JOURNAL OF ECONOMIC THEORY 39, 368–399 (1986)

Dynamic Limit Pricing and Internal Finance*

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Received February 7, 1985; revised October 2, 1985

This paper examines a model of dynamic limit pricing with a profit-maximizing fringe constrained to finance new investment from internal finance. In a differential game, the dominant firm controls price, thereby determining the current earnings of the fringe, while the fringe chooses its optimal retention ratio. If market growth is less than the discount rate, an important feature of the solution is that price must eventually drop to the fringe long-run cost of production. If market growth is initially rapid, the dominant firm is much more aggressive in limiting fringe growth. *Journal of Economic Literature* Classification Numbers: 022, 611. © 1986 Academic Press, Inc.

1. INTRODUCTION

In this paper we examine the optimal pricing strategy of a dominant firm or a group of joint profit-maximizing oligopolists facing expansion by a competitive fringe. The problem is of considerable interest because most concentrated industries consist of a large number of fringe firms alongside one or more dominant firms.[1] Furthermore, expansion by the competitive

* This paper has greatly benefitted from many suggestions and comments by Mark Satterthwaite, Ron Braeutigam, Richard Caves, Steve Erfle, Steven Matthews, John Panzar, and Bill Rogerson and also from the research assistance of Scott McShan. We also gratefully acknowledge the financial support of the National Science Foundation and the J. L. Kellogg Graduate School of Management at Northwestern University (Ken Judd) and the Sloan Foundation.

[1] For some examples of highly concentrated industries with a large number of fringe firms, see Scherer [29, p. 62]. Some examples from Scherer of industries in 1972 with four-firm concentration ratios of 90 or greater with a large number of fringe firms include flat glass (11), cereal breakfast foods (34), turbines and turbine generations (59), and electric lamps (103).

0022-0531/86 $3.00

fringe appears to be an important source of "entry," since full-scale entry by new firms into significant oligopolistic markets appears to be a fairly rare event.[2]

The problem of a dominant firm facing expansion by a competitive fringe was first examined by Gaskins [14]. He labeled the pricing strategy of the dominant firm "dynamic limit pricing." We believe a new formulation is in order because of two developments: (1) Gaskins' model has received widespread application at the theoretical, empirical, and policy levels and (2) the strategic assumptions underlying his model have come under telling criticism in recent years. We believe that our formulation handles the basic criticisms of Gaskins' approach yet continues to yield a rich set of predictions about dominant firm pricing strategy.

Gaskins' model has become widely known and used by both economists and non-economists for further theoretical modeling,[3] empirical research in industrial organization,[4] and policy analysis.[5] Some reasons for this wide range of application can be found in Scherer's [29, pp. 236–243] excellent description of the model and its predictions. Scherer notes that the model "is compelling not only because it yields rich predictions, but also because these predictions appear to be consistent with a good deal of what we know about American industrial history" [p. 239]. Scherer gives several examples, including the pricing strategies of U.S. Steel, American Viscose, American Can, Xerox, IBM, Alcoa, and General Motors.

Along with the many applications have come some telling criticisms of Gaskins' formulation. These criticisms center around the ad hoc nature of the fringe expansion equation and the game theoretic foundation of the

[2] For a discussion of modes of entry and expansion, see Scherer [29, p. 248].

[3] To cite but a few of the theoretical extensions of Gaskins' model, Brock [2] includes technological progress, Lee [22] adds non-price policies and learning by doing, DeBondt [5] includes scale effects and Encaoua and Jacquemin [9] incorporate non-price policies. Flaherty (11) investigates a duopoly model where a fringe firm gradually grows to a size comparable to the dominant firm. However, she assumes Cournot competition with adjustment costs in output.

[4] At the empirical level, Gaskins' model clearly demonstrates the possibility of a feedback relationship between price and market structure—the choice of a pricing policy affects market share over time, as well as market share determining pricing policy. While the vast majority of industrial organization studies continue to be cross-sectional, a few recent studies are dynamic, and more are likely to follow. Brock [2], for example, estimates Gaskins' model econometrically for the computer industry; while Martin [23] includes a concentration equation based on Gaskins' model in a system of simultaneous equations. Martin finds that a dynamic specification of concentration is critical to the specification of the profitability equation.

[5] Gaskins' model has seen application at the policy level, including frequent citations in law journals. It appears that a number of lawyers as well as economists interested in antitrust issues are familiar with the model, including Dunfee and Stern [7], Easterbrook [8], and Kaplow [20].

model. In particular, Gaskins' fringe expansion equation is not based on any maximization behavior on the part of the fringe. Some of the criticisms of the game theoretic foundations of the model apply with equal force to all but the most recent limit pricing models.[6] It has been pointed out by Friedman [12] and Milgrom and Roberts [26] that under complete information, if established firms' pre-entry actions do not influence post-entry costs or demand, these actions cannot deter entry. The capital investment decision is one example of a pre-entry action which can affect post-entry conditions.[7] We are aware of no previous explanations, however, for how price could deter either entry or fringe expansion under complete information.

