in \mathcal{J}$ such that

$$m^h \cdot e + b^h 1_Y = c^h = \eta^h \cdot e + \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k). \quad (5)$$

Rearranging Equation (5) yields

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + \sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K = 0. \quad (6)$$

Equation (6) implies that the $K + 2$ vectors $e, 1_Y$, and $q^1, \ldots, q^K$ in $\mathbb{R}^Y$ have to be linearly dependent. It appears as if whenever the number of states Y exceeds $K + 2$, this condition cannot be satisfied. For example, if the total number of stocks and bonds $J + K$ equals the number of states Y and there are $J \geq 3$ stocks, then the system (6) has more equations than unknowns. And, in fact, using a standard genericity argument, we can show that Equation (6) does not have a solution unless parameters lie in some measure zero space.

Although agents' portfolios typically do not exhibit two-fund monetary separation in economies with only finite-maturity bonds, we can develop special (nongeneric) but economically interesting conditions that do lead to generalized portfolio separation in such economies. The most general conditions are quite technical and do not provide much economic insight (see an earlier version of this article, i.e., Judd, Kubler, and Schmedders 2009). Here, we report results for two special instances. Section 4.1 shows that portfolios in economies with i.i.d. dividend processes satisfy generalized two-fund separation. In Section 4.2, we illustrate the same property when independent stocks have identical persistence.

4.1 Equilibrium portfolios with IID dividends

We first examine economies with i.i.d. dividends. The assumption of no persistence in dividends may be economically unrealistic, but it is often made in the economic literature because it serves as a useful benchmark.

Proposition 1. Consider an economy with J stocks, a one-period and two-period bond, and $Y \geq J + 2$ dividend states. Suppose further that the Markov transition probabilities are state-independent, so all rows of the transition matrix Π are identical. If all agents have equicautious HARA utility functions, then there exists an efficient equilibrium in which agents' portfolios exhibit exact generalized two-fund separation.

Proof. Under the assumption that all states are i.i.d., the Euler Equations (A6) and (A7) in Appendix 6.2 imply that the price of the two-period bond satisfies $q^2 = \beta q^1$, i.e., the prices of the two bonds are perfectly correlated. Then, condition (6) becomes

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + (\theta_1^h - (1 - \beta)\theta_2^h)q^1 = 0.$$

These equations have the unique solution $\eta^h = m^h, \theta_1^h = b^h, \theta_2^h = \frac{b^h}{1-\beta}$. ∎

For i.i.d. dividend transition probabilities, the solution to the agents' budget equations satisfies

$$b^h 1_Y = \theta_1^h (1_Y - q^1) + \theta_2^h (q^1 - q^2).$$

Two bonds are sufficient to span the consol. Just as for economies with a consol, markets are dynamically complete even when they are nominally incomplete, i.e., when $Y > J + 2$. Observe that the spanning condition is unaffected by the stock dividends.

4.2 Identical persistence across stocks and states

Suppose that each stock in the economy has two different dividend states ("high" and "low") and that dividends are independent across stocks. (The latter condition may require a decomposition of stock payoffs into different independent factors.) Since the individual dividend processes are independent, there is a total of $Y = 2^J$ possible states in this economy. The dividends may vary across stocks, but the stocks' 2×2 dividend transition matrices, Ξ, are identical. The Markov transition matrix Π for the economy is then the J-fold Kronecker product (see Appendix 6.2) of the individual transition matrix for the dividend states of an individual stock, $\Pi = \bigotimes_{j=1}^{J} \Xi$.

Proposition 2. Consider an economy with J independent stocks that each have two (stock-dependent) dividend states with identical transition matrices

$$\Xi = \begin{bmatrix} \frac{1}{2}(1 + \xi_H) & \frac{1}{2}(1 - \xi_H) \\ \frac{1}{2}(1 - \xi_L) & \frac{1}{2}(1 + \xi_L) \end{bmatrix}$$

with $\xi_H, \xi_L \in (0, 1)$. Then, bonds of maturities $k = 1, 2, \ldots, J + 1$ span the consol. In the presence of these $J+1$ bonds, and if all agents have equicautious HARA utilities, there exists an efficient equilibrium in which agents' portfolios exhibit exact generalized two-fund separation. The portfolio weights do not depend on stock dividends and utility functions. They are solely a function of the discount factor β and the parameters ξ_H, ξ_L of the transition matrix.

The proof of this proposition (see Judd, Kubler, and Schmedders 2009) reveals that we can derive closed-form solutions for the bond holdings. The analytical solutions are rather messy, so we instead report the numerical values for a single specification of the model. Table 9 displays the portfolios of the finite-maturity bonds that span one unit of the consol for $\xi = (\xi_H + \xi_L)/2 = 0.2$ and $\beta = 0.95$.

The bond portfolios that exactly span the consol exhibit the same qualitative properties as those bond portfolios that approximately span the consol in Section 3. Again, we observe the endogenous emergence of a laddered portfolio of short-maturity bonds as the number of bonds increases. The weight for the one-period bond quickly converges to one as the number of bonds, $J+1$, grows. The same is true for the other bond weights. The weights for the few bonds with longest maturity, again, fluctuate significantly. As we previously mentioned, the reason for the form of the portfolio is that the bond price vectors q^k become more and more collinear as k grows. The spanning condition then

Table 9
Bond portfolio spanning one unit of the consol, $\xi = 0.2$, $\beta = 0.95$

J	2	3	4	5	6	7	8	9	10
Y	4	8	16	32	64	128	256	512	1024
θ_1	1.176	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
θ_2	−4.569	1.220	0.998	1.000	1.000	1.000	1.000	1.000	1.000
θ_3	25.667	−4.800	1.229	0.998	1.000	1.000	1.000	1.000	1.000
θ_4		25.863	−4.847	1.231	0.998	1.000	1.000	1.000	1.000
θ_5			25.903	−4.856	1.231	0.998	1.000	1.000	1.000
θ_6				25.911	−4.858	1.231	0.998	1.000	1.000
θ_7					25.912	−4.858	1.232	0.998	1.000
θ_8						25.912	−4.859	1.232	0.998
θ_9							25.912	−4.859	1.232
θ_{10}								25.912	−4.859
θ_{11}									25.912

requires increasingly larger (in absolute value) weights on these vectors that also have to alternate in sign.

