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Asymptotic methods for aggregate growth models

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Abstract

We use perturbation methods to compute optimal policy functions in simple continuous- and discrete-time aggregate growth models. We demonstrate that computing the kth degree Taylor expansion of the policy function around the steady state involves solving one quadratic equation and $k-1$ linear equations. We also compute Padé expansions, and show that both Taylor and Padé expansions can provide excellent solutions far from the steady state.

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0. Introduction

The increasing use of dynamic models with optimizing agents has led to sophisticated qualitative analyses of economic problems; unfortunately, quantitative analysis has been limited by the lack of closed-form solutions. This problem often leads analysts to examine only special cases such as linear-quadratic objectives with linear laws of motion. While these special cases may suffice (see Hansen and Sargent (1990) for a discussion of the large variety of economic problems which can be analyzed with linear models), in many cases they are inadequate for a robust analysis of interesting problems.

Approximation methods offer an alternative to closed-form solutions. In this paper we examine *perturbation*, or *asymptotic*, approximation methods. The basic idea of asymptotic methods is to formulate a general problem, find a particular

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case which has a known solution, and then use that particular case and its solution as a starting point for computing approximate solutions to nearby problems, where the approximations are derived using implicit function theorems. These methods are widely used in mathematical physics, particularly in quantum mechanics and general relativity theory, with much success. While economists have often used special versions of perturbation techniques, such as linearizing around a steady state, they have generally not exploited the full range and power of asymptotic techniques.

The method we develop below strictly generalizes linearizing around the steady state. Linearization methods (or, at least, the valid ones) compute asymptotically valid linear approximations to the law of motion in the neighborhood of the steady state. The method we develop below generalizes this by computing a Taylor series expansion for the equilibrium law of motion which begins with the linear approximation but then goes on to add higher-order terms. Some have argued that this approach is impractical; in particular, in his World Congress survey of these methods, Marcet (1994) stated that 'perturbation methods of order *higher* than one are considerably more complicated than the traditional linear-quadratic case...' In this paper, we will demonstrate that the high-order terms in the Taylor expansions produced by perturbation methods are, in fact, *easier* to compute than the linear terms, as well as show that they substantially improve the global quality of the approximation.

In this paper we shall outline asymptotic approximation techniques for deterministic one-sector aggregate growth problems. The technique calculates a local approximation to policy functions based on the steady state of the problem. Such steady states are generally the solution to a system of nonlinear algebraic equations and are easily solved numerically even if they do not have closed-form solutions. We will use local information to calculate linear and higher-order approximations for the equilibrium of the deterministic problem near the steady state.

The result of this method is either a high-degree polynomial or a rational function which approximates the solution around a point. There are many reasons why these approximations may be valuable. First, if the implicit infinite series is a valid expression of equilibrium over a nonzero interval of some variable, then we have solved the model over that interval. Second, if the expansion is asymptotically valid, that is, the approximation error vanishes rapidly as the perturbation variable becomes zero, then we can determine the nature of equilibrium for conditions sufficiently near the deterministic steady state. Third, even if the expansion is not asymptotically valid, experience indicates that they may be of value nonetheless in some numerical procedures.

The purpose of this paper is to acquaint economists with the perturbation approach for two simple, stable growth problems. The key point is that these calculations are far simpler than earlier speculation predicted. In fact, in each of these problems, all calculations are linear except for one step, the linear approximation,

which involves solving a quadratic equation. The presence of a quadratic equation is not surprising to anyone familiar with linearizing around a saddle-point stable steady state. The important fact is that calculating the higher-order terms of the Taylor expansions involves only linear equations. Since linear equations are generally considered to be easier than solving quadratic equations, there is no support for the contention that 'perturbation methods of order *higher* than one are considerably more complicated' than linear approximations; in fact, the higher-order terms are *easier* to compute.

A *Warning* to the reader should be made at this point. The perturbation computations below are strictly formal. Problem-specific sufficient conditions for convergence are available for some of the problems and are discussed in an extensive, but difficult, collection of mathematical papers (see Bensoussan and his references). It is not our intent to reproduce those mathematical developments.[1] The purpose of this paper is to take an economic model of interest, and derive plausible formal expansions which are known to be valid under some restrictions. We then test them using economically intuitive diagnostics to ascertain the quality of the approximations.

We find that these formal series do remarkably well in approximating dynamic equilibria. Since there is nothing special about the functional equations which describe optimal growth problems, we anticipate that this approach will yield good approximations for a wide variety of dynamic economic models.[2]

1. Approximation methods based at a point

Before deriving asymptotic expansions for a growth model, we will review basic approximation theory for functions of a single variable. There are two basic methods, Taylor series and Padé expansions, to approximate a function using only information about the function at a specific point. These methods produce locally good approximations with little effort, and sometimes these methods produce approximations which are useful more globally.

1.1. Taylor series approximation

The most basic local approximation is described by Taylor's Theorem:

[1] Since writing this paper, the authors became aware of Santos (1994) which reviews the relevant issues well.

