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Asset Trading Volume with Dynamically Complete Markets and Heterogeneous Agents

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ABSTRACT

Trading volume of infinitely lived securities, such as equity, is generically zero in Lucas asset pricing models with heterogeneous agents. More generally, the end-of-period portfolio of all securities is constant over time and states in the generic economy. General equilibrium restrictions rule out trading of equity after an initial period. This result contrasts the prediction of portfolio allocation analyses that portfolio rebalancing motives produce nontrivial trade volume. Therefore, other causes of trade must be present in asset markets with large trading volume.

EACH DAY FINDS INVESTORS actively trading assets. However, the Lucas (1978) asset pricing model, the foundation of much of general equilibrium finance theory, says little about volume since it assumes a representative agent (or, equivalently, several identical investors) and has no trade in equilibrium. If markets are complete or can be completed through dynamic trading of the available securities (as in Kreps (1982)), then asset prices evolve as if there is a single agent even when there are several agents with different tastes and income processes. Therefore, representative agent models are generally valuable for a theory of asset pricing with complete markets. This approach says nothing about trading volume, which is unfortunate since data on volume may give us additional information about the operation of asset markets and the underlying tastes of investors. This paper examines equilibrium asset trading in the Lucas model with agent heterogeneity and dynamically complete markets. We characterize equilibrium in a constructive fashion and present an algorithm to compute equilibrium prices and trading volume. We find that trading volume of infinitely lived assets is zero in the generic economy. In general, we find each investor's portfolio is constant over time and states once one controls for the maturing of finitely lived assets.

The intuition is clear and follows directly from linear algebra and market completeness. Suppose, for the sake of simplicity, that the current dividend summarizes all information about future dividends.[1] Then the dividend process is a

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[1] This is not necessary. All that is needed is that the current state of the dividend process is common knowledge.

Markov process where we identify the current state[2] with the current dividend. Suppose that there are S states and S long-lived securities where each security's payoff depends solely on the current dividend. If utility is separable over time with constant discount rate (as assumed in Lucas (1978) and in most of the portfolio allocation literature) then each agent's optimal consumption policy is a function of the exogenous state, and is also a vector of S numbers. If markets are dynamically complete, then the state-contingent dividends from the S long-lived assets are S independent vectors, and any state-contingent consumption plan equals the returns generated by some unique *fixed and constant* combination of the S assets. If an agent's endowment does not equal the fixed portfolio that produces the desired consumption process, he can obtain that fixed portfolio through trading in the initial period. Therefore, any consumption plan can be implemented by some trade-once-and-hold-forever trading strategy. By concavity, there is a unique optimal consumption plan; hence, the trade-once-and-hold-forever strategy that implements the optimal consumption process must be the unique optimal trading strategy. This is true for each agent and for any price process. Therefore, it must hold in equilibrium. We show that this intuition generalizes to a mixture of long- and short-lived assets, finding that the holding of assets of any specific maturity is constant after initial trading when markets are dynamically complete.

Our analysis shows that general equilibrium effects have a substantial impact on asset volume; in fact, they rule out asset trading in the standard Lucas model with heterogeneous agents. This no-trade result is initially surprising. When an agent trades a small set of securities, then, for almost all price processes, he will need to trade assets to implement his desired consumption plan. Therefore, one would expect nontrivial equilibrium trading volume in response to new information about future security returns even with symmetric information and dynamically complete security markets. This intuition is confirmed by the classic continuous-time portfolio analysis in Merton (1971), which predicts infinite transaction volume. The recent literature on portfolio allocation also predicts substantial transaction volume; this includes the Brennan and Xia (2002) analysis of how inflation affects dynamic asset allocation, Viceira's (2001) examination of life-cycle asset allocation, and the Dammon, Spatt, and Zhang (2001) study of asset allocation in the presence of capital gains taxes. These studies are partial equilibrium analyses and argue that trading volume is generally substantial for arbitrary combinations of utility functions and price processes. In these models, investors trade because they have targets for their portfolio allocations, but consumption and price movements push portfolios away from these targets, making trade necessary to get back to the target allocations. In contrast, we show that

[2] We need to be careful to distinguish the two possible meanings of the term "state". In the Arrow security approach, the states differ in both the time and dividend dimensions. In a Markov chain, the term state typically refers only to the current state of information about the current and future value of the dividend. In this paper, we will mean the state of the Markov process describing dividends when we say "state" unless it is clear we mean otherwise. The term *date-event* refers to what is called a "contingent state" in Arrow (1964) and "event" in Debreu (1959).

when markets are complete and investors face equilibrium prices, there will often be no trade in equilibrium, essentially because consumption and prices move together in general equilibrium so as to negate any need for trading to get back to target portfolios. Our results indicate that we need to be careful when interpreting partial equilibrium results. In general, partial equilibrium can tell us about investors' consumption target portfolios and their consumption processes, but partial equilibrium analysis is less reliable in predicting the equilibrium trading volume.