5. Nearly Optimal Portfolios with Bond Ladders

The theoretically derived portfolios that exhibit the exact generalization of two-fund monetary separation in Section 4.2 look just like the numerically computed portfolios in Section 3 that displayed this property approximately. Equilibrium portfolios consist of the market portfolio of stocks and a bond portfolio that generates a safe income stream. In the presence of sufficiently many bonds, the holdings of the short-term bonds are almost equal to the safe portion b^h of the income stream. However, the holdings of long bonds always substantially differ from a constant portfolio, and the implied asset trading volume bears no relation to actual security markets.

We now show that very simple and much more economically reasonable nonequilibrium portfolio strategies, namely portfolios that consist of the market portfolio of stocks and a bond ladder of all bonds available for trade, come very close to implementing equilibrium utility, particularly if the number of finite-maturity bonds available for trade is sufficiently large. In Section 5.1, we state and prove a limit result, which supports the idea that bond ladders are good nonequilibrium investment strategies. Section 5.2 defines a welfare measure for the evaluation of suboptimal portfolios. We apply this measure in Section 5.3 to determine agents' welfare losses from holding bond ladders instead of equilibrium portfolios.

While we do not explicitly model transaction costs, we can motivate the construction of bond ladders as a sensible investment approach in the face of transaction costs. As we have seen, equilibrium investment strategies imply enormous trading volume in the bond markets that would be very costly in the presence of even small transaction costs. On the contrary, bond ladders

minimize transaction costs, since the only transaction costs are those borne at the time the bonds are issued.[13]

5.1 A limit result for bond ladders

The next theorem states that a portfolio with constant stock holdings and constant bond holdings (consistent with the linear sharing rules) yields the equilibrium consumption allocation in the limit as the number of bonds tends to infinity.

Theorem 3. Assume there are Y states, J stocks, and that investors have equicautious HARA utility functions. Suppose the economy has B finite-maturity bonds and that allocations in an efficient equilibrium follow the linear sharing rules $c^h = m^h e + b^h \cdot 1_Y$, $h \in \mathcal{H}$. Define portfolios of $\psi_j^h = m^h$, $\forall j \in \mathcal{J}$, and $\theta_k^h = b^h$, $\forall k$. Then, in the limit as B increases,

$$\lim_{B \to \infty} \left(\sum_{j=1}^J \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^B \theta_k^h (q_y^{k-1} - q_y^k) \right) = c_y^h.$$

Proof. Asset prices for bonds and stocks will not depend on B since we are assuming B is large enough so that the equilibrium implements the consumption sharing rules $c^h = m^h e + b^h \cdot 1_Y$ for all B. The budget constraint (2) yields the consumption allocation that is implied by a portfolio with $\psi_j^h = m^h \ \forall j = 1, \ldots, J$, $\theta_k^h = b^h \ \forall k = 1, \ldots, B$, namely,

$$c_y^h = \sum_{j=1}^J m^h d_y^j + b^h (1 - q_y^1) + \sum_{k=2}^B b^h (q_y^{k-1} - q_y^k)$$

$$= m^h e_y + b^h - b^h q_y^B.$$

The price q_y^B of bond B is given by Equation (A8) (see Appendix 6.2). Because $\beta < 1$, $q_y^B \to 0$ as $B \to \infty$. Thus, $c_y^h \to m^h e_y + b^h$ and the statement of the theorem[14] follows. ∎

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[13] In fact, we may argue that transaction costs are lowest for newly issued "on-the-run" bonds. A large portion of previously issued "off-the-run" bonds is often locked away in investors' portfolios, which results in decreased liquidity of these bonds. For example, Amihud and Mendelson (1991) explain that bonds with lower liquidity have higher transaction costs. The increase of transaction costs for previously issued bonds clearly makes bond ladders even more sensible, relative to equilibrium portfolios.

[14] Note that as B increases, the number of assets $J + B$ will eventually exceed the fixed number of states Y, so the bond price vectors will be linearly dependent. As a result, optimal portfolios will be indeterminate. The theorem examines only one particular portfolio, namely one consisting of a portion of the market portfolio and a bond ladder. To avoid indeterminate optimal portfolios, we could increase the number of states in the limit process in order to keep the number of states and assets identical.

Theorem 3 states that if we have a large number of finite-maturity bonds, then the portfolio that consists of the market portfolio of stocks and a bond ladder comes arbitrarily close to implementing the equilibrium sharing rule. But, real markets do not offer bonds with arbitrarily large maturities. We now check how close portfolios with ladders of a finite number of bonds of maturities $1, 2, \ldots, B$ come in generating efficient equilibrium outcomes. For this purpose, we calculate the agents' welfare losses by using such a portfolio, as opposed to the optimal portfolio.

5.2 Welfare measure for portfolios

We now develop a welfare measure for portfolio strategies and their resulting consumption allocations. Welfare calculations in the context of a general equilibrium model suffer from the following difficulty. Usually agents' lifetime utilities (and the resulting consumption equivalents) are rather insensitive to changes in the consumption allocations. For example, for reasonable model specifications, the utility gain of equilibrium allocations in complete markets, relative to equilibrium allocations in incomplete markets, or even autarchy, is very small. For our model, this feature means that the welfare (utility and consumption equivalent) of an allocation generated by a laddered portfolio will be rather close to both the welfare generated by the equilibrium portfolio as well as to autarchy welfare. As a result, it is difficult to provide a meaningful interpretation for such a welfare comparison.

We can define a more meaningful welfare measure as follows. Instead of comparing the welfare of a laddered portfolio directly to the welfare of an equilibrium portfolio, we compare the respective welfare gains that these two portfolios deliver, relative to autarchy welfare, i.e., to the welfare of an agent's initial endowment of stocks. The ratio of the welfare gain delivered by the laddered portfolio to the welfare gain of the equilibrium portfolio is the proportion of the equilibrium welfare gain that can be delivered by the laddered portfolio. One minus this ratio is then a measure for the welfare loss suffered by investors following a ladder strategy. To formalize this approach, we now define three different consumption equivalents.