[2] As this paper goes to press, the results of this paper are being generalized to growth models with nonseparable utility, uncertainty, elastic labor supply, taxation, heterogeneous agents, incomplete asset structures, and several capital stocks, and to dynamic games. See Judd (1991) for an early treatment of these issues.

Taylor's Theorem. If $f \in C^{n+1}[a,b]$ *and* $x, x_0 \in [a,b]$, *then*

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + R_{n+1}(x),$$

where

$$R_{n+1}(x) = \frac{1}{n!} \int_{x_0}^{x} (x - t)^n f^{(n+1)}(t)\,\mathrm{d}t = \frac{(x - x_0)^{(n+1)}}{(n+1)!} f^{(n+1)}(\xi),$$

for some ξ *between* x *and* x_0.

A Taylor series approximation of $f(x)$ based at x_0 uses derivative information at x_0 to construct a polynomial approximation. It is valid to a high order in the neighborhood of x_0. If f is analytic on $[a,b]$ then this approximation is progressively better on $[a,b]$ as n increases. Generally, this approximation is good only near x_0 and decays rapidly away from x_0.

1.2. Rational approximation

Taylor series approximation constructs a polynomial to approximate a function f. An alternative approximation method is to use the same information to construct a rational function, i.e., a ratio of polynomials. Rational approximation based at a point is called *Padé Approximation.* The (m,n) Padé approximant of f at x_0 is a rational function

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials, the degree of p is m, the degree of q, is n, and

$$0 = \frac{\mathrm{d}^k}{\mathrm{d}x^k}(p - fq)(x_0), \quad k = 0, \ldots, m+n.$$

The polynomial p has $m+1$ coefficients and q has $n+1$ coefficients which must be fixed by these conditions. The $m+n+1$ derivative conditions suffice since $q(x_0)$ can be taken to be 1; if $q(x_0) \neq 1$ and $q(x_0) \neq 0$, dividing both $p(x)$ and $q(x)$ by $q(x_0)$ will yield an equivalent fraction where the denominator is 1 at x_0. One usually chooses $m = n$ or $m = n+1$. If f is increasing or decreasing in the neighborhood of x_0, it is advisable to use $m = n+1$ since the ratio will then be roughly proportional to x.

The problem of computing the coefficients of p and q is a linear problem. This follows from the fact that the conditions can be rewritten as $q(x_0)=1$ and

$$p^{(k)}(x_0) = (fq)^{(k)}(x_0), \quad k = 0, \ldots, m+n,$$

which are all linear in the coefficients of p and q once the derivatives of f at x_0 are computed. The resulting system may be singular, which just implies a multiplicity of solutions; in such cases, we reduce n, the degree of the denominator polynomial, and repeat the process. The resulting system of conditions is 'usually' nonsingular; in fact, it was always nonsingular in the cases we examined.

The costs of the Taylor and Padé approximations (holding fixed the number of coefficients) are comparable. Both procedures have a fixed cost of computing the coefficients. On this score, Taylor approximations have less cost because both must compute the same derivatives of f but Padé approximation requires the solution to a linear equation. The experience is that Padé approximants are better global approximants than Taylor series approximations, that is, the error grows less rapidly as we move away from x_0. There are strong theorems confirming this for analytic functions; see Braess (1986), and Cuyt and Wuytack (1986). For this reason, computers typically use Padé approximants to compute trigonometric, exponential, and other functions.

2. Continuous-time deterministic growth

We will first look at a single-sector, single good, continuous-time optimal growth problem. If k is the capital stock and c is the rate of consumption, then the problem is

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c)\,\mathrm{d}t,$$

$$\dot{k} = f(k) - c,$$

where $u(c)$ is the concave utility function and $f(k)$ is the concave gross production function. This is a very simple problem, assuming an inelastic labor supply, separable utility, and no uncertainty. While these assumptions greatly simplify the calculations, they are not essential for our main points. Once we follow in detail the perturbation method for this simple problem, it is easy to program a symbolic language, such as Mathematica or Macsyma, to deal with the more complex cases. The focus on a single dimension does result in much less notational detail. We leave the problems of several dimensions for a future paper.[3]

[3] Judd and Gaspar (1996) contains an introductory treatment of these extensions.

The general Dynamic Programming equations for the value function, $V(k)$, and policy function, $C(k)$, are

$$\rho V(k) = u(C(k)) + V'(k)(f(k) - C(k)), \tag{1}$$

$$0 = u'(C(k)) - V'(k). \tag{2}$$

It is well-known that this problem has a steady state, k^{ss}, and that it is defined by $\rho = f'(k^{ss})$. However, to get the perturbation analysis started, we will first show that Eqs. (1) and (2) can be used to derive the location of the steady state as well as its local behavior. Differentiate (1) with respect to k:

$$\rho V' = u'C' + V''(f - C) + V'(f' - C'). \tag{3}$$

Since $u'(C(k)) = V'(k)$, (3) implies

$$\rho V' = V''(f - C) + V'f'. \tag{4}$$

At the steady state, k^{ss}, $f(k^{ss}) = C(k^{ss})$. Hence (4) implies that

$$\rho V'(k^{ss}) = V''(k^{ss})(f(k^{ss}) - C(k^{ss})) + V'(k^{ss})f'(k^{ss}) = V'(k^{ss})f'(k^{ss}). \tag{5}$$