The paper is organized as follows. Section I describes the standard model of an infinite-horizon pure exchange economy, and Arrow–Debreu equilibrium and financial market equilibrium are defined. In Section II we present the numerical procedure for computing equilibrium, and show that end-of-period asset holdings are generically constant. Section III illustrates the main points with an example. Section IV concludes.

I. The Asset Market Economy

We examine a standard Lucas asset pricing model with heterogeneous agents and complete asset markets. Time is indexed by $t \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}$. A time-homogeneous Markov process of exogenous states $(y_t)_{t\in\mathbb{N}_0}$ is valued in a discrete set $Y = \{1, 2, \ldots, S\}$. The Markov transition matrix is denoted by Π. A date-event σ_t is the history of states up to time t, that is, $\sigma_t = (y_0, y_1, \ldots y_t)$. Let Σ_t denote the possible histories σ_t up to time t. Let $\Sigma = \cup_t \Sigma_t$ denote all possible histories of the exogenous states.

We assume a finite number of types $\mathscr{H} = \{1, 2, \ldots, H\}$ of infinitely lived agents. There is a single perishable consumption good, which is both produced by firms and included in individual endowments. The firms distribute their output each period to its owners through dividends. Investors trade in securities in order to transfer wealth across time and states. We assume that markets are complete with $J = S$ linearly independent assets traded on financial markets. Without loss of generality and for ease of notation, we assume that each asset is either an infinitely lived (long-lived) asset or a single-period (short-lived) asset. Short-lived assets are in zero net supply.

There are $J^\ell \geq 0$ long-lived assets. Asset j pays a dividend $d^j : Y \to \mathbb{R}_+$, $j = 1, \ldots, J^\ell$, which depends solely on the current state $y \in Y$. In addition, there are $J^s = S - J^\ell \geq 0$ short-lived securities issued in each period. Short-lived asset j issued in period t pays $d^j : Y \to \mathbb{R}_+$, $j = J^\ell + 1, \ldots, S$, in period $t+1$, and then expires. Agent h's portfolio at the end of period t at $\sigma \in \Sigma$ is $\theta_t^h(\sigma) \equiv (\theta_t^{h1}(\sigma), \ldots, \theta_t^{hS}(\sigma)) = (\theta_t^{hl}(\sigma), (\theta_t^{hs}(\sigma)) \in \mathbb{R}^S$, where $\theta_t^{hl}(\sigma) \in \mathbb{R}^{J^\ell} (\theta_t^{hs}(\sigma) \in \mathbb{R}^{J^s})$ denotes agent h's portfolio of long-lived (short-lived) assets. His initial endowment of the long-lived assets prior to time 0 is denoted by θ_{-1}^{hl}. We assume that the agent has zero initial endowment of the short-lived assets and, in order to rule out speculative bubbles, that all infinitely lived assets are in positive net supply. Agent h also has an individual endowment $e^h : Y \to \mathbb{R}_{++}$ of the consumption good at each time. The aggregate endowment of the economy in state y is $e(y) = \sum_{h=1}^{H} (e^h(y) + \theta_{-1}^{hl} d^l(y))$.

Each agent h has a time-separable state-dependent utility function

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t, y_t)\right\}, \tag{1}$$

where c is a consumption process. We assume that the state-dependent utility functions $u_h(.,y): \mathbb{R}_{++} \to \mathbb{R}$ are strictly monotone, C^2, strictly concave, and satisfy the Inada property, that is, $\lim_{c\to 0} u'(c, y) = \infty$ where we let $u'(c, y)$ denote $\frac{\partial}{\partial c}(u(c,y))$. We assume that the discount factor $\beta \in (0, 1)$ is the same for all agents, and that all agents agree[3] on the transition matrix for the dividend process.

Let the matrices

$$e = \begin{pmatrix} e^1(1) & \cdots & e^1(S) \\ \vdots & \ddots & \vdots \\ e^H(1) & \cdots & e^H(S) \end{pmatrix}, \quad d = \begin{pmatrix} d^1(1) & \cdots & d^1(S) \\ \vdots & \ddots & \vdots \\ d^S(1) & \cdots & d^S(S) \end{pmatrix} \tag{2}$$

represent individual endowments and security dividends. The vector of utility functions is $u = (u_1, \ldots, u_H)$. We collect the primitives of the economy with financial markets in the expression $\mathscr{E} = (e, d, \Pi, \beta, U)$.

This is a simple model but includes many features thought to affect trading volume. For example, we include income shocks to individuals[4], a feature present in many models, such as Viceira (2001). We also allow individuals to have different tastes for risk and for the dividend process to have time-varying mean and variance. The presence of these factors makes it a reasonable model to study trading volume with heterogeneous agents. In particular, we can examine the portfolio rebalancing and risk-sharing function of asset trading.