Our dynamic limit pricing formulation is based on the importance of internal finance (retained earnings) to fringe firms. In this respect our model is related to Spence [31], in which internal finance plays the crucial role of the constraint on the expansion of later entrants into a new market. Spence, however, chose to examine capacity, not price, as the control variable of the first entrant. Building on Spence, Fudenberg and Tirole [13] also examine how an early entrant in a market can exploit its headstart by strategic investment when there is an **exogenously** given upperbound on investment.[8]

We set up the dynamic limit pricing problem as a deterministic, noncooperative, differential game between the dominant firm and the competitive fringe. The dominant firm controls price while the fringe firms choose their retention ratio. Fringe firms retain all of their income for investment as long as it is in their long-run interest to do so. The connection between current price and expansion is then obvious—current price determines fringe earnings which in turn determines the maximum possible rate of expansion of their capital stock. Today's pricing decision then does affect the future circumstances that dominant firms face.[9]

We demonstrate several interesting features of the equilibrium outcome. First, in the case where the market's rate of growth is less than the dominant firm's discount rate, if the fringe is initially small (large) the price drops (rises) to fringe marginal cost at some finite time although the fringe

[6] A recent example is Matthews and Mirman [24].

[7] For an analysis of capital investment as a deterrent to entry, see Dixit [6].

[8] It is interesting to note that in an early version of their paper, Fudenberg and Tirole [13] point out that their assumption that current levels of output have no effect on the latter periods of the game may not be appropriate if firms relied on self-financing.

[9] There are documented examples of dominant firms setting low prices to reduce the current earnings and internal finance of fringe firms. A recent example is discussed by McAdams [25], an expert government witness in U.S. v. *IBM*. For a different interpretation of *IBM*'s pricing strategy with respect to its plug-compatible periferal equipment competitors, see Fisher *et al.* [10].

market share continues to rise (fall) forever. This contrasts with Gaskins' analysis where the price only approaches the fringe marginal cost asymptotically, and then so only in the case of zero growth.[10] We also examine the case where the market's rate of growth is greater than the dominant firm's discount rate and the game lasts for a finite time, a case not examined elsewhere in the literature. We believe this to be an important case given the fact that demand for many goods grows very rapidly immediately following introduction. In the rapid growth case we find that equilibrium goes through as many as five qualitatively distinct stages. Most interesting is the finding that the dominant firm will be more aggressive during periods of rapid market growth. While our analysis is explicitly open-loop, we argue that our equilibrium must be similar to a closed-loop equilibrium, and show that the long-run steady state outcomes cannot differ.

The next section of the paper is a brief review and a critique of Gaskins' formulation. In section three we summarize recent research supporting the dominance of internal finance and we derive our expansion equation. We set up our general model in section four. In Sections 5, 6, and 7, we present solutions for different assumptions about market rates of growth and contrast our results with Gaskins'. Finally, Section 8 is a discussion of generalizations and alternative solutions to our formulation of dynamic limit pricing.

2. Background

Dynamic limit pricing differs from static limit pricing in that it allows more general strategies on the part of dominant firms. Firms following a static limit pricing strategy either charge the short-run profit-maximizing price and allow their market shares to decline, or they set price at the limit price and preclude all entry. Gaskins argues that there is no justification for this dichotomy; rather maximization of the present value of future profits entails a balancing between current profits and future market share.

In Gaskins' formulation the optimal pricing strategy maximizes:

$$V = \int_0^{\infty} (p(t) - c_d)\, q(p(t), t)\, e^{-rt}\, dt \tag{1}$$

[10] In a comment on Gaskins [14], Ireland [16] shows that Gaskins' expansion equation can be modified such that price always asymptotically approaches fringe marginal cost. None of the basic shortcomings of Gaskins' approach, however, are dealt with.

where V is the present value of the dominant firms' profit stream, $p(t)$ is product price, c_d equals average total cost of production (assumed to be constant over time), $q(p(t), t)$ is the dominant firms' output, and r is the dominant firms' discount rate.

Gaskins assumes that the level of dominant firms' current sales can be decomposed into additive univariate functions of price and time, such that

$$q(p(t), t) = f(p(t))\, e^{\gamma t} - x(t) \tag{2}$$

where $f(p(t))$ is the market demand curve, γ is the market growth rate, and $x(t)$ is the output of the competitive fringe which is assumed to be fixed at any point in time. The net effect of fringe expansion, $\dot{x}$, is to shift the dominant firm's residual demand curve laterally.