Since $V' = u' > 0$, we have k^{ss} defined by

$$\rho = f'(k^{ss}). \tag{6}$$

As we develop the approximation, we will use the familiar example of Cobb–Douglas production and homothetic utility functions to illustrate these techniques. If $f(k) = Ak^{\alpha}$, then $f'(k) = A\alpha k^{\alpha-1}$. So, (6) implies

$$k^{ss} = [\rho/(\alpha A)]^{1/(\alpha-1)}. \tag{7}$$

If $u(c) = c^{1+\gamma}/(1+\gamma)$, then $u'(c) = c^{\gamma}$. So, $f(k^{ss}) = C(k^{ss})$ implies

$$C(k^{ss}) = A(k^{ss})^{\alpha} \tag{8}$$

and (1) implies that

$$V(k^{ss}) = \frac{(A(k^{ss})^{\alpha})^{\gamma+1}}{(\gamma+1)\rho} \tag{9}$$

and (2) implies that

$$V'(k^{ss}) = u'(C(k^{ss})) = (A(k^{ss})^{\alpha})^{\gamma}. \tag{10}$$

Our goal is to compute the Taylor series expansion of these functions around the steady state. Specifically, we want to compute the coefficients of the following:

$$C(k) \doteq C(k^{ss}) + C'(k^{ss})(k - k^{ss}) + C''(k^{ss})(k - k^{ss})^2/2 + \cdots$$

$$V(k) \doteq V(k^{ss}) + V'(k^{ss})(k - k^{ss}) + V''(k^{ss})(k - k^{ss})^2/2 + \cdots$$

We have so far computed $k^{ss}, V(k^{ss}), C(k^{ss}), f'(k^{ss})$, and $V'(k^{ss})$. We want to compute more information about V and C. We next move to $C'(k^{ss})$ and $V''(k^{ss})$. At this point we must make a critical assumption:

Assumption 1. $V(k)$ and $C(k)$ are C^∞ at k^{ss}.

This assumption is clearly excessive, but not unrealistic if we also assume that $u(c)$ and $f(k)$ are also C^∞. Assumption 1 can be proved for this problem with concave u and f by extending the analysis in Judd (1985). Note that all we need is sufficient differentiability at the steady state. This indicates that these calculations are possibly valid even if there are kinks at points away from the steady state, a feature of some models with nonconcave production functions. With Assumption 1, we can proceed to compute the desired derivatives.

We first differentiate (4) with respect to k, yielding

$$\rho V'' = V'''(f - C) + V''(f' - C') + V''f' + V'f''. \tag{11}$$

We also differentiate (2) with respect to k:

$$0 = u''C' - V''. \tag{12}$$

At the steady state, $f = C$ and $\rho = f'$. Hence, at k^{ss}, (11) implies

$$0 = -V''C' + V''f' + V'f''. \tag{13}$$

Substituting (12) into (13) yields

$$0 = -u''(C')^2 + u''C'f' + V'f''. \tag{14}$$

Hence $C'(k^{ss})$ must solve (14):

$$C' = \frac{u''f' \pm \sqrt{(u''f')^2 + 4u''V'f''}}{2u''} \tag{15}$$

$$= \frac{\rho}{2}\left(1 \pm \sqrt{1 + \frac{4u'f''}{u''f'f'}}\right)$$

where we use the fact that $\rho = f'(k^{ss})$ and $u'(C(k)) = V'(k)$. Since $u'' < 0$, (15) has two solutions. However one is negative. Since $C' > 0$ is known, we choose

the positive root. For our choice of functional forms, $u''(c)=\gamma c^{\gamma-1}$ and $f''(k)=\alpha(\alpha-1)Ak^{\alpha-2}$. So, since $C(k^{ss})=f(k^{ss})$,

$$C'(k^{ss})=\frac{\rho}{2}\left(1+\sqrt{1-\frac{4(1-\alpha)}{\gamma\alpha}}\right), \tag{16}$$

$V'''(k^{ss})$ is computed directly from (12).

To demonstrate the ease with which higher-order terms can be calculated, we next move to $C''(k^{ss})$ and $V'''(k^{ss})$. Differentiate (11) and (12) with respect to k:

$$\rho V''' = V''''(f-C)+2V'''(f'-C')+V''(f''-C'') + V'''f'+2V''f''+V'f''', \tag{17}$$

$$0=u'''(C')^2+u''C''-V'''. \tag{18}$$

At k^{ss}, $f=C$ and $\rho=f'$. So (17) reduces to

$$0=2V'''(f'-C')+3V''f''-V''C''+V'f'''. \tag{19}$$

Look closely at (18) and (19). At $k=k^{ss}$, we know that $c=C(k^{ss})$, which in turn implies that we know the steady-state values of $u''', u'', C', f', f'', V''$, and f'''. The only unknowns in (18) and (19) are $C''(k^{ss})$ and $V'''(k^{ss})$. In fact, we can write (18) and (19) as a *linear* system:

$$\begin{pmatrix} u'' & -1 \\ V'' & -2(f'-C') \end{pmatrix}\begin{pmatrix} C'' \\ V''' \end{pmatrix}=\begin{pmatrix} -u'''(C')^2 \\ 3V''f''+V'f''' \end{pmatrix}. \tag{20}$$