Define a utility vector v^h by $v_y^h = u^h(c_y^h)$ for a consumption vector c^h, where c_y^h is the consumption of agent h in state $y \in \mathcal{Y}$. Next, define

$$V_{y_0}^h(c^h) = \sum_{t=0}^{\infty} \left(\beta^t \Pi^t\right)_{y_0 \cdot} v^h = \left([I - \beta\Pi]^{-1}\right)_{y_0 \cdot} v^h.$$

to be the total discounted expected utility value over the infinite horizon when the economy starts in state y_0. Now, we can define $C_{y_0}^{h,*}$ to be the consumption equivalent of agent h's equilibrium allocation $m^h e + b^h$, which is defined by

$$\sum_{t=0}^{\infty} \beta^t u^h (C_{y0}^{h,*}) = V_{y0}^h (m^h e + b^h \cdot 1_Y) \iff$$

$$C_{y0}^{h,*} = (u^h)^{-1} \left((1 - \beta) V_{y0}^h (m^h e + b^h \cdot 1_Y) \right).$$

Similarly, we define a consumption equivalent $C_{y0}^{h,B}$ for the consumption process that agent h can achieve by holding the market portfolio of all stocks and a laddered portfolio of bonds of maturity $1, 2, \ldots, B$. Recall from the proof of Theorem 3 that such a portfolio with stock weights m and bond weights b supports the allocation $me + b \cdot 1_Y - bq_y^B$. The agent chooses the optimal weights $\hat{m}^h$ and $\hat{b}^h$ subject to the infinite-horizon budget constraint,

$$\max_{\{m,b\}} V_{y0}^h (me + b \cdot 1_Y - bq_y^B)$$

$$\text{s.t.} \left([I_Y - \beta \Pi]^{-1} (P \otimes ((me + b \cdot 1_Y - bq_y^B) - c^h)) \right)_{y0} = 0,$$

where $c^h (= m^h e + b^h \cdot 1_Y)$ denotes equilibrium consumption. The prices in the budget constraint are given by the equilibrium prices. We denote the consumption equivalent from this portfolio, which is optimal given the restrictions imposed on the agent, by

$$C_{y0}^{h,B} = (u^h)^{-1} \left((1 - \beta) V_{y0}^h (\hat{m}^h e + \hat{b}^h \cdot 1_Y - \hat{b}^h q_y^B) \right).$$

For the welfare comparison of the portfolio with a bond ladder to an agent's equilibrium portfolio, we compute the welfare gain of each of these two portfolios, relative to the welfare of the agent's initial endowment of stocks. For this purpose, we also define a consumption equivalent $C_{y0}^{h,0}$ for the consumption vector that would result from constant initial stock holdings $\psi^{h,0} \equiv \psi_j^{h,0}$ for all $j \in \mathcal{J}$. Since in our examples we take sharing rules as given, we need to calculate supporting initial stock endowments $\psi^{h,0}$ by solving the budget equations

$$\left([I_Y - \beta \Pi]^{-1} (P \otimes ((m^h e + b^h \cdot 1_Y) - \psi^{h,0} e)) \right)_{y0} = 0, \ h = 1, \ldots, H.$$

Again, the prices in the budget equation are the equilibrium prices. We denote the consumption equivalent from this initial portfolio by

$$C_{y0}^{h,0} = (u^h)^{-1} \left((1 - \beta) V_{y0}^h (\psi^{h,0} e) \right).$$

The welfare gain, compared with autarchy, of the equilibrium portfolio is $C_{y0}^{h,*} - C_{y0}^{h,0}$. Similarly, the welfare gain over autarchy consumption of the allocation that is generated by the laddered portfolio is $C_{y0}^{h,B} - C_{y0}^{h,0}$. Now, we

can define a measure for the welfare loss of the laddered portfolio relative to the optimal portfolio by

$$\Delta C_{y0}^{h} = 1 - \frac{C_{y0}^{h,B} - C_{y0}^{h,0}}{C_{y0}^{h,*} - C_{y0}^{h,0}} = \frac{C_{y0}^{h,*} - C_{y0}^{h,B}}{C_{y0}^{h,*} - C_{y0}^{h,0}}.$$

The term ΔC_{y0}^{h} denotes the relative loss in the welfare gain of the allocation that is delivered by the laddered portfolio when compared with the welfare gain of the equilibrium consumption. We use this measure for a welfare analysis of portfolios with bond ladders.

5.3 Portfolios with bond ladders

We calculate welfare losses for portfolios with bond ladders and choose some of the same model specifications as before. We use the power utility functions from Section 3.1 with the resulting linear sharing rules

$$c^{1} = \left(\frac{1}{2} - b\right) \cdot e + b \cdot 1_Y \ \text{ and } \ c^{2} = \left(\frac{1}{2} + b\right) \cdot e - b \cdot 1_Y.$$

As before, we normalize stock dividends so that the expected aggregate endowment equals one and both agents consume on average half of the endowment. The dividend vectors of the $J = 4$ independent stocks are as follows:

stock	1	2	3	4
high d	1.05	1.08	1.12	1.15
low d	0.95	0.92	0.88	0.85

The economy starts in state $y_0 = 7$ (since $c_7^1 = c_7^2 = 0.5$). The transition probabilities for all four stocks are those of Section 4.2, i.e., all four stocks have identical 2×2 transition matrices. Markets are complete with $J + 1 = 5$ bonds. We set $\xi = 0.2$, so the persistence probability[15] for a stock's dividend state is 0.6. The discount factor is $\beta = 0.95$. The equilibrium portfolios for this economy are reported in the column for $J = 4$ in Table 9. We vary the utility parameters b and γ. Table 10 reports the maximal welfare loss (always rounded up) across agents, $\Delta C = \max_{h \in \{1,2\}} \Delta C_7^h$. (We performed these welfare calculations with standard double precision. Numbers that are too close to computer machine precision to be meaningful are not reported and, instead, are replaced by "≈ 0". Again, we write 1.4 (-4) for 1.4×10^{-4}, etc.)