Gaskins argues that if fringe firms view current product price as a proxy for future price then expansion will be a monotonically nondecreasing function of current price. He then assumes that expansion is a linear function of current price, given by

$$\dot{x}(t) = k_0 e^{\gamma t}(p(t) - \bar{p}) \qquad x(0) = x_0, \bar{p} \geqslant c_d \tag{3}$$

where $\bar{p}$ is the limit price, k_0 is the response coefficient at time 0 $(k > 0)$, and x_0 is the initial output of the competitive fringe. Gaskins also assumes that the response coefficient $k(t) = k_0 e^{\gamma t}$ is a growing exponential function of time. He argues that increasing disposable income should cause a proportional increase in the quantity of resources available to the fringe for investment in any particular market.

Equations (1), (2), and (3) allow the optimal pricing strategy of dominant firms to be solved analytically using the mathematics of optimal control. The objective is to choose $p(t)$ to maximize (1) subject to (2) and (3), where $x(t)$ is the state variable.[11] The necessary conditions[12] for an optimal $p(t)$ can be used to obtain a system of differential equations describing the time path of prices and fringe market shares. If

[11] The Hamiltonian for Gaskins' model is given by

$$H = (p(t) - c_d)(f(p)\, e^{\gamma t} - x(t))\, e^{-rt} + z(t)\, k_0\, e^{\gamma t}(p(t) - \bar{p})$$

where $z(t)$, the costate variable, is the shadow price of an additional unit of rival entry at any point in time. The first term in the equation is the change in present value accruing from current sales. The second term is the product of $z(t)$ and $\dot{x}(t)$, which is the effect of current entry on future profits.

[12] The necessary conditions in Gaskins' formulation are: (i) $\dot{x}^*(t) = k_0\, e^{\gamma t}(p^*(t) - \bar{p})$, $x^*(0) = x_0$; (ii) $\dot{z}^*(t) = -(\partial H/\partial x)(x^*(t), z^*(t), p^*(t), t)$; $\dot{z}^*(t) = (p^*(t) - c_d)\, e^{-rt}$, $\lim_{t \to \infty} z^*(t) = 0$; (iii) $\partial H/\partial p(t) = ((f(p)\, e^{\gamma t} - x^*(t)) + (p^*(t) - c_d) f')\, e^{-rt} + z^*(t)\, k_0^{\gamma t} = 0$.

$w(t) = x(t)\,e^{-\gamma t}$ is the normalized size of the fringe, the resulting system of equations is

$$\dot{w}(t) = k_0(p(t) - \bar{p}) - \gamma w(t), \qquad w(0) = x_0, \tag{4}$$

$$\dot{p}(t) = \frac{k_0(\bar{p} - c_d) - r(f(p) - w(t) + f'(p)(p(t) - c_d)) + \gamma w(t)}{-2f'(p) - f''(p)(p(t) - c_d)}. \tag{5}$$

Equations (4) and (5) define two possible optimal price trajectories, depending on the initial size of the fringe, its cost disadvantage in relation to the dominant firm, and other factors. If the dominant firm is initially large, it will price initially above the steady-state level and lower it gradually over time, thereby causing the fringe to gain market share until the steady state is reached. This is the strategy which is consistent with a number of corporate histories described by Scherer [29] and with the empirical findings of Caves *et al.* [4] on the decline of dominant firms during the early decades of this century. If the dominant firm is initially small, it initially sets price below the steady-state level and raises it gradually over time, thereby causing the fringe to lose market share until the steady state is reached. In both cases the present value of the profit stream is maximized by balancing the contributions of current price to profits with the loss or gain of future profits from the loss or gain of market share. For further details, we refer the reader to the original paper.

The weak point in Gaskins' formulation is that fringe firms (the entrants) are not treated as rational, maximizing economic agents. As Milgrom and Roberts [26, p. 444] point out, this is common to most of the existing limit pricing literature. In addition, a number of issues can be raised about the exact specification of the fringe expansion equation, $\dot{x}(t) = k_0 e^{\gamma t}(p(t) - \bar{p})$. One issue is the response coefficient, k_0. A priori nothing is known about this parameter which is unfortunate since $\dot{x}(t)$, $\dot{p}(t)$, and the steady-state values of market share and price critically depend on its magnitude.[13] A second issue is the justification for the response coefficient growing at an exponential rate γ. Gaskins' justification, that increasing disposable income should cause a proportional increase in resources available to fringe firms in all industries, seems tenuous in an economy where new industries are emerging and competing for resources, some have matured, and others are declining. Another issue is why fringe expansion

[13] Gaskins provides a numerical example at the end of his paper for a given demand curve and a given response coefficient. We recomputed the steady-state values of market share and price, along with the price trajectories for a range of response coefficients and demand parameters. We find that the results are very sensitive to the selection of the response coefficient and the demand parameters. Plausible results for any given demand curve can be obtained only by experimenting with the selection of k_0.

does not depend on the present size of the fringe, as well as price. One would expect that the larger the fringe, the greater $\dot{x}$, other things equal. Finally, is there any justification for a positive, much less a linear, relationship between fringe expansion and price? We return to these issues in the next section.

