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Cyclical and Chaotic Behavior in a Dynamic Equilibrium Model, with Implications for Fiscal Policy*

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1. Introduction

The most vexing question in macroeconomics is the issue of why economic activity fluctuates. Our belief as to why economies fluctuate largely determines our attitudes toward social efforts to stabilize economic activity. Therefore, for both positive and normative reasons, macroeconomists have an intense interest in determining the cause of fluctuations.

This paper presents a formalization of a Schumpeterian view as to why fluctuations naturally occur. In a model with no exogenous disturbances, random or deterministic, we show that fluctuations may arise because innovative entrepreneurs pursue temporary rents. There will be times when valuable investment opportunities will appear, but the pursuit of those opportunities in one period often eliminates them in later periods. Booms in investment thus naturally lead to busts. However, natural economic forces make investment profitable again at a later date, leading to another boom and bust cycle of investment.

We will use a simple model adapted from Judd's (1985) examination of patents and the evolution of innovation. Within a similar model, we will elaborate on the nature of the instabilities that arise and the value of stabilizing such an economy with standard macroeconomic fiscal policy tools. The model is one in which investment takes the form of inventing and introducing a new product. In the short run, a firm will be safe from imitation and will reap monopoly rents from the new product. However, this monopoly rent will persist only until imitation occurs. The model is therefore one in which markets are often noncompetitive. The instabilities are due to these market failures since a first-best allocation would display no such fluctuations. It is very natural to assume this kind of imperfection given the substantial product differentiation present in the economy.

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The instabilities of our model arise from a simple mechanism. If there are few goods currently available, then the demand for any particular good, existing or potential, is high (since we assume that all goods are substitutes), and the incentive to invent a new good is also substantial. Even if many new goods are invented today, the demand for one more new good is not reduced much since new, temporarily monopolized, goods are all sold at high monopolistic prices.

However, when these goods are imitated in later periods, they will sell at marginal cost. This low price, together with the abundance of goods, reduces and possibly eliminates the incentive to innovate. This bust in investment will continue until other natural economic forces, such as population growth, productivity growth, or obsolescence of existing goods, revives the demand for new goods.

The basic mechanism generating our dynamics is somewhat reminiscent of the early work by Goodwin (1951) on trade cycle theory. Goodwin modified Samuelson's (1939) multiplier-accelerator model by making the accelerator nonlinear. This allowed him to obtain self-sustained bounded oscillations in economic variables that are asymmetric over the cycle, whereas Samuelson's model was linear, could only exhibit either damped or explosive oscillations, and had symmetric sinusoidal motions over the cycle. In Goodwin's theory, as in ours, investment eventually creates its own bust, and a significant period of investment inactivity renders capital accumulation profitable again. Like most of his contemporary writers, however, Goodwin relied on agent expectations that are repeatedly and systematically fooled along the solution path of the economy. Needless to say, our paper uses the modern approach of agent maximization and rational expectations.

Our analysis also bears some resemblance to more recent papers. Hart (1982) and Roberts (1986) both use imperfectly competitive markets to produce interesting models, but their analyses are static. Schleifer (1986) generates erratic dynamics, as would a repeated game version of Roberts (1986), but such instabilities are primarily due to multiplicity of equilibria. In contrast, our model has a unique perfect foresight equilibrium. This allows us to do precise comparative dynamics exercises concerning the impact and desirability of stabilization. Since the Schleifer model is the most similar, we will discuss it further below.

Other recent work has demonstrated the possibility of erratic dynamics, but has generally had to rely on unappealing assumptions concerning tastes and/or preferences. Grandmont (1985) obtains chaotic equilibrium paths in overlapping generations models, but only when saving is a decreasing function of the interest rate near the steady rate. The optimal growth models in Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) display chaos but, at least in the one-dimensional case, seem to require heavy discounting by agents. The chaos in Boldrin and Deneckere (1990) relies on factor intensity reversals as well as high discount rates. Woodford (1987) shows that capital market imperfections may lead to complicated dynamics but needs low factor substitutability.

This paper demonstrates that chaos may arise even when preferences are CES and agents do not discount the future, assumptions which generally preclude complicated dynamics in optimal growth models. It therefore shows that elements of market power may be important sources of erratic economic dynamics.

Normative questions cannot fruitfully be examined in most of the papers mentioned above. In Grandmont (1985), stabilizing fiscal policies only have a redistributive impact on social welfare since the equilibria are Pareto efficient.[1] Pareto efficiency also characterizes the chaotic equilibrium paths displayed in the capital theoretical models of Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986). No such efficiency can be presumed here because of the elements of market power we introduce. We determine the value of some possible stabilization schemes within our model. These normative exercises are highly model-specific and admittedly not as robust as its positive aspects. However, carrying out normative exercises will allow us to test some standard arguments about the desirability of "stabilization" within a fully elaborated economy.

2. The Model

We will use the discrete-time version of Judd (1985). At any point in time, the preferences of the representative agent over a continuum of possible goods is

$$U = \int_0^\infty u(x(V))dV$$

where $x(V)$ is the consumption of the good of variety V per period, and $u(x)$ is the utility flow from consuming a good at rate x. We will assume a CES specification of tastes:

$$u(x) = x^c,\ 0 < c < 1.$$

This specification was chosen mainly for its tractability. Alternatively, a quadratic functional form would have been equally tractable. We focus on the CES utility only in order to economize on space.

At the beginning of each period, the set of goods already in existence has Lebesgue measure V_n. These goods are sold competitively at their marginal cost of 1. Labor is the sole factor of production and is supplied inelastically,

[1] In contrast, nondegenerate sunspot equilibria are necessarily inefficient, and hence there may be room for stabilization policy in such models [see, e.g., Grandmont (1986)].

with total supply normalized at unity. New goods cost F units of labor to invent per measure one set of goods, and are sold exclusively by their inventor for one period of time. The set of new goods has measure V_p. If we let x_n represent the consumption of a competitively priced good and x_p that of a monopolized good, then the social budget constraint is given by

$$x_nV_n + x_pV_p + FV_p = 1 \tag{2.1}$$

since all competitive goods enter the economy symmetrically, as do all monopolized goods.

In order to allow the possibility of fluctuations in output, V_ℓ of the goods will be leisure goods. They "sell" at marginal cost since they are not monopolized. Therefore, labor supply and measured output equal $1 - x_nV_\ell$. Note that this approach to introducing leisure goods would be equivalent to adding, the term $\gamma\ell^c$ to our utility function, where ℓ represents time spent on leisure, and γ is a weighting factor.