The determinant of the matrix in (20) is $-2u''(f'-C')+V''$, which is negative since (16) implies that $f'<C'$ at k^{ss}. Hence, (20) has a unique solution. Using (12) to eliminate V'', (20) implies that the steady state values of C'' and V''' satisfy

$$C''=\frac{2(\rho-C')u'''C'C'+3u''C'f''+u'f'''}{u''(3C'-2\rho)}$$

and

$$V'''=\frac{u'''C'C'C'+3u''C'f''+u'f'''}{3C'-2\rho}.$$

In the case of $u'(c)=c^{\gamma}$ and $f(k)=Ak^{\alpha}$, we have $u''(c)=\gamma c^{\gamma-1}$, $u'''(c)=\gamma(\gamma-1)c^{\gamma-2}$, and $f'''=\alpha(\alpha-1)(\alpha-2)Ak^{\alpha-3}$. The solution for C'' becomes

$$C''=\frac{2(\rho-C')(\gamma-1)C'C'C^{-1}+3\alpha(\alpha-1)Ak^{\alpha-2}C'}{3C'-2\rho} + \frac{C\alpha(\alpha-1)(\alpha-2)Ak^{\alpha-3}/\gamma}{3C'-2\rho},$$

where k is the steady-state value, that is, the value such that $A\alpha k^{\alpha-1}=\rho$. If $A=\rho/\alpha$, then the steady-state value for k is 1.

Now we move to C''' and $V^{(4)}$. Differentiate (17) and (18) with respect to k:

$$\begin{aligned}\rho V^{(4)} &= V^{(5)}(f-C)+3V^{(4)}(f'-C')\\ &\quad +3V'''(f''-C'')+V''(f'''-C''')\\ &\quad +V^{(4)}f'+3V'''f''+3V''f'''+V'f^{(4)}. \end{aligned} \tag{21}$$

$$0=u^{(4)}(C')^3+3u'''C'C''+u''C'''-V^{(4)}. \tag{22}$$

At k^{ss}, (21) reduces to

$$\begin{aligned}0 &= 3V^{(4)}(f'-C')+3V'''(f''-C'')+V''(f'''-C''')\\ &\quad +3V'''f''+3V''f'''+V'f^{(4)}. \end{aligned} \tag{23}$$

Again, in (22)–(23), the only unknowns are C''' and $V^{(4)}$. Furthermore, (22)–(23) is linear in these unknowns. A check of the linear system shows that, again, there is a unique solution for C''' and $V^{(4)}$.

One could go on forever with these calculations. Note that after the first step the procedure is rather mechanical, since the later calculations are linear in nature. In particular, the general linear system is

$$\begin{pmatrix} u'' & -1 \\ V'' & -n(f'-C') \end{pmatrix}\begin{pmatrix} C^{(n)} \\ V^{(n+1)} \end{pmatrix}=\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

for some forcing terms A_1 and A_2. Since $-nu''(f'-C')+V''$, the determinant, is always negative, the system is always determinate.

The validity of this procedure, that is, a proof that $C(k)$ is smooth and has the steady state derivatives computed above, can be demonstrated by a trivial extension of the computation of C' of $C(k)$ in Judd (1985). The key fact is that the steady state is stable. The main issue, however, is how good this approximation is. We will address that below after we show how to apply this procedure to discrete-time problems.

We should also point out that the optimal control nature of the problem examined above is not essential. Equilibrium models which do not solve optimal control problems can also be analyzed with these methods. For example, if there were a constant tax on capital income, τ, then the steady-state capital stock satisfies functional equations similar to those above; see Judd (1985) for details and for a demonstration that $C(k)$ is C^∞ when u and f are C^∞. Similarly, models with money in the utility or production functions could be analyzed with these perturbation methods.

3. Discrete-time growth

We will next show how to apply these ideas to discrete-time problems. Again we take the simple optimal growth problem

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$k_{t+1} = f(k_t - c_t).$$

In this formulation, we are assuming that k_t is the capital stock in hand at the beginning of period t, and that out of it must come today's consumption, c_t, with the remaining capital, $k_t - c_t$, used in production, and with the resulting output, $f(k_t - c_t)$, serving as the beginning-of-period capital stock in period $t+1$.