As expected, the relative welfare losses decrease to zero as the number B of bonds increases. However, the losses do not monotonically decrease to zero. Recall that the equilibrium portfolios exhibit holdings close to b for the one-period bond but exhibit very different holdings for bonds of other short

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[15] As always, we performed various robustness checks. We obtained qualitatively identical results for reasonable changes of the persistence parameter ξ and the discount factor. Further results can be obtained from the authors.

Table 10
Welfare loss from bond ladder

γ	1		3		5		10	
$B\backslash b$	0.05	0.3	0.05	0.3	0.05	0.3	0.05	0.3
1	1.4 (−4)	1.4 (−4)	1.4 (−3)	1.4 (−3)	7.2 (−3)	7.2 (−3)	4.3 (−2)	4.8 (−2)
2	5.0 (−6)	5.0 (−6)	3.0 (−3)	3.0 (−3)	1.3 (−2)	1.3 (−2)	6.7 (−2)	7.5 (−2)
5	2.4 (−10)	2.4 (−10)	3.2 (−3)	3.2 (−3)	1.4 (−2)	1.4 (−2)	7.1 (−2)	8.1 (−2)
10	8.3 (−13)	6.3 (−13)	2.6 (−3)	2.6 (−3)	1.2 (−2)	1.2 (−2)	6.7 (−2)	7.5 (−2)
30	≈ 0	≈ 0	7.7 (−4)	7.7 (−4)	5.1 (−3)	5.1 (−3)	4.2 (−2)	4.7 (−2)
50	≈ 0	≈ 0	1.6 (−4)	1.6 (−4)	1.4 (−3)	1.4 (−3)	1.7 (−2)	1.9 (−2)
100	≈ 0	≈ 0	1.2 (−6)	1.2 (−6)	1.3 (−5)	1.3 (−5)	2.7 (−4)	2.8 (−4)

The table reports maximal welfare losses across agents for portfolios with bond ladders. Numbers that are too close to computer machine precision to be meaningful are not reported and, instead, are replaced by "≈ 0." We abbreviate figures such as 1.4×10^{-4} by 1.4 (−4).

maturity. A trivial bond ladder of length 1 prescribes a bond holding that is not too far off from the equilibrium holding of approximately b. On the contrary, a bond ladder of length 5, e.g., forces a portfolio upon an agent that is very different from the equilibrium portfolio in the holdings of these bonds. At the same time, the ladder consists of too few bonds for the limiting behavior of Theorem 3 to set in. These facts result in the increased welfare losses for $B = 5$ and $\gamma \geq 3$. So, the agent would prefer to just hold the one-period bond instead of the ladder with five bonds. Once the ladder gets long enough, the welfare losses monotonically decrease to zero. Observe that welfare losses continue to decrease even after sufficiently many bonds are present to ensure market completeness. With $J = 4$ stocks and $Y = 16$ states, only twelve bonds are needed to complete the markets. The addition of more long-term bonds improves the performance of bond ladder strategies even though the new bonds do not improve the span of the traded assets. The longer the time to maturity of the longest bond, the smaller are both its prices across states and the standard deviation of these prices. The decreasing reinvestment risk results in smaller welfare losses of the bond ladder. Thus, redundant bonds play an important role in improving investors' welfare.

Table 11 reports the restricted portfolio weights $(\hat{m}^1, \hat{b}^1)$ for Agent 1. The last row in the table shows the coefficients of the linear sharing rule, which correspond to the holdings of stocks and the consol in an economy with a consol. The agent's holdings considerably deviate from these coefficients even when the welfare loss is already very small. For example, if $\gamma = 5$, $b = 0.3$, and $B = 50$, the holdings are $(\hat{m}^1, \hat{b}^1) = (0.290, 0.229)$ instead of $(m^1, b) = (0.2, 0.3)$ even though the welfare loss is less than 0.14%. This deviation is caused by the reinvestment risk in the longest bond. So, even though a ladder of, e.g., fifty bonds comes very close to implementing the equilibrium allocation, it uses portfolio weights different from the stock and consol weights to do so.

We recalculated all numbers in Tables 10 and 11 for various sets of parameters. The results do not change qualitatively. Similarly, changing the discount factor does not result in qualitatively different results.

Table 11
Optimal portfolio weights ($\hat{m}^1$, $\hat{b}^1$) for Table 10

$B \backslash b$	γ 1		3		5		10	
	0.05	0.3	0.05	0.3	0.05	0.3	0.05	0.3
1	(0.497,0.062)	(0.482,0.370)	(0.499,0.021)	(0.496,0.129)	(0.500,0.013)	(0.500,0.077)	(0.501,0.006)	(0.507,0.036)
2	(0.495,0.052)	(0.470,0.311)	(0.498,0.019)	(0.492,0.111)	(0.500,0.011)	(0.498,0.067)	(0.501,0.005)	(0.505,0.031)
5	(0.489,0.050)	(0.433,0.300)	(0.496,0.020)	(0.476,0.118)	(0.498,0.012)	(0.488,0.072)	(0.500,0.006)	(0.501,0.034)
10	(0.480,0.050)	(0.380,0.300)	(0.491,0.023)	(0.447,0.137)	(0.495,0.015)	(0.469,0.088)	(0.498,0.007)	(0.491,0.042)
30	(0.461,0.050)	(0.265,0.300)	(0.473,0.035)	(0.336,0.210)	(0.479,0.027)	(0.376,0.161)	(0.488,0.016)	(0.432,0.095)
50	(0.454,0.050)	(0.223,0.300)	(0.460,0.043)	(0.260,0.260)	(0.465,0.038)	(0.290,0.229)	(0.474,0.029)	(0.346,0.172)
100	(0.450,0.050)	(0.202,0.300)	(0.451,0.049)	(0.205,0.297)	(0.451,0.049)	(0.209,0.293)	(0.453,0.047)	(0.218,0.284)
(m^1, b)	(0.45,0.05)	(0.2,0.3)	(0.45,0.05)	(0.2,0.3)	(0.45,0.05)	(0.2,0.3)	(0.45,0.05)	(0.2,0.3)

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Table 12
Welfare loss from bond ladder in calibrated economy

B	$\Delta C_{y_0}^h$	$\hat{m}^1$	$\hat{b}^1$
1	5.8 (−2)	0.367	0.294
2	1.8 (−2)	0.366	0.177
5	1.7 (−3)	0.365	0.104
10	1.2 (−3)	0.362	0.088
30	8.4 (−4)	0.347	0.111
50	4.0 (−4)	0.332	0.139
100	3.3 (−5)	0.309	0.183
(m^1, b^1)		0.3	0.2

The table reports the welfare losses and restricted portfolio weights for Agent 1 for the calibrated economy in Section 3.2. We abbreviate figures such as 5.8×10^{-2} by 5.8 (−2).