The one-period monopoly enjoyed by inventors of new goods represents imitation lags. These may occur for legal reasons, such as patent laws, or for technical reasons which force imitators to take time to succeed. In our model we take this imitation lag as exogenously determined, not affected by the evolution of the economy nor by economic policies. While this is not strictly realistic, it abstracts from elements which are not essential to the results below.

In order to keep innovation from ending, we assume that some of the nonleisure goods become obsolete. Specifically, we assume that of the $V_n(t) + V_p(t)$ goods marketed at the end of period t, only $(V_n(t) - V_\ell + V_p(t))(1 + \delta)^{-1}$ retain their marketability in period $t + 1$. This represents shocks to demand—people lose interest in some goods each period. Such shocks are assumed independent, and independent of past consumption. If a good ever becomes obsolete, it remains so forever. This presents no problem, since we assume an infinite number of potential goods.

The variety of goods therefore obeys

$$V_n(t + 1) = (V_n(t) - V_\ell + V_p(t))/(1 + \delta) + V_\ell. \tag{2.2}$$

We assume that innovators are profit-maximizers. The CES specification of utility implies that the demand for each good has a price elasticity of demand equal to $-\sigma = (c - 1)^{-1}$. The monopoly price is therefore equal to c^{-1}.

If there are V_n goods selling at 1 and V_p goods selling at c^{-1}, then the profits gross of innovation costs to a monopoly seller of a new good equal

$$\pi = \frac{(c^{-1} - 1)d}{V_n + d^cV_p} \tag{2.3}$$

where $d \equiv c^\sigma$.

We assume that innovators can enter freely. They choose to innovate products which have not yet become obsolete, paying only the initial cost of innovation, F. Therefore, innovation will continue until profits net of innovation costs are zero. Hence, V_p is given by

$$V_p(t) = \max\left\{0, \frac{1-c}{F} - \frac{V_n(t)}{d^c}\right\} \tag{2.4}$$

Putting (2.2) together with (2.4) yields a first-order difference equation for V_n:

$$V_n(t+1) = (1+\delta)^{-1}\left(V_n(t) - V_\ell + \max\left\{0, \frac{1-c}{F} - \frac{V_n(t)}{d^c}\right\}\right) + V_\ell \equiv h(V_n(t)) \tag{2.5}$$

It is this dynamic equation which we examine.

Figure 14.1 represents equation (2.5). Note the nonlinear structure. If $V_n(t) < d^c(1-c)/F$, then $V_n(t+1)$ is a decreasing function of $V_n(t)$. This is somewhat surprising. For, while innovation investment is decreasing in current be-

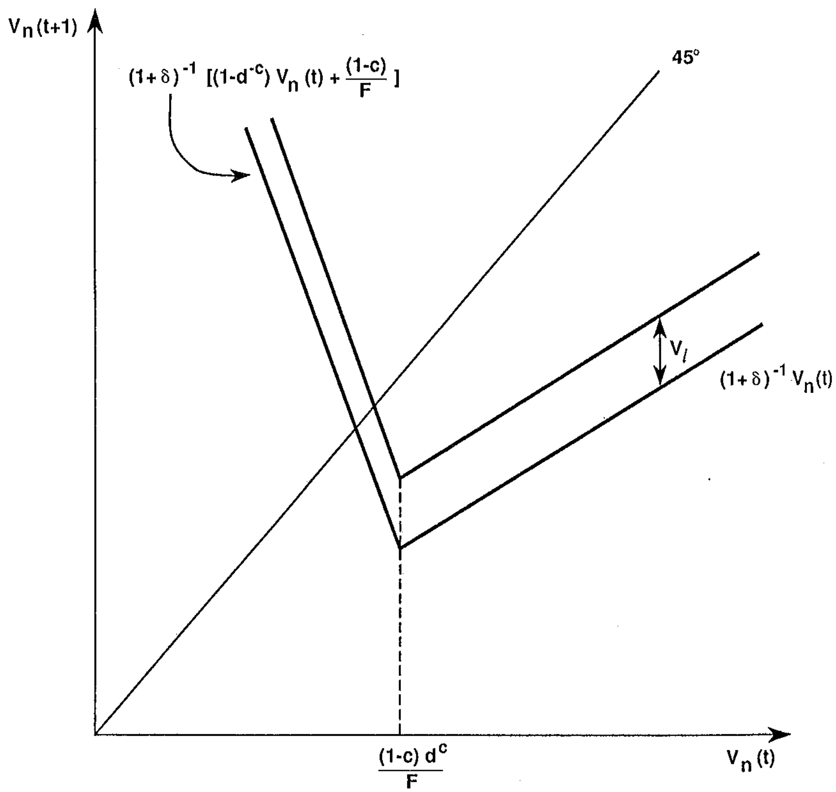


Fig. 14.1. The dynamics of per capita variety, $V_n(t)$

ginning-of-period variety, the number of goods which become obsolete is also decreasing in current variety. Equation (2.5) implies that the net effect of an increase in current end-of-period variety and the increase in subsequent obsolescence is a decrease in the next period's terminal variety.

At $V_n = d^c(1-c)/F$, the system changes character. For greater V_n, no innovation occurs. The equation of motion is then governed only by the obsolescence factor. Hence, an increase in current variety increases the next period's variety.

For $V_\ell \le d^c(1-c)/F$, the steady-state level of V_n is

$$V^* = \frac{(1-c)/F + \delta V_\ell}{\delta + d^{-c}}$$

There are a variety of interesting features to this model. First, direct computations show that for $V_n < \hat{V} = d^c(1-c)/F$

$$x_n = d^{-c}F/(1-c)$$
$$U = 1$$

whereas, for $V_n > \hat{V}$,

$$x_n(V_n) < x_n(\hat{V}),\ x_n'(V_n) < 0$$
$$U(V_n) > U(\hat{V}),\ U'(V_n) > 0$$

In particular, note that consumption of any good selling at marginal cost, leisure or nonleisure, is constant during any period with innovation, as is the utility flow realized in that period. In periods with substantial variety and no innovation investment, x_n falls with variety—an increase in variety causes agents to spread labor resources uniformly over all goods—and utility rises with variety. Measured output, $1 - x_n V_\ell$ rises with V_n since x_n falls. Measured output is lowest during periods of innovation.

The dependence of the equation of motion on the number of leisure goods is particularly simple. From (2.5) we see that the number of leisure goods is just an additive element of h. In Figure 14.1, the lower example of h has no leisure goods; this implies that the right arm goes through the origin. The higher example models an economy with leisure goods, and the right arm has a positive intercept on the vertical axis equal to the number of leisure goods. This indicates that the presence of leisure will not affect the dynamics of our model by much.