A. Arrow–Debreu Equilibrium

We define the Arrow–Debreu equilibrium for an economy $\mathscr{E}$ which would arise in a world with an Arrow contingent security for every date-event. We take consumption at time 0 to be the numeraire. The price of the consumption good at σ is denoted $p(\sigma)$; similarly, we define $y(\sigma)$ and $c(\sigma)$. Let $\omega^h(y) = e^h(y) + \theta_{-1}^{h\ell} d^\ell(y)$ denote the initial endowment of agent $h \in \mathscr{H}$ in state y.

DEFINITION 1: *An Arrow–Debreu equilibrium for an economy $\mathscr{E}$ is a collection of prices $(\bar{p}(\sigma))_{\sigma\in\Sigma}$ and consumption plans $(\bar{c}(\sigma))_{\sigma\in\Sigma}$ satisfying the following conditions:*

(1) $\sum_{h=1}^H \bar{c}^h(\sigma) = \sum_{h=1}^H \omega^h(y(\sigma))$ *for all* $\sigma \in \Sigma$.

(2) *For each agent, the consumption plan $\bar{c}$ maximizes $U_h(c^h)$ given the lifetime budget constraint* $\sum_{\sigma\in\Sigma} \bar{p}(\sigma)\omega^h(y(\sigma)) = \sum_{\sigma\in\Sigma} \bar{p}(\sigma)c^h(\sigma)$.

[3] Conventional rational expectations assume that the agents know the true transition probabilities. The only thing we need for our analysis is that agents agree.

[4] More precisely, we allow shocks to income of agent types, not purely idiosyncratic shocks.

Bewley (1972) proved, under slightly different assumptions on the fundamentals, the existence of an Arrow–Debreu equilibrium and the first and second welfare theorem for the economy $\mathcal{E}$. Mas-Colell and Zame (1991) generalized the results to our framework.

B. Financial Market Equilibrium

It is unreasonable to assume a separate security for each time and state contingency. Instead, the Lucas model and our generalization forces agents to use the S available assets to achieve the desired consumption stream. Let $q_t(\sigma) \equiv (q_t^1(\sigma), \ldots, q_t^S(\sigma))^T = (q_t^l(\sigma), q_t^s(\sigma))^T$ be the ex-dividend price of assets in period t at node σ. At each node $\sigma_t = (y_0, \ldots, y_t)$, agent h faces a budget constraint[5],

$$c_t^h = e^h(y_t) + \theta_{t-1}^{hs} d^s(y_t) + \theta_{t-1}^{hl}(q_t^l + d^l(y_t)) - \theta_t^h q_t, \tag{3}$$

where $d^l(y_t)$ ($d^s(y_t)$) is the column vector of the payoffs of the long-lived (short-lived) assets at y_t. The notion of a financial market equilibrium is defined as follows.

Definition 2: *A financial market equilibrium for an economy $\mathcal{E}$ is a process of portfolio holdings $\{(\bar\theta^1, \ldots, \bar\theta^H)\}$ and asset prices $\{(\bar q, \ldots, \bar q^J)\}$ satisfying the following conditions:*

(1) $\sum_{h=1}^H \bar\theta_t^h = \sum_{h=1}^H \theta_{-1}^h$ *for all* $t > 0$.

(2) *For each agent* h:

$$\begin{aligned} (\bar\theta^h, c^{-h}) &\in arg\max_{\theta,c} U_h(c) \\ c_t^h &= e^h(y_t) + \theta_{t-1}^{hs} d^s(y_t) + \theta_{t-1}^{hl}(q_t^l + d^l(y_t)) - \theta_t^h \bar q_t \\ \sup_{\sigma\in\Sigma} &|q(\sigma)\theta^h(\sigma)| < \infty. \end{aligned}$$

It is well known that in the absence of speculative bubbles, if there are as many assets as states, then there is generically a one-to-one correspondence between Arrow–Debreu equilibria and efficient financial market equilibria; we make this point precise below. The following existence theorem is proven in Kubler and Schmedders (2003).

Theorem 1: *For a generic set of short-lived asset dividends, the economy $\mathcal{E}$ has at least one efficient financial market equilibrium.*

II. A Theorem on Constant Portfolios

In this section, we show that for a generic set of short-lived assets' dividends every contingent market equilibrium is equivalent to a financial market equilibrium in which the end-of-period portfolio holdings of each agent is constant

[5] We now drop the σ arguments to make the expressions less clumsy.

after an initial adjustment in period 0. We prove this in a constructive fashion. The constructive proof also provides an algorithm for the computation of equilibria in our model.