5.4 Welfare losses in the calibrated economy
Table 12 reports the welfare losses and restricted portfolio weights for Agent 1 for the calibrated economy in Section 3.2. The welfare losses decrease to zero as the length B of the bond ladder increases. The portfolio weights for the optimal combination of the market portfolio and bond ladder deviate considerably from the corresponding weights in the linear sharing rule for short bond ladders but then converge to the sharing rule values as the length B increases.

6. Concluding Discussion

In light of our results, we conclude this article with a reexamination of the asset allocation puzzle. Finally, we argue that some limitations of our analysis, which are common in the literature, do not diminish the relevance of our key results.

6.1 On the asset allocation puzzle
Our analysis of investors' portfolios allows us to contribute to a recent discussion on the two-fund paradigm. In our discussion of two-fund separation in Section 2.1, we mentioned that various notions of this concept exist. The notion that most people now have in mind when they talk about two-fund separation is monetary separation, as defined in a static demand context by Cass and Stiglitz (1970). That is, people typically refer to the separation of investors' portfolios into the riskless asset and a common mutual fund of risky assets. To this day, and despite the early criticism of Merton (1973), this and other static results are often applied to dynamic contexts. Canner, Mankiw, and Weil (1997) document recommendations from different investment advisors who encourage conservative investors to hold a higher ratio of bonds to stocks than aggressive investors.[16] They point out that this financial planning

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[16] Canner, Mankiw, and Weil (1997) consider portfolios consisting of stocks, bonds, and cash, with cash being treated as the riskless asset. They document that investment advisors recommend conservative (and even

advice violates the two-fund monetary separation theorem, and they call this observation the "asset allocation puzzle." This apparent puzzle received considerable attention. Brennan and Xia (2000), Campbell and Viceira (2001), and Bajeux-Besnainou, Jordan, and Portrait (2001), among other papers, offer a resolution. In their models, the two-fund separation theorem does not hold and the optimal ratio of bonds to stocks increases with an investor's risk aversion, which coincides with the recommendations of typical investment advisors. All three papers argue that the investment horizon is important and stress that the application of the classical static results to a dynamic problem can likely lead to misleading results.

We can give a different resolution of the asset allocation puzzle based on the results of our analysis. Cash is not a riskless asset in a dynamic world. Moreover, if the investment horizon exceeds the longest available bond maturity, then investors do not have access to a safe asset. In the absence of a safe asset, we cannot expect portfolios to exhibit the classical notion of two-fund monetary separation (in its narrow static sense). Instead, in our model with many bonds, all investors, independent of their wealth and risk aversion, use the available finite-maturity bonds to generate the safe portion of their consumption stream. Investors with a higher demand for a safe consumption stream, such as more risk-averse investors, take larger positions in all bonds. For example, investors who hold nearly optimal portfolios that consist of the market portfolio and a bond ladder would make larger investments in bonds of all maturities, the larger their demand for a safe income stream becomes. Therefore, we argue that we should view bonds as part of the portfolio that generates a safe stream of income even though their prices fluctuate over time. Only the stocks should be viewed as risky assets. And, with this view, two-fund monetary separation reemerges. All investors invest some portion of their wealth in the market portfolio of stocks and the remaining portion in a portfolio of all available bonds that (approximately) delivers a safe income stream. This interpretation of our results thus reconciles the fact that bond investments are increasing in investors' risk aversion with the two-fund separation paradigm.

6.2 Limitations and implications

Similar to many other analyses of bonds in the literature, we do not account for all characteristics of bonds that a sensible investor needs to be aware of. First, we assume that the bonds in our model have no credit risk. We thus completely abstract from the possibility that the bond issuer may default. Second, all bonds in the model have a no-call provision. Strickland et al. (2009) emphasize that

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moderately risk averse) investors to hold a significant fraction of their wealth—beyond what liquidity needs would require—in cash assets. In addition, the recommended relative portions of stocks and bonds depend strongly on investors' risk attitudes. Advisors treat bonds as risky relative to cash, so that the risky portfolio consists of both stocks and bonds. The fact that the recommended ratio of these assets depends on the investor's risk aversion violates two-fund (monetary) separation.

a laddered bond portfolio should ideally consist of only non-callable bonds. Third, we abstract from tax consequences of bond investments and thus do not distinguish between taxable, tax-deferred, or tax-exempt bonds.

In addition, and again similar to much other work, we do not account for inflation. As a result, the bonds in our model should be interpreted as inflation-protected bonds. Such bonds exist; for example, the U.S. Treasury has been issuing Treasury Inflation-Protected Securities (TIPS) since 1997. Hammond (2002) advocates that investors should buy such inflation-protected bonds. Our model implies that there should be TIPS with long maturities since those are the key to a bond ladder's effectiveness.[17]

Again, similar to much other work, we do not analyze life-cycle implications for optimal portfolio choice. In a model with infinitely lived agents, we cannot examine this issue. An appropriate model might be an overlapping generations general equilibrium model. In such a model, investors' portfolio decisions will be affected by their remaining lifetime. We conjecture that a young investor with a life expectation exceeding the longest available bond maturity may still invest in ladder-like bond portfolios, while an older investor will treat bonds very differently, since he will not hold long-term bonds until maturity and thus has to account for fluctuations in the capital value of these bonds. Certainly life-cycle (and human capital) considerations for optimal portfolio decisions are an important topic for future work.