3. Dynamic Behavior

Figure 14.1 displays a graph depicting the first-order difference equation governing this economy. The equation of motion has two distinct regions. If the measure of existing goods at the beginning of the period exceeds $\hat{V}$, then profits from innovation are not sufficient to encourage innovation. Therefore, no innovation occurs and variety falls because of obsolescence. If there are fewer than $\hat{V}$ goods, innovation will occur. In fact, as the number of goods at the beginning of the period falls, the profits to innovation rise so much that the total number of goods at the beginning of the next period increases.

The critical feature for our purposes is the nature of the steady state, V^*. Note that it can easily be unstable. Because the right arm of $h(\cdot)$ lies below the 45-degree line, the dynamics of the system is confined to a compact region. Nevertheless, the steady state at V^* may be unstable, implying oscillations around V^*. The dynamics of our model can be quite complex, as we see below. We will examine them more precisely for the case $V_\ell = 0$. The general case offers no difference in character, just notation.

Recall from Section 2 that the difference equation governing the dynamic behavior of the economy when $V_\ell = 0$ is given by:

$$(1 + \delta)V_n(t + 1) = V_n(t) + \max\left(0, \frac{(1 - c)}{F} - \frac{V_n(t)}{d^c}\right) \equiv h(V_n(t))(1 + \delta)$$

Assuming the steady state $V^* = \dfrac{(1 - c)}{(d^{-c} + \delta)F}$ to be unstable, i.e., $\dfrac{d^{-c} - 1}{1 + \delta} > 1$ (where $d = c^{1/(1-c)}$), we obtain Figure 14.2.[2]

Let A be the square with side length η, enclosed by dashed lines in Figure 14.2. A is a trapping region; all points that follow the difference equation $h(\cdot)$ end up in this square after a finite number of iterations, and remain there forever after. In order to analyze the dynamics of the system, we may restrict attention to region A. In fact, it will be convenient to turn this figure 180 degrees clockwise, and to normalize conditions such that $\eta = 1$.[3] We then obtain Figure 14.3.

Defining $a = d^{-c} - 1$ and $b = (1 + \delta)$, we obtain

$$\alpha = 1 - b/a$$

$$\beta = 1 + 1/a - 1/b$$

We may also show that $\alpha < \beta$,[4] implying that the interval $I_1 = [0,\alpha]$ maps into the interval $I_2 = [\alpha,1]$. Observe also that the interval I_2 maps onto $I_1 \cup$

[2] A necessary condition for V^* to be unstable is $c > 1/2$. Indeed, V^* is unstable if and only if $\delta < f(c) - 2$, where $f(c) = d^{-c}$. Observe that f is monotone on $(0,1)$ with $f(0) = 0$, and $\lim_{x\to 1} f(x) = e$. Since $1/2$ is the unique value of c for which $f(c) = 2$, we obtain the desired result.

[3] This normalization is innocuous as it affects only the amplitude and not the qualitative nature of the dynamics. To obtain the correct scaling, one only needs to multiply by

$$\eta = \frac{\delta(1 - c)(1 - d^c)}{F(1 + \delta)^2}.$$

[4] α is less than β whenever $b(1 + b) > a$, a condition which may be rewritten as $d^{-c} - 1 < (1 + \delta)(2 + \delta)$. But since $d^{-c} \leq e < 3$, our claim follows.

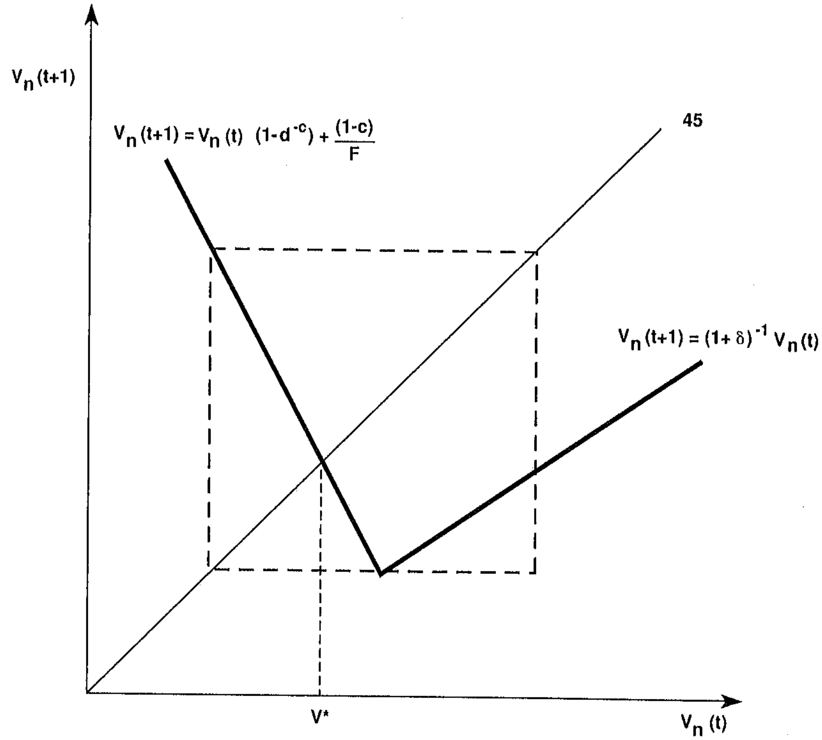
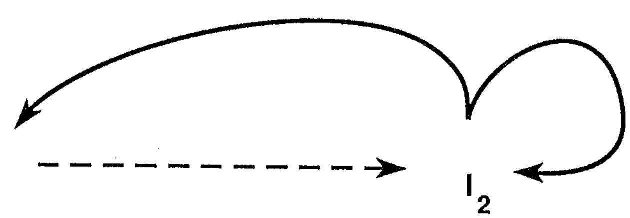


Fig. 14.2. The difference equation $V_n(t + 1) = h(V_n(t))$

I_2. Letting the symbol $\dashrightarrow$ indicate "map into," and $\rightarrow$ "map onto," we obtain the following diagram:



This diagram makes it clear that any cycle must spend at least as much time in I_2 as in I_1. In fact, the same reasoning is true for any path $x_n = h^n(x)$ in A. Since the map g: $[0,1] \rightarrow [0,1]$ in Figure 14.3 is just a (rescaled) rotation of the map h in Figure 14.2, we will, in the sequel, refer to it as h as well. This should not lead to confusion.