The solution can be expressed as a policy function, $C(k)$, satisfying the Euler equation

$$u'(C(k)) = \beta u'(C(f(k - C(k))))f'(k - C(k)). \tag{24}$$

At the steady state, k^{ss}, we have $f(k^{ss} - C(k^{ss})) = k^{ss}$, implying

$$u'(C(k^{ss})) = \beta u'(C(k^{ss}))f'(k^{ss} - C(k^{ss})), \tag{25}$$

which in turn implies

$$1 = \beta f'(k^{ss} - C(k^{ss})) \tag{26}$$

which uniquely determines k^{ss}. Furthermore

$$k^{ss} = f(k^{ss} - C(k^{ss})). \tag{27}$$

Again, we assume that C has as many derivatives as necessary. Araujo and Scheinkman (1977) demonstrates this for locally stable steady states, such as this one. Taking the derivative of (24) with respect to k implies[4]

$$\begin{aligned} u''(C(k))C'(k) = {} & \beta u''(C(f(k - C(k))))C'(f(k - C(k))) \\ & \times f'(k - C(k))[1 - C'(k)]f'(k - C(k)) \\ & + \beta u'(C(f(k - C(k))))f''(k - C(k))[1 - C'(k)]. \end{aligned} \tag{28}$$

At $k = k^{ss}$, this reduces to (we will now drop all arguments)

$$u''C' = \beta u''C'f'(1 - C')f' + \beta u'f''(1 - C'). \tag{29}$$

[4] To help understand the expression, we use parentheses to denote function composition and brackets to denote a multiplicative factor, but for this expression only.

This is a quadratic equation with the solution

$$C' = \frac{1}{2}\left(1 - \beta - \beta^2 \frac{u'}{u''} f'' + \sqrt{\left(1 - \beta - \beta^2 \frac{u'}{u''} f''\right)^2 + 4 \frac{u'}{u''} \beta^2 f''}\right). \quad (30)$$

If we take another derivative of (28) and set $k = k^{ss}$, we find

$$\begin{aligned} u''C'' + u'''C'C' = {} & \beta u'''(C'f'(1 - C'))^2 f' + \beta u''C''(f'(1 - C'))^2 f' \\ & + 2\beta u''C'f'(1 - C')^2 f'' + \beta u' f'''(1 - C')^2 \\ & + \beta u' f''(-C''), \end{aligned} \quad (31)$$

which is a linear equation in the unknown C''. This analysis can continue to compute higher order terms; however, it is clear that the discrete-time case has much greater algebraic complexity than the continuous-time case. We will stop with C''.

We again point out that equilibrium with distortions, such as those which arise from the presence of money or taxes, can be analyzed using these perturbation methods. The key ingredient is having the equilibrium decision rules expressed as determinate solutions to well-behaved functional equations.

4. Global quality of asymptotic approximations

The ideas underlying the foregoing asymptotic methods validate only a local concept of approximation. For example, we can be confident about the asymptotic expansion of $C(k)$ around the steady state only for a sufficiently small neighborhood of the steady state. We will next examine the global properties of the asymptotic expansions. In particular, we will try to determine the range over which the expansions are good. There are several reasons for this. If the asymptotic expansions are good over a broad range, we may not need to use the much slower standard numerical procedures. Even if the expansions are good over only a small range, they may still serve as good initial guesses for standard numerical procedures and significantly improve their performance.

We need to compute an index indicating how well the approximating series does at each value of k. We first discuss the discrete-time case. If the computed series was the exact solution, then the Euler equation, (24), would hold exactly. We will judge the quality of the approximation by how much the series fails in satisfying (24). We do not want to just calculate the error in (24) since that error depends on the units chosen for consumption and capital. However, we can rewrite (24) and define the Euler equation error as

$$R(k) = 1 - (u')^{-1}(\beta u'(C(f(k - C(k))))f'(k - C(k)))/C(k). \quad (32)$$

This expresses the Euler equation error at k as a fraction of consumption at k under the consumption rule $C(k)$. This expression is actually a residual, that is, the deviation from zero which occurs when we substitute the solution into the original operator equation. It is unit-free, and will serve as our error index for the discrete-time case.

Once we have a candidate solution, we want to check its quality.[5] A direct procedure is to check how much, if at all, the approximation violates the Euler equation. First we should understand what a deviation from zero means in economic terms. Consider (32). It is a difference, relative to $C(k)$, between consumption at a capital stock k and what that consumption would be if an optimizing agent knew that tomorrow he will use the consumption rule C, and that personal and aggregate wealth will both be $f(k - C(k))$. Therefore, our residual equation applied to the approximate solution is the one-period optimization error relative to current consumption.

This approach to checking accuracy expresses the resulting errors in economic terms, essentially in terms of how irrational agents would be in using the approximate rule. If one found that this relative optimization error were about 0.1, then we would know that the approximation implies that agents make 10% errors in their period-to-period consumption decisions, a magnitude which few economists would find acceptable. However, if this index were 0.000001, then the approximation implies that agents make only a \$1.00 mistake for every \$1,000,000 they spend. Few economists would seriously argue that real-world agents do better than this. While such an approximation, C, may not be the mathematically exact equilibrium decision rule, it is hard to argue that it is unacceptable as a description of human behavior. In fact, many would argue that it is as compelling a description of behavior as the exact mathematical solution to (24).