3. Internal Finance and the Expansion Equation

Similar to Spence [31], we assume that fringe expansion is constrained by the availability of internal finance. Corporations may finance expansion with internal finance or with debt and new share issues, sources of external finance. It is well known, however, that internal finance has been the dominant source of finance historically[14] as well as during the post-World War II era. New share issues accounted for only 8% of all new equity finance for nonfinancial corporations (a large part originating from public utilities) over the period 1970–79,[15] while the debt/capital ratio in manufacturing in 1981 was under 20%.[16] Given the above figures it is not surprising that retention ratios for small firms tend to be extremely high. For example, during the last decade, corporations under 5 million in assets had **average** retention ratios of over 80% while corporations between 5 and 25 million in assets had average retention ratios of over 75%.[17] It is not at all uncommon for fringe firms in growing industries to pay no dividends (i.e., retain 100% of income) for long periods of time.

Many explanations exist for this pattern of finance. An excellent overview is contained in Meyers [27], where he argues for the "pecking order" theory of finance—internal finance followed by debt and new share issues. The usual explanations for low debt/capital ratios include the costs of financial distress,[18] agency costs,[19] the personal tax advantages of equity,

[14] See Butters and Lintner [3] for a review of the historical importance of retentions as a source of finance for expansion.

[15] See King and Fullerton [21, Table 6.15].

[16] After corrections for inflation, King and Fullerton [21, p. 239] report a debt/capital ratio of 0.198 for 1981. Furthermore, the debt/capital ratio varies little across firms in different size categories.

[17] Retention ratios for corporations by asset size appear in the Internal Revenue Service, *Statistics of Income, Corporate Income Tax Returns*, 1970–1979, Table 5.

[18] Financial distress refers to the set of problems that arise whenever a firm has difficulties in meeting its principal and interest obligations. Bankruptcy is the most extreme form of financial distress. For an in-depth discussion, see Haley and Schall [15, p. 377].

[19] Agency costs arise from the efforts of creditors of the firm to ensure that the firm honors its contractual obligations. These costs result from the attempts by creditors to modify or control firm decisions and the failure to make some investments due to the pricing of debt contracts.

and flotation costs.[20] Recent theoretical foundations for the paucity of new share issues include the design of the corporate tax system and assymetric information between managers and existing and potential shareholders.[21] The former explanation is briefly discussed below.

The United States and a number of other countries employ what is known as a "classical" tax system. Among the provisions of this system is that capital gains are taxed at the personal level at a favorable rate compared to dividend and interest income. A number of recent studies[22] have examined the cost of equity finance under the classical tax system. In each study, internal finance is shown to dominate new share issues. The basic intuition is that no tax savings occur from the issue of new shares, while tax savings do occur when earnings are retained because a dividend tax is avoided for a lower tax on capital gains. Given the typical shareholders' marginal tax rate, the tax advantage of internal finance over new share issues appears to be quite large.

The importance of internal finance is clearly a reason why current expansion and future capacity of the fringe is a function of current price. A high current price established by the dominant firm increases the internal finance available for the purchase of capital and the expansion of output.

Let the revenue of the fringe net of all operating expenses and taxes be denoted by $R(p(t), x(t))$, where x is the output of the fringe. The fringe obviously will not retain 100% of its income in all time periods—eventually it will choose to pay some dividends. Let $u(t)$ be the fraction of earnings each fringe firm chooses to retain. Further, assume, as does Gaskins, constant returns to scale and a capacity contraint.[23] Then the expansion equation of the fringe can be written as

$$\dot{x}(t) = R(p(t), x(t)) \cdot u(t) \cdot J \tag{6}$$

[20] This is particularly true for small issues of debt or new shares—and therefore especially relevant to fringe firms—because the transaction costs tend to be largely fixed costs.

[21] A second theoretical foundation for the dominance of internal finance is presented by Myers and Majluf [28] and summarized by Myers [27]. In their model, firms have assets in place as well as potential investment opportunities. Managers have inside information both on the true value of existing assets and the investment opportunities. Their objective is to maximize the value of existing shares. Myers and Majluf show that if manager's inside information is unfavorable, the firm will always want to issue new shares even if the only good use for the funds raised is to put them in a bank. But if management acts this way, its decision to issue will signal bad news to both old and new shareholders. The conditions for a rational expectations equilibrium indicate that firms may pass up positive net present value opportunities if they have to be financed by new share issues.

[22] See Auerbach [1] for a review of the literature.

[23] This type of cost function is commonly used in theoretical work in industrial organization. See, for example, Spence [30] and Dixit [6].

where J is the physical output-dollar value of capital ratio.[24] A useful way to think about the expansion equation is that if $K(t)$ is the dollar value of the capital stock of the fringe at time t, then $x(t) = K(t)\,J$, and thus $\dot{x}(t) = \dot{K}(t)\,J$. $\dot{K}(t)$ is just the net revenue of the fringe when $u = 1$.