The above observation allows us to derive a condition under which all noncritical cycles (cycles not containing the critical point α) are unstable. Let $[x]$

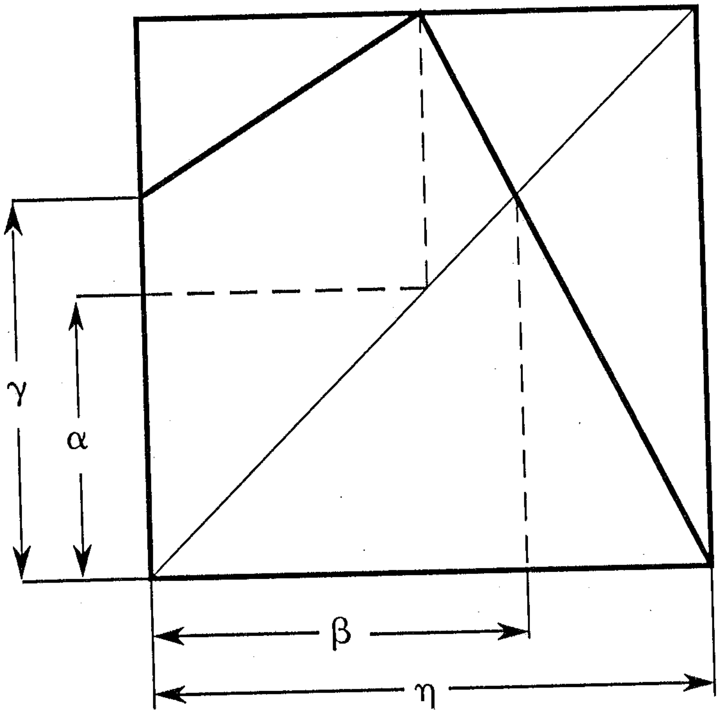


Fig. 14.3. The restriction of the map h to the trapping region

denote the orbit through x, then $[x]$ is unstable if $|(h^n)'(x)| > 1$. But $|(h^n)'(x)| = \prod_{k=0}^{n-1} |h'(h^k(x))|$, and h' is either equal to $h'|_{I_1}$ or $h'|_{I_2}$ if x is noncritical. Thus,

$$|(h^n)'(x)| = |h'|_{I_2}|^{k}\,|h'|_{I_1}|^{\ell}$$

where $k + \ell = n$, with $k \geq \ell$, and

$$h'|_{I_1} = \frac{1}{b},\quad h'|_{I_2} = -\frac{a}{b}$$

Hence, $|(h^n)'(x)| \geq \left[\dfrac{a}{b^2}\right]^k > 1$ if $b < \sqrt{a}$. It is easily seen than when $b < \sqrt{a}$, then any critical orbit must be unstable as well (expand $h^n(x) - h^n(\alpha)$). Thus, if $b < \sqrt{a}$, *all* cycles are unstable. In particular, for all but a countable set of initial points x, the iterates of x, $h^n(x)$, will not display any periodic behavior, even if one iterates long enough.[5] They will continually jump around

[5] The condition $b < \sqrt{a}$ implies that h^n can have at most finitely many ($2^n - 1$, to be precise)

from I_1 to I_2, in an apparently unpredictable fashion. Under this condition, then, we may say that the map h displays chaotic dynamics. We summarize these results in Theorem 1:

Theorem 1. If $\sqrt{a} < b < a$, then for all but a countable set of initial conditions x in $[0,1]$, the trajectories generated by h from x, $\{h^n(x)\}_{n=0}^{\infty}$, are completely aperiodic. That is, trajectories are neither cyclic nor asymptotic to any periodic orbit.

In fact, one may also prove the following, somewhat stronger theorem:

Theorem 2. If $b < \sqrt{a}$, then $h(\cdot)$ has an absolutely continuous invariant measure that is ergodic. This measure is unique, and has a density which is of class C^∞. Moreover, the support of the measure is $[0,1]$.

Proof. For piecewise C^2 functions h: $[0,1] \to [0,1]$ that satisfy the expansiveness condition $\inf|(h^n)'| > 1$ for some positive integer n, Lasota and Yorke (1973, Theorem 3) prove the existence of absolutely continuous invariant measures. Observe that our map satisfies this condition with $n = 2$. For the same class of functions, Li and Yorke (1978) prove that if h and its derivative exhibits a total of m discontinuities, there can be at most m invariant measures. Since our map h has one such discontinuity point, its invariant measure is unique, and also has full support. The invariant measure is absolutely continuous, and thus ergodic. Finally, Szewc (1984, Theorem 7.2) shows that if the map h can be extended to a map of class C^r on the closure of every component of continuity, then the density is of class C^{r-1}. □

Let us explain why Theorem 2 is somewhat stronger than Theorem 1. The ergodicity of the h-invariant measure μ implies that the empirical distributions $\mu^n(x)$ (generated by the iterates $h^j(x)$, $j = 0, 1, \ldots, n-1$, and assigning probability $1/n$ to each $h^j(x)$) converge weakly to μ for μ-almost every x. Thus, for a set of initial conditions x of full μ-measure (and here also of full Lebesgue measure), trajectories starting at x will look rather erratic since they eventually fill the whole support.[6] In particular, as claimed in Theorem 1, trajectories will be completely aperiodic. Observe, however, that there is also some regularity to the way in which these trajectories jump around in the unit interval. Indeed, the fraction of time that a randomly chosen trajectory must spend in any interval $A \subset [0,1]$ approaches $\mu(A)$ as n goes to infinity! This

fixed points. Thus, h has at most a countable number of cycles. Since the set of preimages of any given cycle is countable, and since there are only countably many cycles, all but a countable set of trajectories will be completely aperiodic.

[6] It should be emphasized that this version of chaos (which might be termed "ergodic chaos") *is* observable, in contrast to the weaker notion of topological chaos (also often referred to as chaos in the sense of Li and Yorke (1975)), for which the chaotic set may have Lebesgue measure zero, and thus essentially be unobservable. For a further discussion of this issue, we urge the reader to consult Grandmont (1984, p. 15) or Melese and Transue (1986).

long-term predictability stands in sharp contrast to the short-term erratic behavior that most trajectories display.[7]

We now turn to the case: $\sqrt{a} < b < a$.

Theorem 3. If $\sqrt{a} < b < a$, then h has an attractive period-2 cycle. Moreover, all points in [0,1] (except for the fixed point of h), are attracted to this stable orbit.

Proof. When $\sqrt{a} < b < a$, we have $r \equiv ab^{-1} >$, $s \equiv b^{-1} < 1$ and $rs < 1$. The set of turning points for h^2 is:

$$T(h^2) = \{x \in (0,1): h^k(x) = \alpha \text{ for some } 0 \leq k < 2\}$$

where α is the critical point of h [see, e.g., Preston (1983) Proposition 2.1., p. 320]. Thus $T(h^2) = \{\alpha, h^{-1}(\alpha)\}$, where $h^{-1}(\alpha)$ is a singleton in I_2 (this follows from $\beta > \alpha$). h^2 is a piecewise linear function connecting the points $(0, h^2(0))$, $(\alpha, 0)$, $(h^{-1}(\alpha), 1)$ and $(1, h(0))$. Furthermore, the slope of h^2 equals $-rs$ on $J_1 \equiv [0,\alpha)$, s^2 on $J_2 \equiv (\alpha, h^{-1}(\alpha))$ and $-rs$ on $J_3 \equiv (h^{-1}(\alpha), 1)$. h^2 is illustrated in Figure 14.4.