We proceed in three steps. First, the welfare theorems tell us that the equilibrium allocation is the solution of a representative agent's maximization problem over h consumption goods. The artificial representative agent has a separable utility function

$$E\left\{\sum_{h=1}^{H}\lambda^h\sum_{t=0}^{\infty}\beta^t u_h(c_t,y_t)\right\},\tag{4}$$

where the λ^h are the Negishi weights (see Negishi (1960)). Each choice of the Negishi weights λ^h, $h=2,\ \ldots,\ H$ implies a Pareto efficient allocation which corresponds to an equilibrium (with transfers) with security prices $\bar{q}$ and portfolio choices $\bar{\theta}^h$ for all agents $h=1,\ \ldots,\ H$. In fact, once we have computed a Pareto efficient consumption allocation we will be able to give closed-form solutions for both asset prices and portfolios. An equilibrium corresponds to a set of Negishi weights such that each agent's consumption process equals his wealth when valued at the prices implied by those weights.

A. Equilibrium Solution

The following lemma is the basis for showing that all relevant economic variables exhibit time homogeneity. It follows directly from the first welfare theorem.

LEMMA 1: *For an economy $\mathcal{E}$ every efficient equilibrium exhibits time-homogeneous Markovian consumption processes for all agents.*

Proof: The lemma follows from Pareto efficiency and time separability, and is well known in the general equilibrium finance literature. In particular, section 20 in Duffie (1988) develops these points in the same manner we use here. ■

The recursive property of equilibrium proven in Lemma 1 is the key to all of our results.[6] First, note that the artificial representative agent's utility function can be rewritten as

$$E\left\{\sum_{t=0}^{\infty}\beta^t\left(\sum_{h=1}^{H}\lambda^h u_h(c_t,y_t)\right)\right\},\tag{5}$$

where the λ^h are state- and time-independent weights on individual utilities.

Second, we take advantage of recursivity in our notation. We change the notation and express the dependence of all variables on the exogenous state through a subscript. The notation here will be inconsistent with the notation above, but will

[6] In this paper we focus solely on efficient equilibria. To our knowledge there are no (non)-existence results for inefficient equilibria.

allow us to exploit the recursive nature of any equilibrium. For example, c_y^h will denote the consumption of agent h in state y.

We introduce some other useful notation. The state in a current period is denoted by y and the random variable of the subsequent state is denoted by y_+ We will use a circle, $\circ$, to denote element-wise multiplication of vectors. Specifically, if x, $y \in R^S$ then

$$x \circ y = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_S y_S \end{pmatrix} \in R^S. \tag{6}$$

Finally, I_S is the $S \times S$ identity matrix.

With these model definitions and our notation, we are now ready to solve for equilibrium. A three-step process will compute the Negishi weights, individual consumption processes, asset prices, and individual portfolios.

Step 1 computes the Negishi weights. At the optimal solution to the representative agent's optimization problem, the derivatives $u_h'(c_y^h, y)$ at each state y are collinear across agents; that is,

$$u_1'(c_y^1, y) = \lambda^h u_h'(c_y^h, y), h = 2, \ldots, H. \tag{7}$$

Since marginal utilities are collinear, we can replace Arrow-Debreu prices with the marginal utility of agent 1. Therefore, we define $p_y = u_1'(c_y^1, y)$ to be the price of consumption in state y, and we let $p = (p_y)_{y \in Y} \in R^S$ be the vector of prices.

The budget constraint for each agent h states that the present value of his consumption must equal the present value of his initial endowment. Let V_y^h be the present value of consumption for agent h if the economy starts in state $y \in Y$. We can compute V_y^h by solving the recursive equation

$$V_y^h = p_y c_y^h + \beta E\{V_{y+}^h | y\}, \; y \in Y. \tag{8}$$

In matrix terms, (8) implies $V^h = p \circ c^h + \beta \Pi V^h$ and has the unique solution $V^h = [I_s - \beta\Pi]^{-1}(p \circ c^h)$. Let W_y^h denote the present value of agent h's endowments and portfolio dividends if the economy starts in state $y \in Y$; W_y^h is the solution to

$$W_y^h = p_y \omega_y^h + \beta E\{W_{y+}^h | y\}, \; y \in Y. \tag{9}$$

The unique solution to (9) is $W^h = [I_S - \beta\Pi]^{-1}(p \circ \omega^h)$.

If the economy starts in the state $y_0 \in Y$ at period $t = 0$, then the budget constraint for the Arrow-Debreu model requires that $V_{y_0}^h = W_{y_0}^h$ for $h = 1, \ldots, H$. Due to Walras' law, it actually suffices to require this last equation for the first $H - 1$ agents only. So, we require

$$\left[I_S - \beta \prod\right]^{-1} (p \circ (c^h - \omega^h))_{y_0} = 0 \tag{10}$$

for $h=1, \ldots, H-1$. Market clearing requires that

$$\sum_{h=1}^{H} c_y^h = \sum_{h=1}^{H} \omega_y^h, \quad \forall y. \tag{11}$$

The system of equations (7, 10, 11) has $HS+(H-1)$ unknowns, HS unknown state-contingent, agent-specific consumption levels c_y^h, and $H-1$ Negishi weights λ^h. Any solution to (7, 10, 11) is an equilibrium state-contingent consumption c_y^h for agent $h=1,\ldots,H$ in state $y \in Y$.