The above expansion equation completely excludes sources of external finance. The assumption of no external finance is stronger than necessary. Rather, what is needed is that external finance, in particular new share issues, be sufficiently more costly than internal finance. This point is discussed in more detail in section eight. The absence of debt finance in the expansion equations is not a substantive limitation. If debt can be increased by some fixed finite amount for every additional dollar of new internal finance, then a multiplier equal to the ratio (debt + equity)/equity could be included without any change in the results. We do not include debt finance, but note that it could be easily incorporated in J.

Before proceeding to the general model, it is appropriate to compare our fringe expansion equation with that of Gaskins', $\dot{x}(t) = k_0 e^{\gamma t}(p(t) - \bar{p})$. As long as the fringe is operating, Eq. (6) can be rewritten as

$$\dot{x}(t) = (p(t) - c_f)\,x(t) \cdot u(t) \cdot J$$

where c_f is the fringe non-capital unit costs of production up to the capacity constraint $x(t)$. Clearly, our fringe expansion equation displays a linear relationship between **both** $\dot{x}$ and p and $\dot{x}$ and x. Quite simply, fringe revenue varies proportionately with both price and output. Also of interest is that something analogous to Gaskins' k appears in our expansion equation; taking the partial derivative of $\dot{x}$ with respect to p, one obtains $x(t)\,uJ$. Our "response coefficient" depends on the current size of the fringe, the fringe retention rate, and the physical output-capital ratio. What is especially important is that the parameter J is knowable a priori—that is, for individual industries one could determine what the response coefficient is at any moment in time. It is apparent that our response coefficient will increase over time as long as $\dot{x}(t) > 0$. Our formulation does not, however, provide any economic justification for Gaskins' assumption that $k(t)$ grows exponentially over time in every industry at some common rate γ.

[24] It should be noted that $1/(Jp)$, not $1/J$ is the conventional capital-value of output ratio. Since $\dot{x}(t)$ is expressed in physical units of output, not in dollar value of output, J must also be expressed in physical units of output per dollar of capital per period of time. This presents no problem for applications as long as the distinction between $1/(Jp)$ and $1/J$ is kept in mind. As an example, suppose the after-tax income of the fringe is \$15,000,000 and $p = \$10,000$ (e.g., output is automobiles) and $1/Jp = 3$ (the average value in the U.S.), then $J = 1/\$30,000$ and therefore $\dot{x} = 500$.

4. The General Model

We shall determine the nature of the open-loop equilibrium in a dynamic game between the dominant firm and the competitive fringe. In this game the dominant firm chooses a price path, $p(t)$, and the fringe firms choose their reinvestment rate, $u(t)$. Since all fringe firms have access to the same constant returns to scale technology, we can assume without loss of generality that each fringe firm chooses the same $u(t)$. Equilibrium is any pair of $p(t)$ and $u(t)$ such that each is a best reply to the other. In examining the Nash open-loop equilibria, we are implicitly assuming that at some initial time the players simultaneously make irreversible decisions concerning $p(t)$ and $u(t)$. While closed-loop equilibrium analysis is preferable since it allows continuous and sequential decision-making, it is intractable. In this problem the two equilibrium concepts will be seen to be similar. In particular, they must have the same steady states. Therefore, we examine the open-loop equilibrium.

Both players make their choices in order to maximize discounted profits, with the dominant firm taking into account its impact on fringe capacity. Let $f(p)$ be demand at $t=0$, and x_0 the fringe capacity at $t=0$. We assume that the interest rate is $r>0$ and that market demand grows at the rate of $\gamma \geqslant 0$. Recall that c_f is the variable marginal cost for the fringe up to the capacity constraint x, where the absolute capacity constraint x can be increased by J units per dollar of profits. We assume that there is no possibility of leasing the equipment and that there is no resale value to the equipment, presumably due to *ex post* firm specificity of the equipment and high costs of monitoring care of leased equipment.

c_d is the marginal and average cost of production for the dominant firm. We assume that the fringe and dominant firms' costs are not too dissimilar. In particular $c_f + rJ^{-1}$, the fringe firms' long-run cost of production, is assumed to be less than the dominant firm's monopoly price. Assumption 1 states this formally.

Assumption 1. *$(p-c_d)\,f'(p)+f(p)<0$ for all p less than c_f+rJ^{-1}.*

This avoids the trivial case where the fringe firms are pushed out of the market even if the dominant firm acts like a monopolist. c_d may be interpreted in a number of ways. First, one could assume that the dominant firm is using a technology different from the fringe firms, one without marginal capacity costs, i.e., there is a large initial set-up cost, but thereafter costs are proportional to output. This is not an absurd assumption since the dominant firm operates at a different scale of production and possibly uses a different technology.