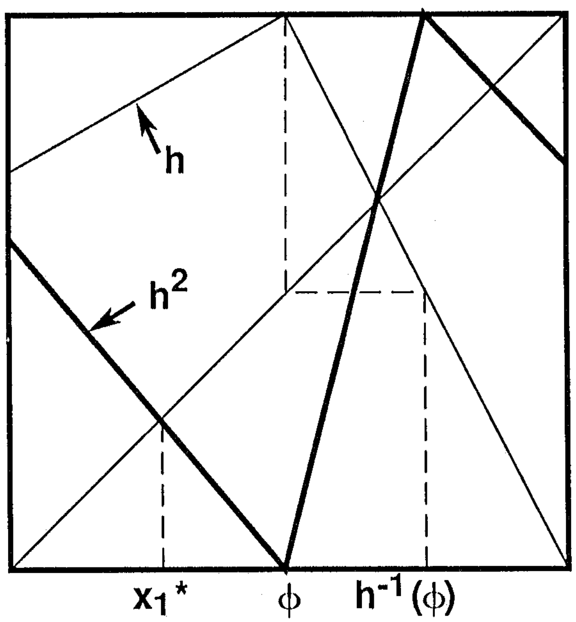


Fig. 14.4. The graph of h^2 when $\sqrt{a} < b < a$

[7] In addition, it can be shown that the function h has a positive Lyapounov exponent, and hence exhibits sensitive dependence on initial conditions. For a single explanation of this concept, see Deneckere and Pelikan (1986) and Grandmont (1984).

Since the graph of h^2 cuts the 45-degree line somewhere in the interval $(0,\alpha)$, and since h^2 has a slope of rs in absolute value, $[x_1^*]$ is a stable orbit.[8] In fact, every point in $[0,\alpha]$ is attracted to this fixed point of h^2, i.e., $(h^2)^n(x) \to x_1^*$, and every point in $[h^{-1}(\alpha), 1]$ is attracted to x_2^*. Furthermore, all points in (α, V^*) are mapped into J_1 after one iteration, and all points in $(V^*, h^{-1}(\alpha))$ are mapped into J_2 after one iteration (i.e., two iterations of h). □

We now treat the hairline case $rs = 1$.

Theorem 4. When $a = b^2$, all points in [0,1] except the fixed point of h get mapped into an unstable period-4 cycle. Furthermore, there is a continuum of period-4 cycles.

Proof. Recall that

$$h(x) = \begin{cases} ab^{-1} - ab^{-1}x & \text{on } I_2 \\ \beta + b^{-1}x & \text{on } I_1 \end{cases}$$

Note that since $h(\beta) = \alpha$, the orbit $0 \to \beta \to \alpha \to 1$ represents a period-4 cycle. Hence, we obtain Figure 14.5.

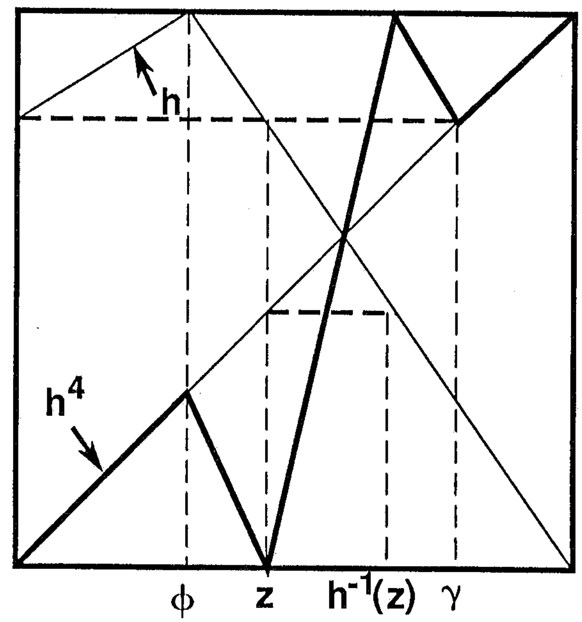


Fig. 14.5. The graph of h^4 when $b^2 = a$

[8] x_1^* can be calculated to be $x_1^* = \alpha a/(a + b^2) = (a - b)/(a + b^2)$.

Let us compute the set of turning points of h^4:

$$T(h^4) = \{x \in (0,1) \colon h^n(x) = \alpha \text{ for some } 0 \leq n < 4\}$$

By the previous argument, it follows that

$$T(h^4) = \{\alpha,\beta\} \cup \{z, h^{-1}(z)\}$$

where z is the unique element in I_2 s.t. $h(z) = \beta$. Observe that $0 < \alpha < z < h^{-1}(z) < \beta < 1$.

Since any element in $[0,\alpha]$ must traverse I_1 and I_2 twice, the slope of h^4 on $[0,\alpha)$ is $(rs)^2 = 1$. Similarly, any element in $[\beta,1]$ spends an equal fraction of time in I_2 and I_1, and h^4 has slope 1 on $(\beta,1]$. This proves the existence of a continuum of period-4 points. Some further computations yield $(h^4)'(x) = -r^3s = -a^3b^{-4}$ on $(\alpha,z) \cup (h^{-1}(z), \beta)$, and $(h^4)'(x) = a^4b^4$ on $(z, h^{-1}(z))$. We illustrate h^4 in Figure 14.5. It immediately follows that each point in $[0,1]$ except for $V^* = h(V^*)$ gets attracted to a period-4 cycle. □

Only one case remains to be treated, namely where $a = b$ (when $b > a$, $V^* = h(V^*)$ is stable and all points are attracted to it). But this remaining case is trivial, as our map then becomes as shown in Figure 14.6.

Thus, there is a continuum of unstable period-2 cycles, and every initial point gets attracted to one of them. As b crosses a, a flip bifurcation occurs: the steady state loses stability and gives birth to period-2 cycles.