Most modern work on portfolio choice examines pure asset demand instead of equilibrium portfolios. Asset price or return processes are exogenously given and are not determined by equilibrium conditions. Instead, we employ a general equilibrium model in order to enforce a consistency between investors' preferences, dividends, and the prices of all securities. We regard our general equilibrium model to be an excellent expositional tool for our analysis. It would certainly be possible to do a similar analysis with exogenously specified nonequilibrium price processes in our model with many states and bonds.

We believe that neither our choice of model nor the limitations of our analysis diminish the relevance of our bond ladder results. Any sensible analysis of bonds with many different maturities—whether in the presence or absence of inflation or in the face of equilibrium prices or exogenously specified price processes—will also imply that long-term bonds are nearly perfect substitutes. Naturally, "optimal" portfolios of such bonds will likely exhibit the implausibly large long and short position of the nearly dependent bonds.

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[17] Hammond (2002) writes:

> *In fact, it might be more appropriate to think of inflation bonds, not as one of the portfolio's risky assets, but rather as the closest we can get to the theoretical riskless asset.*

Our analysis refines this statement. It is not an inflation-indexed bond of fixed maturity that gets closest to the theoretical riskless asset, but actually a bond, ladder of inflation-indexed bonds of varying maturities. Moreover, the longer the maturity of the longest-maturity bond, the better the bond ladder replicates the theoretical riskless asset. This finding has clear policy implications: It is beneficial for investors to have access to inflation-indexed bonds with very long maturities. Our results support the U.S. federal government's commitment to inflation-indexed bonds (Hammond 2002).

We expect that in these circumstances bond ladders will, again, serve as both simple and nearly optimal bond investment strategies for investors who want to generate a safe income stream. The introduction of redundant bonds that increase the set of available maturities further reduces the reinvestment risk of ladders and thus helps investors generate a stream of safe payoffs. In sum, the features of our analysis that make bond ladders an attractive investment strategy are robust to sensible variations of the modeling framework. It is, therefore, no surprise that we observe laddered bond portfolios as a popular investment strategy on financial markets.

Appendix

A. Equilibrium in Dynamically Complete Markets

A.1 Consumption allocations, asset prices, and portfolios

We describe Judd et al.'s (2003) three-step numerical procedure, which we use for computing equilibria in our model. In the first step, we need to solve a nonlinear system of equations in order to determine consumption allocations. In the second step, we substitute the allocations into the agents' Euler equations. The solution to these equations delivers the security prices. In the third step, we substitute the consumption allocations and asset prices into agents' budget equations. We solve these equations in order to determine equilibrium portfolios.

Step 1: Consumption allocations.   Financial markets are dynamically complete. Thus, equilibria are Pareto efficient, and the welfare theorems imply that each equilibrium allocation is the solution of a representative agent's utility maximization problem over H consumption goods. The (artificial) representative agent has a separable utility function

$$E\left\{\sum_{h=1}^{H} \lambda^h \sum_{t=0}^{\infty} \beta^t u_h(c_t)\right\},$$

where the λ^h are the Negishi weights. Without loss of generality, we normalize the weights such that $\lambda^1 = 1$. Each choice of Negishi weights $\lambda^h, h = 2, \ldots, H$ implies a Pareto efficient allocation that corresponds to an equilibrium (with transfers). In turn, every equilibrium implies Negishi weights so that the equilibrium consumption allocation is the optimal solution to the representative agent's utility maximization problem.

As we mentioned before in Section 1.2, Pareto efficiency also implies that the consumption allocation is time-homogeneous, i.e., the allocation only depends on the current exogenous Markov shock $y \in \mathcal{Y}$. Thus, as is done throughout the article, we can denote the consumption of agent h in state y by c_y^h. At the optimal solution to the representative agent's utility maximization problem, the marginal utilities of all agents are collinear, i.e.,

$$u_1'(c_y^1) = \lambda^h u_h'(c_y^h), \qquad h = 2, \ldots, H, \ y \in \mathcal{Y},$$

and we can normalize Arrow-Debreu prices to be equal to the marginal utility of Agent 1. Therefore, we define $P_y = u_1'(c_y^1)$ to be the price of consumption in state y, and denote by $P = (P_y)_{y \in \mathcal{Y}} \in \mathbb{R}_{++}^Y$ the price vector.

The (Arrow-Debreu) budget constraint for each agent h states that the present value of her consumption must equal the present value of his initial endowment. Let V_y^h be the present value

of consumption for agent h if the economy starts in state $y \in \mathcal{Y}$. We can compute V_y^h by solving the recursive equation

$$V_y^h = P_y c_y^h + \beta E \left\{ V_{y+}^h | y \right\}, \qquad y \in \mathcal{Y},$$

where $E \left\{ V_{y+}^h | y \right\}$ denotes the conditional expectation of the present value in the next period given that the economy is currently in state y. In matrix terms, these equations can be written as $V^h = P \otimes c^h + \beta \Pi V^h$ and have the unique solution $V^h = [I_Y - \beta \Pi]^{-1} (P \otimes c^h)$, where $\otimes$ denotes the element-wise multiplication of vectors and I_Y denotes the $Y \times Y$ identity matrix. Let W_y^h denote the present value of the dividends of agent h's initial portfolio if the economy starts in state $y \in \mathcal{Y}$; W_y^h is the solution to

$$W_y^h = P_y \left(\sum_{j=1}^{J} \psi_j^{h,0} d^j \right) + \beta E \left\{ W_{y+}^h | y \right\}, \qquad y \in \mathcal{Y}.$$

The unique solution to these equations is $W^h = [I_Y - \beta \Pi]^{-1} (P \otimes \sum_{j=1}^{J} \psi_j^{h,0} d^j)$.