Step 2 computes asset prices using the Euler equations of agent 1 and the normalization $p_y = u'_1(c, y)$. For a long-lived asset j, the Euler equations for agent 1 imply

$$q_y^j p_y = \beta E\{p_{y+}(q_{y+}^j + d_{y+}^j)|y\}, y \in Y, \tag{12}$$

which is a system of S linear equations in S unknowns. The solution is

$$q^j \circ p = \left[I_S - \beta \prod\right]^{-1} \beta \prod (p \circ d^j). \tag{13}$$

For a short-lived asset j, the Euler equations for agent 1 are

$$q_y^j p_y = \beta E\{p_{y+} d_{y+}^j | y\}. \tag{14}$$

Therefore, the price for short-lived asset j in state y is

$$q_y^j = \frac{\beta E\{p \circ d_{y+}^j | y\}}{p_y} = \frac{\beta \prod_y (p \circ d^j)}{p_y}. \tag{15}$$

Step 3 computes the equilibrium portfolios. We show that they are also recursive; that is, depending solely on the state y and any state to which the dividend process can move from y. The result is well known if we had only long-lived assets; for example, see section 20 in Duffie (1988). Here we state and prove the generalization to our case with short- and long-lived assets.

LEMMA 2: *In a financial market equilibrium with time-homogeneous Markovian consumption processes, the equilibrium portfolio holdings of all agents must also be time-homogeneous Markovian processes.*

Proof: To simplify the notation, we omit the agent superscript h in this proof. By assumption the dividend process is Markovian. Lemma 1 and Step 2 imply that both the agents' consumption processes and the asset price processes must be Markovian in an efficient financial markets equilibrium. Therefore, an agent's utility maximization problem with these Markovian dividend and price processes can be equivalently stated as a stationary dynamic programming problem.

$$V(w, y) = \max_{\theta}\{u(c_y) + \beta E(V(w^+, s))\}, \tag{16}$$

where

$$c_y = e_y + w - \theta \cdot q_y$$
$$w_s^+ = \theta^l \cdot (q_s^l + d_s^l) + \theta^s \cdot d_s^s. \tag{17}$$

Define $D_1 = (q^1 + d^1, \ldots, q^{J^\ell} + d^{J^\ell}, d^{J^{\ell+1}}, \ldots, d^S)^\top$. Kubler and Schmedders (2003) show that D_1 has full rank generically in the dividends of the short-lived assets. Thus, there is a one-to-one relationship between portfolios and the S-vector of next-period wealth w^+. Moreover, the strict concavity of the utility function implies that there exists a policy function determining the unique optimal choice $\theta = \Theta(w, y)$. In equilibrium, the dynamic programming problem yields the equilibrium consumptions. At every date-event node σ an agent's wealth w equals the present value of his future excess consumption, that is; the difference between the present value of his future consumption and his future endowments. The present values of future consumption and future endowments are both Markovian; therefore, wealth must also be Markovian. Hence, at most S, different wealth values w_y, $y = 1, \ldots, S$ can appear in equilibrium, and only S different portfolios $\theta_y = \Theta(w_y, y)$ can appear in equilibrium. ■

Lemma 2 implies that we can write θ_y^h for the portfolio of agent h at node σ with $y = y(\sigma)$. Assume that the transition matrix Π has no zero elements; therefore, all states can be reached in one transition from each state. Then the budget constraint implies that if the current state is y and the previous state was z, then the end-of-period portfolio in state z must finance the consumption and investment choices in state y. This collection of budget constraints for agent h across the various states implies

$$\theta_z^{h\ell}(q_y^\ell + d_y^\ell) + \theta_z^{hs}(d_y^s) = c_y^h - e_y^h + \theta_y^h q_y, \forall y, z \in Y. \tag{18}$$

Equation (18) is a collection of S^2 equations for the S^2 unknown end-of-period portfolios. The key fact is that the right-hand side of (18) is the allocation of wealth across consumption and investment in state y and, because of recursivity, cannot depend on z, whereas the left-hand side of (18) depends strongly on the previous period's state. Recursivity implies that an agent must have the proper resources in state y to carry out the stationary plan for current and future consumption no matter what the state was in the previous period even though the initial wealth in state y depends on the portfolio at the end of the previous period. This clearly puts strong constraints on the possible values of θ_z^h, the end-of-period portfolio of type h agents in state z. In fact, we will see that, generically, this is possible only if θ_z^h is independent of the state z. We now present the details.