Summarizing, we obtain the following scenario when we vary b (or δ) and

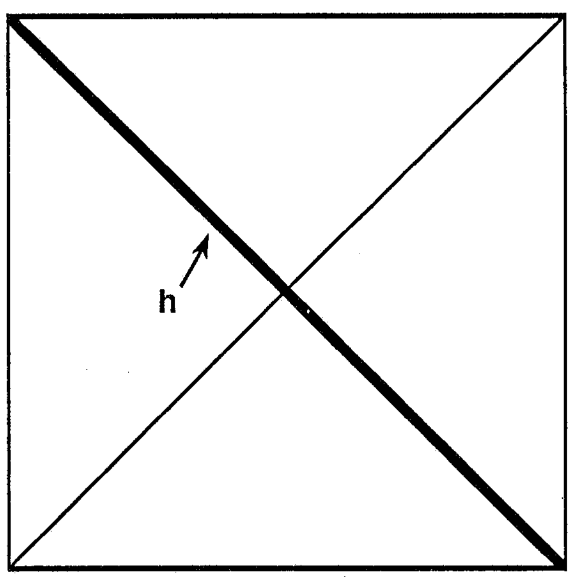


Fig. 14.6. The graph of h when $b = a$

$a > 1$ (or $c > 1/2$). When b is large, there is a damped oscillation, and all initial points are attracted to the steady state. As b crosses a, the steady state becomes unstable and a continuum of unstable period-2 cycles appears, to which all trajectories not starting at the steady state are attracted. When $\sqrt{a} < b < a$, there is a unique stable period-2 cycle to which all trajectories are asymptotic. As b crosses $\sqrt{a}$, the period-2 cycle loses stability and a continuum of unstable period-4 cycles is created. All points except for the fixed point of h (and its inverse images outside the trapping region) are attracted to one of these period-4 cycles. When $1 < b < \sqrt{a}$, chaos obtains. Finally, when $b = 1$, a continuum of unstable steady states is present.

At this point it will be valuable to summarize our results in Theorem 5.

Theorem 5. If $b > a$, then the steady state is stable. If $b = a$, there is a continuum of two-cycles. If $\sqrt{a} < b < a$, there is a unique stable two-cycle. If $b = \sqrt{a}$, there is a continuum of unstable four-cycles. If $1 < b < \sqrt{a}$, the system is chaotic, i.e., it is completely aperiodic and $h(\cdot)$ has an absolutely continuous invariant measure which is ergodic.

Theorem 5 states that there are three distinct types of dynamic behavior, ignoring the knife-edge cases. If goods are poor substitutes or the rate of obsolescence is high, the system converges to the steady-state level of variety and innovation. As goods become more substitutable and obsolescence slows, we find the emergence of stable two-cycles. In this regime, the economy oscillates between periods of high and low rates of investment. Finally, as goods become highly substitutable, the system breaks into a chaotic dynamic. It oscillates between periods of high investment and no investment, but according to no regular pattern.

While the analysis above was done for the case of $V_\ell = 0$, Theorem 4 also applies to the case $V_\ell > 0$, as long as $V_\ell < (1 - c)d^c/F$. Indeed, as can be seen from Figure 14.1, the only thing that changes when $V_\ell > 0$ is the size of the trapping region.[9] After rescaling such that $\eta = 1$, the same analysis as above applies.[10]

The nature of the cycle is roughly similar to that typically observed. In our model, periods of high investment relative to trend (corresponding to V_n below the steady state) do lead to periods of consumption above trend, whereas periods of consumption above trend may not lead immediately to periods of high

[9] The size of the trapping region can be shown to equal

$$\eta = \frac{\delta(1 - c)}{(1 + \delta)^2 F}(1 - d^c) - \frac{\delta}{(1 + \delta)^2}(1 + d^{-c})V_\ell.$$

[10] Of course, when the fraction of leisure goods becomes too large ($V_\ell \geq (1 - c)d^c/F$), the unique fixed point of h will lie on the right arm of Figure 14.1. All points will then be mapped onto the stable branch after at most one iteration, and converge to the stable steady state. In this steady state, only leisure goods are consumed.

investment and low consumption. Therefore, in this limited sense, investment booms tend to lead to periods of high consumption.

Labor supply movements are also easily determined if the amount of leisure goods is not too large. Since consumption of nonmonopolized goods, including leisure time, is falling in variety during innovation periods, labor supply and measured output are smallest when variety is smallest. In our simple first-order model, investment and consumption are negatively correlated with labor supply and output.

Since we have only a first-order difference system, our model cannot display a realistically rich autoregressive structure. However, in the more general (and far less tractable) continuous time model, one does find, for example, investment leading consumption [see Judd (1985)]. Therefore, the dynamic behavior induced by this kind of model is generally qualitatively similar to actual business cycles, indicating that the mechanisms we study here cannot be written off as implausible on comovement grounds. We next show that these effects are also quantitatively significant by business cycle standards.

We should compare our analysis in some detail to that of Schleifer. His is a model of cost-reducing innovation as opposed to product innovation. He also assumes that a firm which innovates has only one period before imitation eliminates rents from innovation. At that point, any resemblance ends. In Schleifer, ideas for innovation arrive exogenously and at no cost to the innovator; also, there is no imitation of an idea until it is used. In his model, firms with innovative ideas have an interest in all using their ideas at the same time since that is when the present value of profits is highest. In contrast, our innovators have no such timing preferences since there is an endogenous amount of innovation, the profitability of the marginal innovation is zero in equilibrium, and the dynamic market structure elements become the dominant consideration.

As is typical in such coordination models, there are many equilibria; given the multiplicity of equilibria, Schleifer's model also has sunspot equilibria which model the Keynesian view that investment is sensitive to self-fulfilling expectations on the part of firms. Such sunspot constructions have been used often to "explain" the excess volatility of economic life. The approach to the excess sensitivity problem taken here is quite different in spirit, following the chaotic dynamics literature. Our model is one in which elements of innovation and imperfect competition *necessarily* generate (for some values of the parameters) dynamics which are "too" volatile, not just include such a possibility in an infinite set of equilibria. This model shows that we can generate chaotic dynamics with standard and simple classes of tastes and technology, an improvement over much of the chaotic dynamics literature which relies on strong income effects, high discount rates, and/or capital intensity reversals.

Even in cases where chaos does not arise, the dynamic behavior of the economy differs substantially from the optimal solution. Since the equation of motion is always downwardly sloped at the steady state, as in Figure 14.2, the convergence to the steady state will always be oscillatory; if the economy were run optimally, convergence to the steady state would be monotonic. Furthermore, any shock to the economy, such as an unexpectedly large amount of obsolescence would start a damped oscillation. Therefore, in all cases, our equilibrium with imperfect competition will have substantially different qualitative dynamic behavior.

The importance of imperfect competition for understanding macroeconomic observations has been argued before. In particular, Hall (1986) has argued that models of monopolistic competition (where price does not equal marginal cost) help us understand price movements in response to business cycle shocks. This analysis goes one step further by arguing that the market structure itself generates dynamics similar to business cycles.