If the dominant firm uses a technology similar to that of the fringe firms, it would also have a capacity choice, and we should take care that we remain consistent with the imperfect capital market assumptions crucial to our analysis. We should also apply the restrictions concerning leasing and resale to the dominant firm. Nevertheless, we may often ignore the dominant firm's capacity choice in this case. If investment is irreversible, but the dominant firm's sales are always growing, then the irreversibility is not binding and c_d is the long-run marginal cost, i.e., short-run marginal costs plus the opportunity cost of capacity. If the dominant firm's sales are declining, then c_d is the marginal variable cost, since the capital costs are sunk and unrecoverable. We will assume that c_d is constant through time. This means our analysis applies to two cases: when there is no marginal capacity cost for the dominant firm or when we find the dominant firm's sales to be monotonic in equilibrium.

Recall that w is the fringe capacity expressed as a proportion of market size, that is, $w(t) = x(t)\, e^{-\gamma t}$. w is the state variable of interest to both players, the dominant firm wanting to keep it low and the fringe possibly wanting to increase it. The evolution of w is given by

$$\dot{w} = (p - c_f)\, s(p, w)\, uJ - \gamma w \tag{7}$$

which is derived from the fringe expansion equation, (6), where $s(p, w)$ is fringe supply normalized for market growth and revenue equals $(p - c_f)\, s$. Since the fringe firms are profit-maximizing, we define $s(p, w)$ to be w if $p > c_f$ and zero otherwise. At $p = c_f$, the fringe is indifferent between producing and not producing and we assume they produce zero in this case to keep the dominant firm's problem well-behaved. The dominant firm's problem is

$$\operatorname*{Max}_{p(t)} \int_0^F e^{\gamma t}(f(p) - s(p, w))(p - c_d)\, e^{-rt}\, dt$$

s.t.

$$\dot{w} = (p - c_f)\, s(p, w)\, uJ - \gamma w$$

where $F \leqslant \infty$ is the time at which the game ends. Let η be the dominant firm's shadow price for w. By the Pontryagin maximum principle

$$\dot{\eta} = r\eta + (p - c_d)\, s_w(p, w) - \eta u(p - c_f)\, J \tag{8}$$

where the current-value Hamiltonian (using $r - \gamma$ as the discount rate) is

$$H(w, p, \eta) = \begin{cases} (p - c_d)(f(p) - s) + \eta((p - c_f)\, suJ - \gamma w), & p > c_f, \\ (c_f - c_d)\, f(p) - \eta\gamma w, & p \leqslant c_f, \end{cases} \tag{9}$$

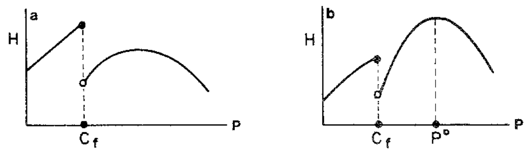

FIG. 1. Dominant firm's objective.

and p is chosen to maximize H:

$$p \in \arg\max_{p} H(w, p, \eta). \tag{10}$$

Since marginal revenue to the dominant firm is positive for $p < c_f$,

$$p = c_f \qquad \text{or} \qquad 0 = (p - c_d) f' + f(p) - w + \eta u w J. \tag{11}$$

The corner choice, $p = c_f$, cannot be ruled out since at that price the fringe shuts down, causing the Hamiltonian to look like the graph in either Fig. 1a, where the corner choice of c_f is the solution, or Fig. 1b, where the optimal p, p^0, is above c_f.

Figure 1 shows clearly that the assumption that the fringe shuts down at $p = c_f$ makes the objective of the dominant firm upper semicontinuous and is necessary for the existence of a solution to (10).

Each fringe firm will maximize the present value of its net cash flow, income minus retentions, taking prices as given. Since each firm is a price-taker, the fringe acts in the aggregate as a profit-maximizing price-taker. Therefore, the competitive fringe collectively solves the problem

$$\operatorname*{Max}_{u(t) \in [0,1]} \int_0^F e^{\gamma t} (p - c_f)\, s(p, w)(1 - u)\, e^{-rt}\, dt$$

s.t.

$$\dot{w} = (p - c_f)\, suJ - \gamma w.$$

If λ is the shadow price for w from the point of view of a fringe firm, then its evolution is described by

$$\dot{\lambda} = r\lambda - (p - c_f)\, s_w(p, w)(1 - u) - \lambda(p - c_f)\, uJ \tag{12}$$

and the decision rule for a fringe firm is

$$u = \begin{cases} 1, & \lambda > J^{-1} \\ \in [0, 1], & \lambda = J^{-1} \\ 0, & \lambda < J^{-1}. \end{cases} \tag{13}$$

The fringe decision rule is bang–bang since both the payoff and equation of motion are linear in the control, u.

DEFINITION. Open-loop equilibrium is any pair of strategies $(p(t), u(t))$ together with some functions of time, λ, η, and w, which satisfies (7), (8), (10), (12), and (13), given an initial fringe size equal to w_0.

5. CASE I: $\gamma < r$

We first examine the slow growth case where the rate of market growth is less than the interest rate. We consider the case of an infinite horizon, exactly the situation examined by Gaskins. We first investigate the steady state of the equilibrium equations and then construct an equilibrium which converges to that steady state.