The system (18) is equivalent to

$$\begin{array}{ll} \theta_z^{h\ell}(q_s^\ell + d_s^\ell) + \theta_z^{hs}(d_s^s) = \theta_y^{h\ell}(q_s^\ell + d_s^\ell) + \theta_y^{hs}(d_s^s) & y, z, s \in Y, z, y \neq s \\ \theta_y^{h\ell}(q_y^\ell + d_y^\ell) + \theta_y^{hs}(d_y^s) = c_y^h - e_y^h + \theta_y^h q_y & y \in Y \end{array}. \tag{19}$$

Recall that $D_1 = (q^1 + d^1, \ldots, q^{J^\ell} + d^{J^\ell}, d^{J^\ell+1}, \ldots, d^S)$. Equation (19) is equivalent to:

$$(\theta_y^{h\ell} - \theta_z^{h\ell}, \theta_y^{hs} - \theta_z^{hs})D_1 = 0, \qquad \forall y, z \in Y \tag{20}$$

$$\theta_y^{h\ell} d_y^\ell + \theta_y^{hs}(d_y^s - q_y^s) = c_y^h - e_y^h, \qquad \forall y \in Y, \tag{21}$$

where the row vector $q^j = (q_1^j, \ldots, q_S^j)$ denotes prices of asset j across states $y \in Y$. Similarly, $d^j = (d_1^j, \ldots, d_S^j)$ denotes the row vector of dividends of asset j across all states. We have argued above that the $S \times S$-matrix D_1 has (generically) full rank S. Then equations (20) imply that $\theta_y^h = \theta_z^h$, for all states y, $z \in Y$. Hence, we can define the state-independent portfolio vector $\Theta^h \equiv \theta_y^h$ for all $y \in Y$ and equations (21) become

$$(\Theta^{h\ell}, \Theta^{hs})D_2 = c^h - e^h, \tag{22}$$

where $D_2 = (d^1, \ldots, d^{J^\ell}, d^{J^{\ell+1}} - q^{J^{\ell+1}}, \ldots, d^S - q^S)^T$. The $S \times S$-matrix D_2 has full rank S if and only if D_1 has full rank. So, for generic dividends of the short-lived assets, the system (22) implies that for all $h = 1, \ldots, H$

$$(\Theta^{h\ell}, \Theta^{hs}) = (c^h - e^h)D_2^{-1}. \tag{23}$$

Substituting the equilibrium prices $\bar{q}$ into equation (23) leads to the equilibrium portfolio holdings $\bar{\Theta}^h$ for all agents $h = 1, \ldots, H$.

Furthermore, there is a special case that deserves attention. Suppose that there are no short-lived assets and that the dividend payoff matrix d is nonsingular. Being nonsingular is a generic property, so its assumption is natural. The absence of short-lived assets implies that asset prices q do not appear in (23). Then, the nonsingularity of d implies that (23) is surely solvable. This is the key special case that captures the basic intuition of the result.

We summarize our findings for the equilibrium portfolios in the following theorem.

THEOREM 2: *Consider an economy $\mathscr{E} = (e, d, \Pi, \beta, u)$. For any choice of u, e and β if Π has no zero entries and the rows of d^l, the dividends for the long-lived assets, are linearly independent, then for a generic set of dividends of the short-lived assets, d^s, in every Pareto-efficient equilibrium, the end-of-period portfolio of each agent is constant after the initial round of trading. Therefore, the trading volume for each infinitely lived security is zero after the initial round of trading. Furthermore, if there are no short-lived assets, then the portfolio of each agent is constant after the initial period.*

Theorem 2 shows that there will be no trade in assets after an initial period. This is a strong result that relies on many assumptions. We assume infinitely lived agents with additively separable utility and a common discount factor. An overlapping generations model will produce trade as may models with nonseparable preferences. The result is still surprising, since we may think that heterogeneous risk preferences would motivate some portfolio rebalancing in equilibrium even in this simple model.

B. Complete Markets with S–1 Short-lived Assets

For infinite-horizon economies with only short-lived assets, we can use Theorem 2 to obtain another surprising result about equilibrium portfolios.

THEOREM 3: *Consider an economy $\mathscr{E} = (e, d, \Pi, \beta, u)$ with S–1 short-lived assets. Assume Π has no zero entries. Furthermore, assume the economy starts in state y_0 at $t = 0$ and that all short-lived assets have zero payoff in state y_0. For any choice of u, e and β, there is a generic set of dividends of the (short-lived) assets, d, such that there exists a financial market equilibrium that is equivalent to the Arrow–Debreu equilibrium. In those cases, the financial market equilibrium is complete with $S - 1$ securities and the end-of-period portfolio of each agent is constant over time and states.*