Above we showed that imperfect competition can be a source of macroeconomic fluctuations. Next, we investigate whether these effects can be nontrivial. We also pursue the implications for typical macroeconomic policy.

4. Volatility and Stabilization

Once we have a model with volatility in investment, we should address two questions. First, we should determine whether the volatility is quantitatively significant. Second, we should ask ourselves whether standard macroeconomic policies can be used to improve the economy's performance. To address these issues we resort to numerical simulation, as many of the cases will generate chaotic dynamics which cannot be described in a tractable closed form. In these simulations we allow c to vary between $1/2$ and 1 since $c < 1/2$ implies stability. Also, δ will vary between .05 and .5. If $\delta < .05$, then there is practically no obsolescence and the system will usually be stable; if $\delta > .5$, then over one-half the goods become obsolete during their monopolized stage. Keeping $\delta \in [.05, .5]$ focuses on the cases that are most sensible and most likely to generate interesting dynamics.

The volatility in investment is significant in our model. When the economy converges to a two-cycle, alternating between periods of no investment and positive investment, the coefficient of variation in investment is 1. This is because the mean and the standard deviation of investment then both are equal to one-half of the positive investment level. One surprising result of our simulations is that even when the economy does not settle down to a stable cycle, the coefficient of variation is indistinguishable from 1. More precisely, we allowed the equilibrium difference equation to run for 300 periods and used the final 200 as a sample of the ergodic distribution. Coefficients of variation for the chaotic cases varied between .95 and 1.05. This volatility is substantial and exceeds observed volatility in national income accounts. Therefore, if our mechanism is at work at all, it will contribute significantly to fluctuations in investment.

It is common in macroeconomics to ask if government policy can be usefully manipulated to reduce undesirable fluctuations. This question must have a negative answer if one confines attention to Pareto optimal paths, as in Grandmont (1985). In our model there is no presumption of efficiency since some markets are monopolized. Thus, the desirability of stabilization then becomes an open question.

First, observe that any dynamic allocation can be achieved by an appropriate system of innovation subsidies financed by lump-sum taxes. In particular, the first-best optimal allocation can be so achieved. Second, note that the first-best allocation is globally asymptotically stable, converging to the unique steady state from any initial condition.[11] The optimal allocation can thus be achieved by appropriate instruments. Furthermore, it is convergent.

It is not surprising that an intervention which directly attacks the market failure will enhance economic performance. However, if we do not have the appropriate instruments, or choose not to use them, it is not clear that stabilization remains a desirable objective. In standard macroeconomic analyses where the government is assumed to have a mean-variance objective (often unrelated to the preferences of the individual agents), stabilization is often found to be desirable. Here, on the other hand, we have a structural model. Social welfare is given by an explicit welfare function which displays an aversion to excessive fluctuations, rather than a simple mean-variance criterion.

Another interpretation of our analysis below is that the mean-variability tradeoff implicit in stabilization schemes is sensitive to the tools used. The subsidy scheme discussed above implicitly has a low sacrifice in the "mean" performance relative to the stabilization gains. In fact, there is a gain in the mean because of the subsidy's correction of the undersupply of variety.

We next consider a simple form of stabilization which is more characteristic of standard macroeconomic policies: the government can levy lump-sum taxes on agents and discard the real resources so acquired. Such "expenditures" are allowed to be conditioned on the current state of the economy, $V_n(t)$. They act

[11] The proof of this is standard and hence omitted. If individuals discount the future, this result is trivial. In our analysis we will assume no discounting to avoid inessential complications which would arise in those cases where chaos arises. Even then the result holds. Using the overtaking criterion, one derives a pair of equations governing the vector (x_t, V_{t-1}). That system is saddle-point stable, and the convergent path overtakes nonconvergent paths.

essentially to reduce the labor resources available to the private sector. Therefore, if we let G represent these expenditures, our difference equation becomes

$$V_n(t+1) = (1+\delta)^{-1}\left(V_n(t) - V_\ell + \max\left\{0, \frac{(1-c)(L-G(t))}{F} - \frac{V_n(t)}{d^c}\right\}\right) + V_\ell \tag{4.1}$$

$$G(t) = g(V_n(t), t)$$

where $g(\cdot, \cdot)$ represents government expenditure expressed as a function of total current variety and calendar time.

One particular form of stabilization policy would be

$$G(V_n) = \begin{cases} \gamma(V^* - V_n), & \text{if } V_n < V^* \\ 0 & , \text{ otherwise} \end{cases} \tag{4.2}$$

where V^* is the steady-state level of V_n without any government expenditure. Note that government expenditure is high in those states where investment is greatest and output smallest, i.e., when $V_n < V^*$. Also note that the steady state is unaltered; we do this since we want to focus on a policy that is stabilizing only, not one that alters the mean as well. For any γ, the above policy is stabilizing, in the sense that government expenditure is greatest when output is lowest. The correlation with investment is admittedly not realistic, but that is an unfortunate consequence of using a first-order difference system.

The resulting equation of motion for the economy is represented in Figure 14.7. The dynamic system is unchanged except when $V_n(t)$ is less than V^*. In that region the response of $V_n(t+1)$ to $V_n(t)$ is reduced. This produces a kink at V^*, as represented in Figure 14.2 with the broken left piece. The result is a continuous equation of motion, even at the kink at V^*. It is straightforward to show that if γ is sufficiently large, the resulting system converges to the steady state. In general, if the product of the left and right derivatives of the equation of motion at V^* is less than one, the system is stable.

In order to keep the analysis tractable, we assume that individuals do not discount the future. This is done since it yields a particularly sharp result for our model.

Theorem 6. Complete stabilization is possible, but never desirable, if the discount factor is one.