The Steady State

Theorem 1 enumerates the possible steady states of our equilibrium for the general model with $\gamma < r$.

THEOREM 1. *If $\gamma < r$, the steady state of any open-loop equilibrium is* (i), (ii), *or* (iii):

(i) $w^{SS} = 0$ *and u arbitrary*;

(ii) *if $c_f + rJ^{-1} \geqslant c_d$, then*

$$\eta^{SS} = (c_f + rJ^{-1} - c_d)/(\gamma - r)$$

$$\lambda_{SS} = J^{-1}$$

$$p^{SS} = rJ^{-1} + c_f$$

$$u^{SS} = \gamma/(p^{SS} - c_f)\, J = \gamma/r$$

$$\begin{aligned} w^{SS} &= ((p^{SS} - c_d) f'(p^{SS}) + f(p^{SS}))/(1 - \eta^{SS} u^{SS} J) \\ &\equiv (r - \gamma)((rJ^{-1} + c_f - c_d) f'(p^{SS}) + f(p^{SS}))/(r + \gamma J(c_f - c_d)/r); \end{aligned}$$

(iii) *if $c_f + rJ^{-1} < c_d$, the "dominant" firm is driven out if c_d is the dominant firm's marginal variable cost, and price equals $c_f + rJ^{-1}$ in any case.*

In (i), the fringe does not exist so the dominant firm will set price at the monopoly level. In (ii), the fringe exists and has size w^{SS}. In (iii), price is governed by fringe cost, and the fringe eliminates the dominant firm if it has a sufficiently superior cost structure.

Proof. It is not possible that in a steady state the dominant firm limit prices by setting $p = c_f$. With such a steady-state price, the fringe firms would not invest since the quasi-rent would be zero, implying that the fringe would disappear asymptotically. However, choosing a steady-state price of c_f would not be a best response by the dominant firm to the fringe choice of no investment. Therefore, $p = c_f$ cannot be part of a mutual best response. Hence, in a steady state, the firm must choose a price on the interior of its choice set. Hence, the possible steady states are found by setting all derivatives in the equilibrium equations (7), (8), and (12), equal to zero and using the interior condition in (11). It is straightforward to show that in (ii) the dominant firm will not set price equal to c_f given the computed value for the shadow price η. Hence, (ii) does describe an equilibrium steady state. Cases (i) and (iii) are trivial cases where technology and initial conditions drive the long-run equilibrium. Q.E.D.

Convergence to the Steady State

We next examine the evolution of the game out of steady state. We want to determine whether the game can converge to the steady state and the extent to which the dominant firm can affect this transition. Due to the specific nature of the manipulations involved in this transition analysis, we assume that demand is linear.

ASSUMPTION 2. $f(p) = a - bp$.

A convenient reference price will be the dominant firm's monopoly price,

$$p^* \equiv \frac{a}{2b} + \frac{c_d}{2}.$$

In particular, if there were no fringe as in case (i) above, then the dominant firm would charge p^*. We will also concentrate on the more interesting case where fringe long-run marginal cost exceeds that of the dominant firm.

ASSUMPTION 3. $c_f + rJ^{-1} \geqslant c_d$.

The alternative has a trivial steady state and the analysis of convergence to that condition requires only minor adjustments to the following analysis. In general, the dominant firm's behavior is described by setting

$$p = c_f \qquad \text{or} \qquad p = p^* + \frac{w}{2b}(\eta u J - 1), \tag{14}$$

whichever maximizes the Hamiltonian, H, which is given by substituting $a-bp$ for $f(p)$ in (9).

Next, we will show that for w close to w^{SS}, we can construct an equilibrium path which will converge monotonically to the steady state. We need to first establish that, given this convergence assumption, when w is close, *but not necessarily equal,* to w^{SS}, price and the fringe shadow price of capacity are equal to their steady-state values.

LEMMA 1. *If an equilibrium converges to the steady state, then for t such that $w(t)$ is sufficiently close to w^{SS}, $p(t)=p^{SS}$ and $\lambda(t)=\lambda^{SS}$.*

Proof. To establish this we show that it is inconsistent for price, λ, and fringe capacity to all converge gradually to their steady-state values. Since w cannot jump, it must converge gradually to w^{SS}, if it converges. There would appear to be four combinations of p and λ converging asymptotically to their steady-state values from above or below. However, p and λ must move in the same direction. If λ is above and falling to its steady-state value, J^{-1}, then price must also be falling. This follows from the fact that $u=1$ when λ exceeds J^{-1}, implying via (12) that λ is falling if and only if p also exceeds its steady-state value, c_f+rJ^{-1}. Similarly, if λ is less than and rising to J^{-1}, then p also is less than and rising to its steady-state value.