The philosophy behind this accuracy check is that we should find an ε such that our approximation is an ε-equilibrium. The advantage of this approach is that our approximation to an exact equilibrium becomes reinterpreted as an approximate equilibrium. The disadvantage of focussing on ε-equilibrium is the likely existence of an open set of such equilibria. However, as long as the problem is well-conditioned, something which can be numerically checked, that set is likely to be small, and even negligible for many purposes.

[5] Some might wonder just how accurate we need an approximation to be. In fact, Danthine et al. (1987) have argued that the linear approximation computed in Magill (1977) is adequate for macroeconomic purposes. However, their tests concerned only a few economic variables such as consumption and output. In light of the results in Magill (1977) (and, more generally, in Bensoussan, 1988), this is not surprising. The adequacy of the linear approximation is much less likely once one turns to other economic variables, such as risk premia, the term structure of interest rates, and their correlations since these variables involve higher-order properties of tastes and technology, as documented in Judd (1991). Therefore, we attempt to find approximations which are as accurate as possible given limitations on computer time and space.

This approach to checking accuracy can also be taken when evaluating the answers from standard numerical methods, as is done in Judd (1992). By using the same accuracy index, we can evaluate the asymptotic procedures and other approximation methods in a uniform way.

The continuous-time case has a similar unit-free expression for the error. The true consumption policy function satisfies the differential equation

$$0 = R(k) \equiv (\rho C(k))^{-1} \left(C'(k)(f(k) - C(k)) - \frac{u'(C(k))}{u''(C(k))}(\rho - f'(k)) \right). \tag{33}$$

The right-hand side of (33), $R(k)$, is unit-free, and is the continuous-time version of the residual function defined in (32). We will use it as our error index for the continuous-time case.

In both the continuous- and discrete-time models examined above, the asymptotic expansions do an excellent job over a wide range of capital stocks. We first consider the continuous-time deterministic model. To examine the global quality of the asymptotic approximation, we will take the computed power series and compute how well it solves the defining differential equation at various capital stocks.

In Figs. 1–4 we display the Euler equation errors for our approximations assuming a Cobb–Douglas production function with capital share 0.25, and utility function $c^{1+\gamma}/(1+\gamma)$. More specifically, we plot the base 10 logarithm of the Euler equation error of our approximations against the capital stock. The production function was always adjusted so that the steady state is $k^{ss} = 1$. Since we are plotting the logarithm of the errors, we need to truncate the argument of the logarithm function in order to avoid overflow; hence, we actually plot $\log_{10}(|R(k)| + 10^{-15})$ against k. Therefore, the minimal value possible for the plot is -15. We chose $\gamma = -2.0$ for both the discrete and continuous-time models; the pictures for $\gamma \in [-10.0, -0.2]$ are similar. Fig. 1 displays the accuracy index for the Taylor approximations of degrees 1, 5, 10 and 15 in the continuous-time model, and Fig. 2 displays the Euler equation errors for the (3,2), (5,5) and (8,7) Padé approximations of the continuous-time model. In these figures, $R(k)$ is defined by the right-hand side of (33). Figs. 3 and 4 display the accuracy index for order 1, 5 and 10 Taylor and order (3,2) and (5,5) Padé approximations for the discrete-time model. In Figs. 3–4, $R(k)$ is defined by the right-hand side of (32).

The results are very encouraging and equally good for both the continuous- and discrete-time cases. When the error is of the order of 10^{-12} or less, we must regard it as being negligible since it is practically equal to the machine zero. Values of 10^{-6} are also quite small since it indicates that the Euler equation error is akin to an agent making a one dollar error per million dollars of

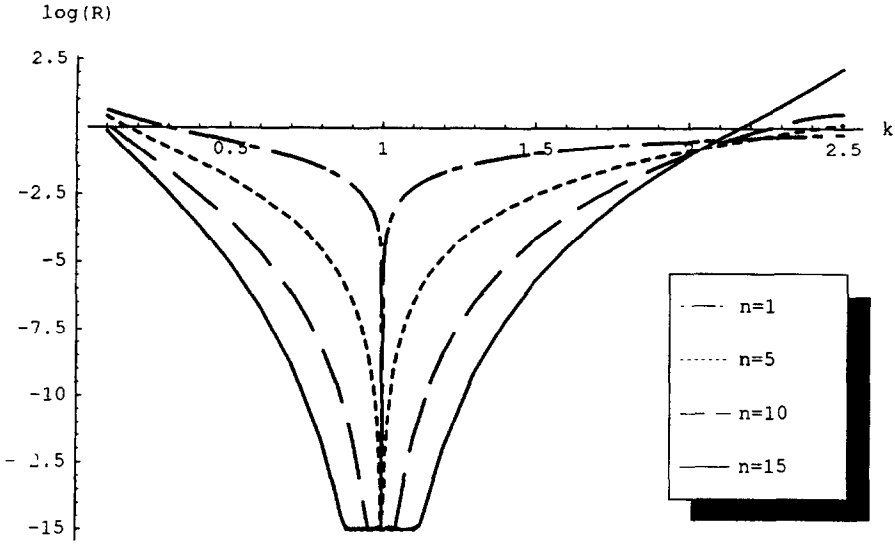

Fig. 1. Residual for Taylor expansion of continuous-time model.