Proof: Consider an infinite-horizon pure exchange economy with a time-homogeneous discrete Markov process of exogenous states and S short-lived assets. Theorem 2 implies that in equilibrium the type h budget constraint at time $t = 0$ is $c^h_{y0} = e^h_{y0} - \bar{\Theta}^h \bar{q}$. Theorem 2 also implies that, in equilibrium, we can write the type h budget constraint in state y_0 at any time $t > 0$ as $c^h_{y0} = e^h_{y0} + \bar{\Theta}^h d_{y0} - \bar{\Theta}^h \bar{q}$. These two equations imply that $\bar{\Theta}^h d_{y0} = 0$ for all h. Now suppose that there is only a single asset with nonzero payoffs in the state y_0. In that case, the last equation implies that every agent has a zero position in that asset in all states. In other words, the agents only trade the other $S - 1$ short-lived assets and still obtain complete markets consumption. This observation completes the proof of the theorem. ■

C. Generalizations

We next present two extensions of Theorem 2. Space limitations prevent us from presenting the details, but the statements of these results help highlight our basic points and indicate the limitations of Theorem 2.

Theorem 2 assumes that all assets are either infinitely lived or one-period bonds. There is some trade in the sense that new bonds are issued in each period. Suppose that there are many risk-free assets with finite maturity exceeding one period. For example, there may be 10- and 20-year bonds. Then the generalization of Theorem 2 says that each investor holds the same amount of each kind of security at the end of each period. When we interpret this result, we must keep in mind that any particular finite-term bond changes maturity over time. For example, a bond maturing in 2020 is a 20-year bond in 2000 but becomes a 10-year bond in 2010. In 2010, an investor may need to sell or buy some 2020 bonds to bring his holdings of 2020 bonds to the desired level for bonds maturing in 10 years. There is portfolio rebalancing, but it is just the maturity rebalancing necessary to neutralize changes in maturity structure that would occur without trade. Similar considerations apply to options. The key general result is that the end-of-period holding of any kind and maturity of security is unchanged over time.

Theorem 2 assumes that Π has no zero entries. While most Π have no zero entries (in the sense of genericity), some natural dividend processes have sparse Π

matrices. For example, if dividends follow a simple random walk, then from each state $y \in Y$, there are only two states $z \in Y$ for which $\Pi_{yz} \neq 0$. In this case, only two assets are needed to dynamically complete the market; see Kreps (1982). Theorem 2 does not apply, and there will be trade in this case. This is clear, since constant holdings of two assets can span only a two-dimensional space of consumption plans, but the efficient allocation will lie in a larger space. The techniques of Theorem 2 can still be used to determine that volume, as long as the consumption allocation is efficient.

III. A Numerical Example

The three steps in our construction of equilibrium can be executed numerically. The first step, computing the Negishi weights and consumption functions, requires solving the nonlinear system of equations (7, 10, 11). The second and third steps, computing the asset prices and the portfolios, respectively, require solving linear systems of equations. For small examples it is possible to implement the algorithm in an Excel spreadsheet utilizing the solver add-in. We find equilibrium allocations, prices, and portfolios in a few seconds, even when we require a relative numerical error[7] of less than 10^{-10}.

We use this algorithm to compute a simple example illustrating the key features of Theorem 2. We construct an example where demand considerations lead us to think that there should be trading over time and, indeed, would cause substantial trade for most price processes. Of course, Theorem 2 shows that equilibrium conditions imply that there will be no trade. This example will help clarify the basic insights of Theorem 2.

Assume $H = 2$ agents with CRRA utility functions, $u(c) = c^{1-\gamma}/(1-\gamma)$. Type 1 agents will be relatively risk tolerant with $\gamma_1 = 0.5$ and type 2 investors will have $\gamma_2 = 4$. The common discount factor equals $\beta = 0.95$, assuming a period of time equal to about a year. Assume $S = 3$ exogenous states. The first asset is long-lived (call it stock 1) and has a dividend vector $d^{s_1} = (1, 1, 0.8)^T$. The second asset (stock 2) is also long-lived and has a dividend vector $d^{s_2} = (0.8, 1, 1)^T$. The third asset is a riskless short-lived bond paying one unit of the consumption good in every state, so $d_y^b = 1$ for $y \in \{1, 2, 3\}$. Both stocks are in unit net supply, and the agents have both an initial endowment of $\theta_{-1}^{1s_i} = \theta_{-1}^{2s_i} = 0.5$ for $i = 1, 2$. The bond is in zero net supply, and the agents have zero initial holding of this security at the beginning of period 0. The agents' individual state-contingent endowments are $e^1 = (1.5, 1, 1)$ and $e^2 = (0.5, 1, 1)$. The Markov transition matrix is

$$\prod = \begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.48 & 0.04 & 0.48 \\ 0.04 & 0.48 & 0.48 \end{bmatrix}.$$

We assume that the economy starts in state 2.

[7] Relative numerical error refers to the consumption equivalent error in the Euler equations. For example, when we say that we had an error less than ε in solving an equation of the form $u_1'(c_y^1, y) - \lambda^h u_h'(c_y^h, y)$, we mean that our solutions satisfied $|u_1'(c_y^1, y) - \lambda^h u_h' 1(c_y^h, y)| < \varepsilon |u_1'(c_y^1, y)|$.