If the economy starts in the state $y_0 \in \mathcal{Y}$ at period $t = 0$, then the budget constraint for the Arrow-Debreu model requires that

$$V_{y_0}^h = W_{y_0}^h, \qquad h = 1, \ldots, H.$$

Due in part to Walras' law, it actually suffices to require this last equation for only $H - 1$ agents. Substituting the expressions for the present values of consumption and portfolio payoffs into the budget constraints leads to the equations

$$\left([I_Y - \beta \Pi]^{-1} (P \otimes (c^h - \sum_{j=1}^{J} \psi_j^{h,0} d^j)) \right)_{y_0} = 0,$$

for $h = 2, \ldots, H$. Market clearing requires that

$$\sum_{h=1}^{H} c_y^h = e_y, \qquad y \in \mathcal{Y}.$$

Recall that $e_y = \sum_{j=1}^{J} d_y^j$ denotes the social endowment in state $y \in \mathcal{Y}$.

In sum, if the economy starts in the state $y_0 \in \mathcal{Y}$ at period $t = 0$, then the Negishi weights and consumption vectors must satisfy the following equations:

$$u_1'(c_y^1) - \lambda^h u_h'(c_y^h) = 0, \qquad h = 2, \ldots, H, \ y \in \mathcal{Y}, \quad \text{(A1)}$$

$$\left([I_Y - \beta \Pi]^{-1} (P \otimes (c^h - \sum_{j=1}^{J} \psi_j^{h,0} d^j)) \right)_{y_0} = 0, \qquad h = 2, \ldots, H, \quad \text{(A2)}$$

$$\sum_{h=1}^{H} c_y^h - e_y = 0, \qquad y \in \mathcal{Y}. \quad \text{(A3)}$$

Equations (A1)–(A3) have $HS + (H-1)$ unknowns, HS unknown state-contingent, agent-specific consumption levels c_y^h, and $H - 1$ Negishi weights λ^h. This system of nonlinear equations has as many equations as unknowns. The welfare theorems imply that this system always has at least one solution. Any solution to Equations (A1)–(A3) yields an equilibrium state-contingent consumption c_y^h for agent $h = 1, \ldots, H$ in state $y \in \mathcal{Y}$.

Step 2: Asset prices. Once we have computed the consumption vectors, we can use the Euler equations of Agent 1 to obtain closed-form solutions for asset prices. For stock j, the Euler equation of Agent 1 states that

$$p_y^j P_y = \beta E \left\{ P_{y+} (p_{y+}^j + d_{y+}^j) | y \right\}, \qquad y \in \mathcal{Y},$$

which is a system of Y linear equations in the Y unknown prices $p_y^j, y \in \mathcal{Y}$. The unique solution of this linear system is the price vector of stock j,

$$p^j \otimes P = [I_Y - \beta \Pi]^{-1} \beta \Pi (P \otimes d^j). \tag{A4}$$

Similarly, the price of a consol is given by

$$q^c \otimes P = [I_Y - \beta \Pi]^{-1} \beta \Pi P. \tag{A5}$$

We calculate the price of finite-maturity bonds in a recursive fashion. For a one-period bond, the Euler equations for Agent 1 are

$$q_y^1 P_y = \beta E \left\{ P_{y+} | y \right\}, \qquad y \in \mathcal{Y},$$

which yield the price of the one-period bond in state y,

$$q_y^1 = \frac{\beta \Pi_{y.} P}{P_y} = \frac{\beta \sum_{z=1}^{Y} \Pi_{yz} P_z}{P_y}, \qquad y \in \mathcal{Y}, \tag{A6}$$

where $\Pi_{y.}$ denotes row y of the matrix Π. Then, the price of the bond of maturity k is

$$q_y^k = \frac{\beta \Pi_{y.} (P \otimes q^{k-1})}{P_y} = \frac{\beta \sum_{z=1}^{Y} \Pi_{yz} P_z q_z^{k-1}}{P_y}, \qquad y \in \mathcal{Y}. \tag{A7}$$

Repeated substitution yields the bond price formula

$$q_y^k = \frac{\beta^k (\Pi^k)_{y.} P}{P_y} = \frac{\beta^k \sum_{z=1}^{Y} (\Pi^k)_{yz} P_z}{P_y}, \qquad y \in \mathcal{Y}. \tag{A8}$$

Step 3: Portfolios. Given the consumption allocations and asset prices, the budget Equations (1) or (2) now determine the asset positions for economies with a consol or finite-maturity bonds, respectively. Observe that the budget equations of agent h are a square linear system of Y independent equations in the Y unknown portfolio variables, which has a unique solution.

A.2 Linear sharing rules

We can easily calculate the linear sharing rules for the three HARA families of utility functions under consideration. Some straightforward algebra yields the following sharing rules (as a function of Negishi weights, which are determined in Step 1 of the numerical procedure).

For power utility functions, the linear sharing rule is

$$c_y^h = e_y \cdot \left(\frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right) + \left(A^h - \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \sum_{i \in \mathcal{H}} A^i \right) = m^h e_y + b^h. \tag{A9}$$

Note that for CRRA utility functions, $A^h = 0$ for all $h \in \mathcal{H}$, the sharing rule has zero intercept, $b^h = 0$, and household h consumes a constant fraction

$$m^h = \left(\frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right)$$

of the total endowment. For CARA utility functions the linear sharing rules are

$$c_y^h = e_y \cdot \frac{\tau^h}{\sum_{i \in \mathcal{H}} \tau^i} + \left(\tau^h \ln(\lambda^h) - \frac{\tau^h}{\sum_{i \in \mathcal{H}} \tau^i} \sum_{i \in \mathcal{H}} \tau^i \ln(\lambda^i) \right), \tag{A10}$$

where $\tau^h = 1/a^h$ is the constant absolute risk tolerance of agent h.