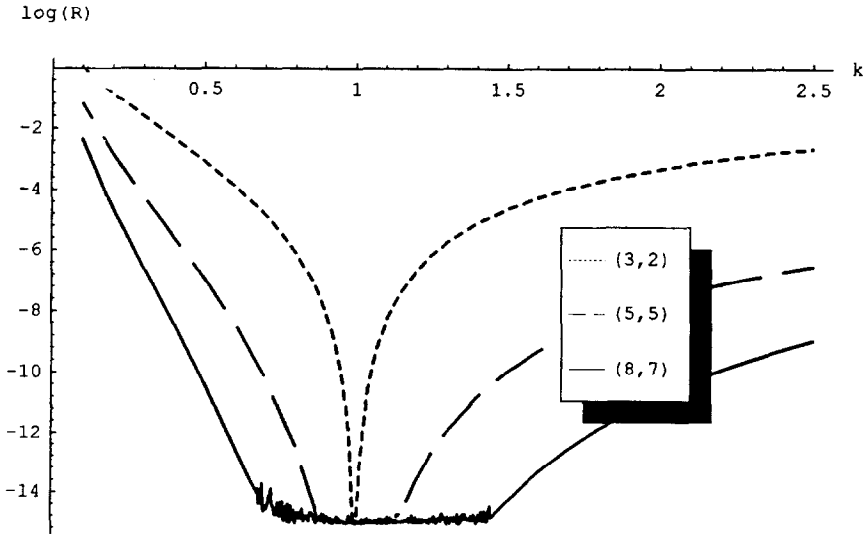

Fig. 2. Residual for Pade' expansion of continuous-time model.

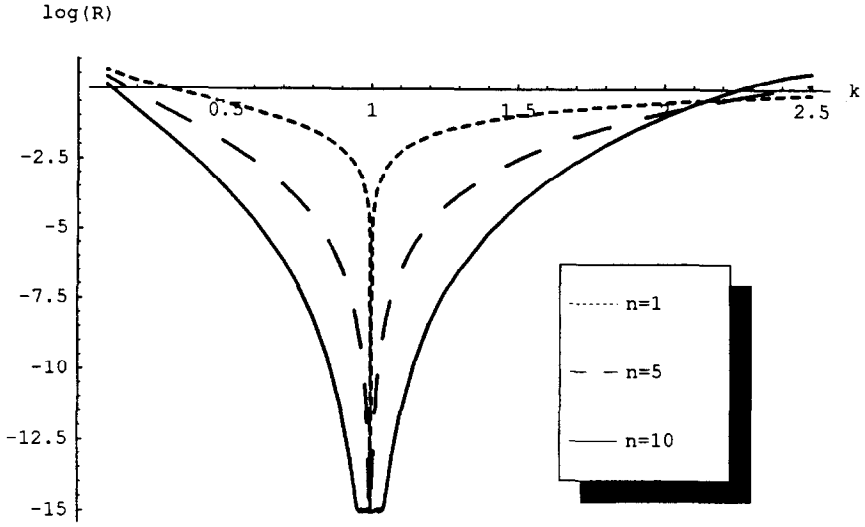

Fig. 3. Residual for Taylor expansion of discrete-time model.

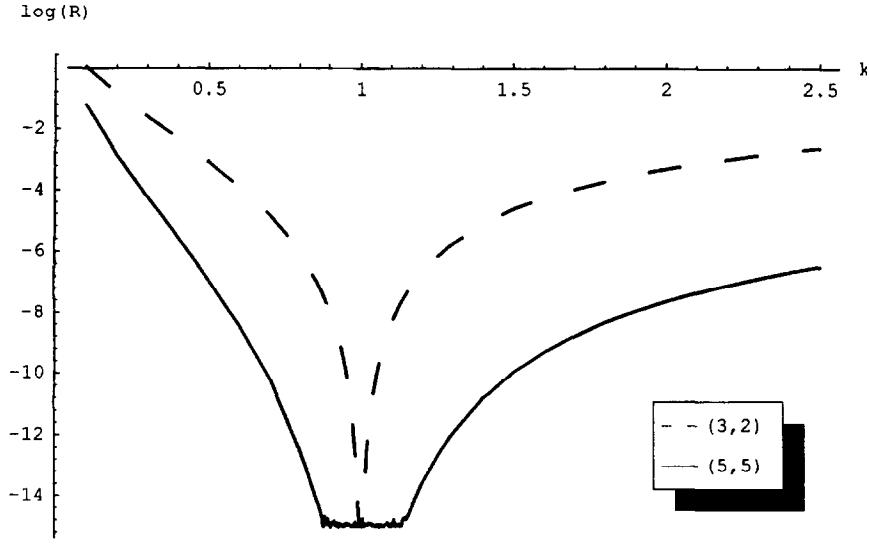

Fig. 4. Residual for Pade' expansion of discrete-time model.

consumption. By this logic, we will regard values for $R(k)$ greater than 10^{-4} to be unacceptable.