This example captures two interesting features of asset markets and income processes. First, the conditional variances of the two stocks' dividends change across the three states. In state 1, stock 1 has a low one-period dividend variance, and stock 2 has greater conditional variance. In state 2, the stocks have equal dividend variance. Finally, in state 3, the two stocks have reversed their roles compared to state 1, making stock 2's dividends less risky in the short run. Standard portfolio rebalancing intuition would have the risk tolerant agent 1 holding the riskier stock 2 in state 1, but then trading it for stock 1 when state 3 occurs, since stock 1 is then riskier. We specify the dividend process here; when we later compute the equilibrium price process, we will check to see if the assets' conditional returns display the same pattern of conditional riskiness.

Second, agents have substantial endowment risk that may lead them to trade assets in response to endowment shocks in order to smooth out consumption. In state 1, agent 1's endowment is more correlated with stock 1's dividends than stock 2's dividends, whereas in state 3, these correlations are almost eliminated. Agent 2 faces the opposite situation. Therefore, standard intuition says that agent 1 (2) should hold more (less) of stock 2 over stock 1 in state 1, but be relatively indifferent in state 3. This example is constructed so that the risk aversion and endowment shock effects work, according to partial equilibrium logic, in the same direction, pushing agent 2 to hold stock 1 in state 1, but to sell it off in state 3.

We first compute the Negishi weights. Without loss of generality, we can fix $\lambda^1 = 1$. Solving the optimality and feasibility conditions shows that $\lambda^2 = 6.8017$. We also find that state-contingent consumptions are

$$\begin{aligned} c^1 &= (2.0351, 2.2162, 2.0351) \\ c^2 &= (1.7649, 1.7838, 1.7649). \end{aligned}$$

The consumption solution allows us to compute asset prices for all assets. The Euler equations of agent 1 imply that the prices of the stocks and the bond are

$$\begin{aligned} q^{s_1} &= (17.5466, 18.2178, 17.4030) \\ q^{s_2} &= (17.4030, 18.2178, 17.5466) \\ q^b &= (0.93097, 0.98971, 0.93097). \end{aligned}$$

The conditional expected stock returns are

$$\begin{aligned} r^{s_1} &= (0.074569, 0.010468, 0.074750) \\ r^{s_2} &= (0.074750, 0.010468, 0.074569) \end{aligned}$$

and the risk-free rates are $r^f = (0.074146, 0.010394, 0.074146)$. The asset price process is similar to the dividend process. The conditional expected excess returns of the two stocks are (in percentages)[8]

[8] This example is grossly inconsistent with observed equity premia, as is any simple Lucas model with moderate risk aversion.

$$r^{s_1} - r^f = (4.23\times10^{-2}, 7.43\times10^{-3}, 6.04\times10^{-2})$$
$$r^{s_2} - r^f = (6.04\times10^{-2}, 7.43\times10^{-3}, 4.23\times10^{-2}).$$

This pattern of excess returns follows the riskiness of the dividends, since each stock has a larger conditional excess return and conditional equity premium in those states where its dividends have greater conditional one-period ahead variance. Therefore, asset 2 is unambiguously riskier than asset 1 in state 1, and vice versa in state 3.

Step 3 of the algorithm tells us that the constant portfolio for agent 1 is

$$\Theta^1 = (\theta^{1s_1}, \theta^{1s_2}, \theta^{1b}) = (-.61529, 1.8847, -5.1778)$$

and the constant portfolio for agent 2 is $(1, 1, 0) - \Theta^1$. The holdings are easily understood in terms of unconditional covariances. Each agent shorts that stock with dividends positively correlated with his endowment process and goes long in the other stock, which has dividends negatively correlated with his endowment process. In particular, agent 1 shorts the first stock and buys the second stock. The holdings of the safe asset correspond to relative risk aversion of the agents, with the more risk-averse agent 2 buying the riskless bond from agent 1. There is no need to trade across states, since the returns of the three assets are linearly independent and portfolio Θ^h of these three assets implements the desired three-dimensional consumption plan of type h households.

IV. Conclusion

Volume is an important aspect of financial markets. We show that the standard intuitions about asset trading and portfolio rebalancing implied by standard partial equilibrium portfolio theory are misleading when we include general equilibrium considerations. Portfolio allocation models predict that an investor will engage in substantial trading, unless the price process lines up precisely with his marginal utility process, a seemingly unlikely occurrence. However, we show that this coincidence will often hold in general equilibrium, and that trading volume is zero in equilibrium in the generic Lucas (1978) model with heterogeneous agents. Therefore, general equilibrium conditions have a substantial impact on trading volume. This result indicates that other factors considered in the literature, such as life-cycle factors, asymmetric information, heterogeneous beliefs, and incompleteness of the asset market, play a significant role in generating trade volume.

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