A.3 Allocations and prices for the example in Section 2.3

Consumption allocations are as follows:

$$c^1 = (0.688, 0.666, 0.666, 0.644)^\top = \frac{6}{11} \, (d^1 + d^2) - 0.425,$$

$$c^2 = (0.678, 0.667, 0.667, 0.656)^\top = \frac{3}{11} \, (d^1 + d^2) + 0.121,$$

$$c^3 = (0.674, 0.667, 0.667, 0.660)^\top = \frac{2}{11} \, (d^1 + d^2) + 0.304.$$

The fluctuations of agents' consumption allocations across the four states are fairly small. The reason for this small variance is the small dividend variance of the two stocks. The state-contingent stock prices are solutions to any agent's Euler equations and are

$$p^1 = (19.43, 19.01, 18.98, 18.58)^\top,$$

$$p^2 = (19.40, 18.98, 19.01, 18.60)^\top.$$

The price vector of the consol is

$$q^c = (19.40, 18.99, 19.01, 18.61)^\top.$$

Bond prices for bonds of various maturity are

$$q^1 = (0.963, 0.946, 0.954, 0.938)^\top,$$

$$q^2 = (0.918, 0.899, 0.906, 0.887)^\top,$$

$$q^5 = (0.790, 0.773, 0.775, 0.758)^\top,$$

$$q^{10} = (0.612, 0.599, 0.599, 0.586)^\top,$$

$$q^{25} = (0.284, 0.277, 0.277, 0.271)^\top,$$

$$q^{50} = (0.079, 0.077, 0.077, 0.075)^\top.$$

B. Kronecker Products

Let A be an $n \times p$ matrix and B be an $m \times q$ matrix. Then, the Kronecker or direct product $A \otimes B$ is defined as the $nm \times pq$ matrix

$$A \bigotimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1p}B \\ a_{21}B & a_{22}B & \cdots & a_{2p}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{np}B \end{bmatrix}.$$

For our purposes, we need the following property of the Kronecker product. If A and B are stochastic (Markov matrices), then $A \otimes B$ is stochastic.

Table A1
Summary statistics for detrended exogenous endowment and dividend process

	μ	ρ	$\sigma(\varepsilon)$
e_t	1.016	0.935	0.0316
div_t	1.237	0.909	0.1049

C. Calibrated Economy: Input Data

Dave and Pohl (2010) estimate consumption and dividend processes for the U.S. economy employing the dataset of Mehra and Prescott (1985) extended until 2004. As a measure of aggregate consumption, they use the sum of per-capita nondurable consumption (deflated by the corresponding nondurable price index) and services consumption (deflated by the services price index). For dividends, they use the total amount of S&P 500 dividends in a year deflated by the average producer price index for that year. They detrend the data to make them stationary and assume a long-term growth rate of 1.58%, as estimated by DeJong and Ripoll (2007). Dave and Pohl (2010) back out endowment (nondividend income) data from the detrended dividend and consumption data. Using the estimate of Cecchetti, Lam, and Mark (1993)—dividends make up about 4% of consumption—and to account for the scaling of the S&P 500, they scale the detrended dividends by a factor of $S = 0.032$. They finally obtain endowment data by subtracting the scaled dividend data from consumption data. Finally, Dave and Pohl (2010) estimate joint AR(1) processes for the logarithm of detrended dividends, d_t, and the logarithm of detrended endowments, e_t.

$$e_{t+1} = (1 - \rho_e)\mu_e + \rho_e e_t + \varepsilon_{te} \tag{A11}$$

$$div_{t+1} = (1 - \rho_{div})\mu_{div} + \rho_{div}div_t + \varepsilon_{tdiv} \tag{A12}$$

Table A1 reports the point estimates. (For standard errors, see Dave and Pohl 2010.)

Dave and Pohl (2010) estimate the covariance of the two error terms to be $Cov(\varepsilon_{te}, \varepsilon_{tdiv}) = 0.001$. (They do not provide a standard error on this estimate.) Since this value is rather tiny and to simplify the generation of a discrete-state Markovian shock process for our general equilibrium model, we assume that the two processes are uncorrelated. Therefore, we can employ the procedure of Tauchen and Hussey (1991) to approximate the two individual autoregressive processes by Markov chains. We use the Matlab implementation of the Tauchen-Hussey procedure accompanying Flodén (2007) in order to obtain two separate Markov chains with five states each. Since the input data for the procedure are in logarithmic units, we exponentiate the computed values for the endowment vector and the dividend vector, respectively. We multiply the dividend vector by $S = 0.032$ in order to obtain dividends in the appropriate proportion to the endowment vector. The resulting payoff vectors as well as the two Markov chains are as follows:

The five-point approximation for the nondividend income (endowment) process is given by

$$(2.579, 2.674, 2.762, 2.853, 2.958).$$

The corresponding transition matrix is as follows:

$$\begin{bmatrix} 0.6161 & 0.3154 & 0.0636 & 0.0048 & 0.0001 \\ 0.2486 & 0.4292 & 0.2593 & 0.0591 & 0.0038 \\ 0.0481 & 0.2491 & 0.4056 & 0.2491 & 0.0481 \\ 0.0038 & 0.0591 & 0.2593 & 0.4292 & 0.2486 \\ 0.0001 & 0.0048 & 0.0636 & 0.3154 & 0.6161 \end{bmatrix}.$$

The five-point approximation for the dividend process is given by

$$(0.087712, 0.098912, 0.11024, 0.12288, 0.13856).$$

The corresponding transition matrix is as follows:

$$\begin{bmatrix} 0.5995 & 0.3261 & 0.0688 & 0.0054 & 0.0002 \\ 0.2396 & 0.4292 & 0.2656 & 0.0615 & 0.0041 \\ 0.0474 & 0.2490 & 0.4072 & 0.2490 & 0.0474 \\ 0.0041 & 0.0615 & 0.2656 & 0.4292 & 0.2396 \\ 0.0002 & 0.0054 & 0.0688 & 0.3261 & 0.5995 \end{bmatrix}.$$

The final 25×25 Markov transition matrix for our model is then the Kronecker product (see Appendix 6.2) of the two individual 5×5 Markov transition matrices.

Dave and Pohl (2010) use their estimated AR(1) processes to match the level of asset prices in a standard representative agent Lucas model. In their calibration exercises, they find the best fit of the model to the data for values of the discount factor β between 0.96 and 0.99 and for values of the risk-aversion parameter γ between 2.5 and 3.5. Thus, we report results for our general equilibrium model with $\beta = 0.97$ and $\gamma = 3$. Our results are robust to changes in these two parameters.

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