First, note that the log residual error is minimal (equal to -15) for capital stocks near $k^{ss} = 1.00$, expressing the fact that the expansion is exact up to

round-off error at the steady state. Next, note that the Taylor series errors remain very small over a moderate range of capital stocks as we move away from the steady state, but takes off exponentially as k comes close to 0 or 2. This rapid deterioration is expected since only local information at the steady state is used. Furthermore, the singularity in f at $k=0$ makes it impossible that the Taylor series for $C(k)$ has a radius of convergence exceeding 1, which in turn implies that the Taylor series will likely deteriorate near $k=2$. Third, the Euler equation errors fall quickly as we increase the number of terms used; at $k=0.5$, each step from $n=5$ to $n=10$ to $n=15$ reduced the error by a factor of nearly 100. Only for k far from the steady state did the errors not fall rapidly.

The small errors indicate that we could make use of the asymptotic series in two ways. First, for capital stocks within 40% of the steady state, the errors for the $n=15$ approximation are so small, of the order of machine error, that it should be a good approximation. In fact, the Euler equation errors associated with asymptotic series created using n derivatives are only slightly larger than the degree n polynomials generated by projection techniques in Judd (1992), indicating that the asymptotic series are almost as good as possible for a fixed degree of flexibility. Second, the errors at 0.3 and 1.8 indicate that the asymptotic series may not be a good approximation over the interval $[0.3, 1.8]$, but the norm of the error over that interval is small enough that one is confident that it is close to a solution. Therefore, it could be used as an initial guess for a standard numerical procedure, such as the projection method discussed above.

Figs. 2 and 4 show that the Padé expansions are even better. Near the steady state, the Padé expansions are almost perfect, but they are excellent even when we move far away from the steady state. In fact, the errors of the $(5,5)$ expansions over the interval $[0.6, 2.0]$ are quite small (less than 10^{-8}) for both the discrete- and continuous-time cases. Also, the $(5,5)$ and $(8,7)$ Padé expansions do quite well even for capital stocks above 2.0, a region where the Taylor expansions are never useful. Although the Padé approximation is a bit more costly to compute, it appears to be well worth the cost if one cares about capital stocks away from the steady state.

These figures report only one case of γ and α. However, we computed these values for many other values of $\gamma \in [-10.0, -0.1]$, and for $\alpha = 0.33$, always finding the same results.

We should note the speed of these procedures. First, Mathematica can be used to compute the coefficients of the, say, fifteenth-order asymptotic series in terms of γ, α, and any other structural parameter. This can be done on a personal computer, but only with large memory and efficient handling of the algebraic expressions. While this sounds inefficient, it is a fixed cost. One could then take the Mathematica-produced expressions for these Taylor series coefficients and write FORTRAN code to compute them for specific values of the structural parameters.

In fact, the simple structure of the algebra for the continuous-time model makes it possible to write a simple FORTRAN program for the coefficients; using it, we can compute the first 100 coefficients of the Taylor expansion for $C(k)$ in a second on a Pentium. Since a linear equation in the unknown coefficients defines the Padé expansion, that also takes little time to compute once we have the Taylor coefficients. Therefore, if one had to compute $C(k)$ for several values of the structural parameters, such as in a comparative dynamic exercise or a maximum likelihood estimation, this procedure is efficient since it results in a low marginal cost per set of structural parameters of computing $C(k)$.[6]

This example shows that high-order asymptotic expansions can be very accurate far away from the central point of the expansion, particularly when Padé expansions are computed. This indicates that they may be competitive with standard numerical procedures or useful in providing them with good initial points.

We must admit that the solutions for these models are nearly linear, implying that the higher-order adjustments are small in magnitude for this model and for these parameters. For many applications, linear approximations are probably adequate for these cases. However, it is unlikely that such will be true of all possible applications and for most interesting models. In fact, one of the puzzles of this model is that the uncertainty version implies equity risk premia far smaller than we see empirically. Also, those who argue for asymmetries in the business cycle are essentially arguing that linear approximations are inadequate. One suspects that many interesting problems will need approximations beyond the linear term. The method described in this paper and its successors show that Taylor expansions are easy to compute despite the apparent belief otherwise, and that they can solve the key equations over nontrivial regions of the state space.

5. Conclusions

In this paper, we have demonstrated that standard perturbation methods can be used to find high-quality approximate solutions to both discrete- and continuous-time aggregate growth problems. We have also demonstrated that these methods appear valid over a far greater range than the local theoretical properties indicate. In fact, they do almost as well as the solutions generated by projection methods, a standard, reliable, and rapid numerical procedure for solving these models. Other examples of these perturbation methods are in Judd (1991). Given the pervasive use of these methods in the physical sciences, one suspects that they can also be used quite extensively in economics, well beyond the simple examples discussed in this paper.

[6] The reader can acquire these programs from the authors by sending a request to judd@hoover.stanford.edu.

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