Proof. Such stabilization can be implemented by a policy which increases government spending as investment increases above the steady-state level, as in the case when one combines (4.1) and (4.2) with γ large. This changes $h(\cdot)$

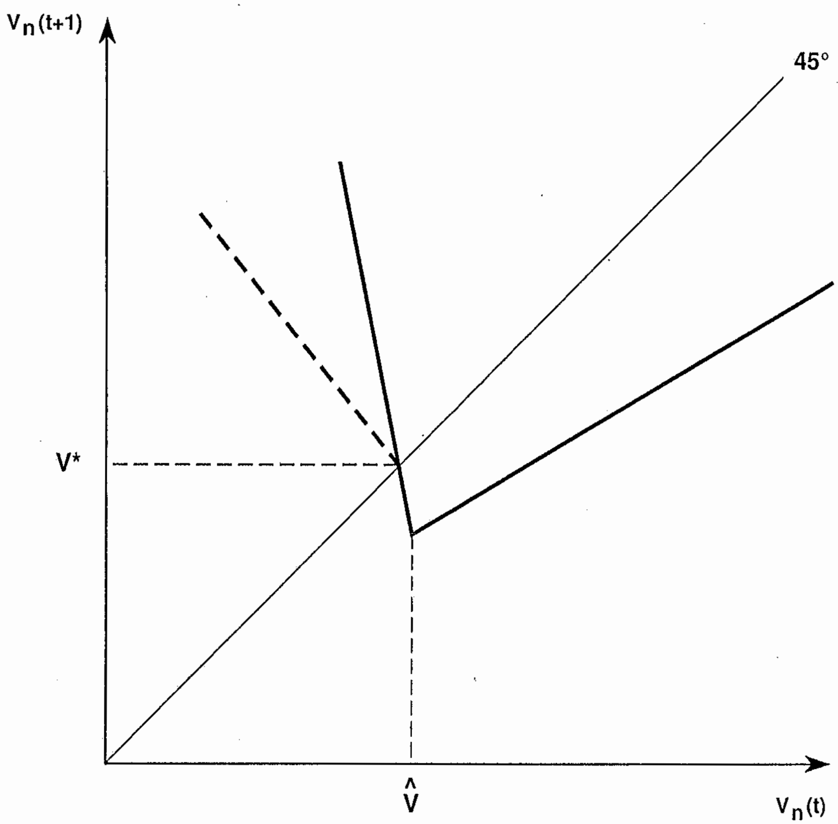


Fig. 14.7. An example of a completely stabilizing, but undesirable, stabilization scheme

to $\hat{h}(\cdot)$ in Figure 14.7, making the steady state stable. In this steady state, no actual government expenditure occurs. Hence, average long-run utility is given by utility at V^*.

Recall that utility is invariant to variety if innovation occurs, and is rising in variety if there is no investment. Since the steady state is at a level of variety where continuous investment occurs, it is better to have some periods with no investment. □

Theorem 6 shows that we would never want to completely stabilize the economy at its steady state if the future is not discounted. Allowing discounting would, of course, be more realistic. However, since the results are clearly continuous in the discount rate, Theorem 5 implies that for large discount factors, complete stabilization will be worse than no stabilization at all. Since this result is demonstrated only for our CES specification, it is obviously of limited application. On the other hand, CES utility is generally regarded as

quite reasonable and is often used in econometric analyses. It cannot be said that Theorem 6 relies on an implausible specification of tastes.

This analysis is offered here primarily as an example of a fluctuating economy which does not want stabilization, even though the first best involves no fluctuations. While the fluctuations do not reflect a first-best allocation, the use of tools which eliminate the fluctuations but do not address the market imperfections that cause them will further reduce utility. Policies imposed on an economy afflicted with distortions may have unintended effects, even when they are trying to address one of the symptoms of the imperfection. Appropriate selection of policies depends crucially on the exact nature of the market imperfection supposedly addressed. These are standard lessons from second-best analysis which are rarely applied in macroeconomic policy analysis.

Theorem 6 addressed the choice between no stabilization and complete stabilization. Surely small amounts of stabilization are also possible. To model such alternatives, we next consider the same specification of government policy, but with γ chosen to be small. Since solutions of the equilibrium are not tractable in the chaotic cases, which would be the most interesting, we use numerical solutions to calculate the welfare impacts of this government policy.

We first assumed that government expenditure had no social value. In this case we found *no case* in which any stabilization was desirable. It is not surprising that stabilization is not desirable if the stabilization policy wastes resources. This was not a problem when we completely stabilized the economy since government expenditure, and waste, are zero at the steady state. To determine whether our negative results on small stabilization policies were due to the waste element, we next assumed that the social marginal utility of government expenditure equaled some fixed fraction of the marginal utility of consumption. In these cases, stabilization was often desired. In fact, when $c = .88$ and $\delta = .1$, a small amount of stabilization was valuable as long as the marginal utility of government consumption exceeded 1/10 of the marginal utility of private consumption. In particular, we found that it was beneficial to increase γ as long as this did not cause the system to completely stabilize at the steady state. As γ is increased, the economy passes from being chaotic to having a stable two-cycle, and long-run average utility rises. However, as soon as γ was made large enough to stabilize the system completely, average utility fell below the $\gamma = 0$ level, as implied by Theorem 6. This pattern of increased government spending being desirable until the system completely stabilized held true in every case where stabilization had any value at all. Furthermore, when stabilization did have value, its effect was equivalent to a 0.5 percent to 1 percent increase in labor endowment. This amount may appear small, but it is of the same order of magnitude as the standard deviation of output over the business cycle.

These examples are not to be taken literally. However, they do indicate that the effects we are discussing are not of negligible importance. They show that the market forces associated with imperfect competition and the pursuit of temporary economic rents can contribute significantly to volatility in macroeconomic indices. While the model presented here is too stylized, abstracting from many critical elements of a real economy, it indicates strongly that imperfect competition can have a nontrivial effect on the dynamic behavior of an economy.

Conclusion

This essay has examined a model of investment, consumption, and labor supply which possessed unique, but possibly erratic, perfect foresight equilibria. The instabilities arose because of standard market imperfections and the presence of temporary monopoly power. While the model abstracted from many elements, such as uncertainty, capital formation, etc., it did provide a laboratory where we could study the interactions of individual investment decisions and the dynamic macroeconomic equilibrium. The pursuit of temporary rents arising from temporary monopoly power is a common activity in our economy, particularly in important "hi-tech" sectors and sectors where firms introduce highly differentiated products which are later imitated by their competitors. While the quantitative importance of the effects studied here can only be determined empirically, our analysis does indicate a possible source of excessive volatility in economic activity. In particular, we were surprised to find that chaotic equilibria naturally arise, even though the framework is simple and the taste specification is one of the simplest and most heavily used ones in econometric analyses of demand.

Given that we have a simple model with fluctuations, one can address issues in stabilization policies which use standard fiscal tools. We showed that complete stabilization was never desired in our model and that slightly stabilizing policies were desirable under some conditions. In general, we found that it was unlikely that stabilization would be beneficial in this economy where the fluctuations were driven by the pursuit of temporary rents by investors. However, such welfare calculations must be viewed carefully since they are surely sensitive to alternative specifications of tastes, technology, and market structure.

We conclude from this preliminary study that the pursuit of temporary rents may contribute to volatility in economic activity even when the first-best allocation converges, and that these effects can be quantitatively significant. Further work incorporating a less stylized intertemporal structure and more elements of reality are needed to make the arguments more compelling. However, the simple version above indicates that such efforts will be valuable in

understanding the dynamics of a modern economy with substantial elements of monopolistic competition.

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