Next suppose that p exceeds p^{SS} and λ is falling to λ^{SS}. Then $u=1$ since λ exceeds J^{-1}. This together with (8) implies that the rate of growth in w would exceed $r-\gamma$, since p exceeds c_f+rJ^{-1}. Since we are examining the slow growth case, $r-\gamma$ is positive and w must hit w^{SS} at some finite time. To stay at w^{SS} at this point, u must fall to its steady-state value. However, (14) shows that price increases when u falls, implying that when w hits w^{SS}, price would have to rise and thereby stay above p^{SS}, a contradiction. Similarly, one can prove that if λ and p were to rise to their steady-state values from below, price would have to drop and stay below p^{SS} when w^{SS} is hit. Therefore, if w is to converge to w^{SS}, λ and p must be at their steady-state values when w is close to w^{SS}. Q.E.D.

With Lemma 1 in hand we can first prove a local existence result.

LEMMA 2. *For w_0 sufficiently close to w^{SS}, there exists a unique equilibrium which converges to the steady state.*

Proof. Since p and λ are constant as w converges to its steady state by Lemma 1, u must be changing as w approaches w^{SS}. To determine the conditions for the equilibrium movements of u and η as w converges to its steady state, we differentiate the price equation, (14), with respect to time.

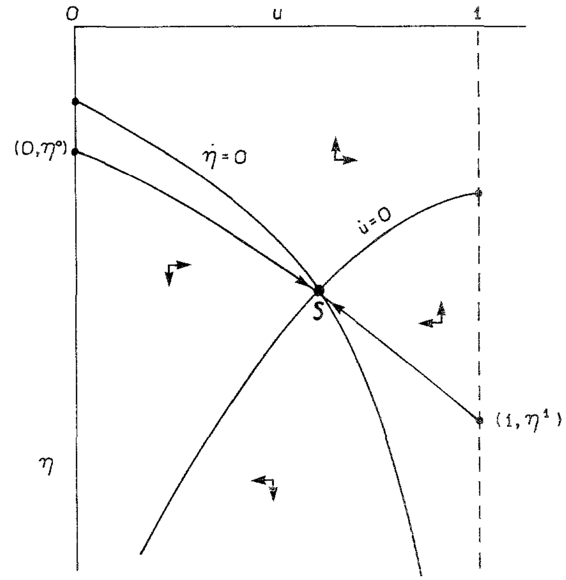

FIG. 2. Phase diagram of equilibrium around steady state of slow-growth case.

(Equations (14) and (7) are used to eliminate w and $\dot{w}$ from the resulting expression.) The result shows that u and η must obey

$$\dot{u} = \frac{r-\gamma-u\eta J(r-\gamma)-Ju(c_f+rJ^{-1}-c_d)}{\eta J} \tag{15}$$

$$\dot{\eta} = r(1-u)\,\eta + c_f + rJ^{-1} - c_d. \tag{16}$$

The phase diagram for this system is presented in Fig. 2. Note the saddlepoint stability of the steady state. If w is close to w^{SS}, then there are unique η and u on the stable manifold such that $c_f + rJ^{-1} = p^* - w(\eta uJ - 1)/2b$ since any hyperbola of the form $\eta u = k$ has a unique intersection with the negatively sloped stable manifold. Therefore, for w near w^{SS}, there is a unique $\eta - u$ pair on the stable manifold of Fig. 2 consistent with w and the pricing formula. As η and u converge to their steady-state values in Fig. 2, the unique corresponding w also converges to w^{SS}. We have thereby demonstrated the existence of an equilibrium path near the steady state which converges monotonically to the steady state.

Q.E.D.

We make no claim of general uniqueness since we have not ruled out cycles. However, Lemma 2 shows that this equilibrium is the only one converging monotonically to the steady state. In this equilibrium, u and η

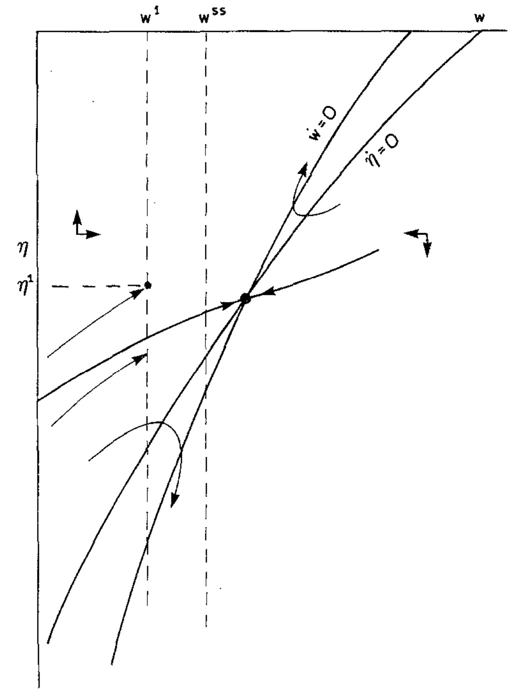

FIG. 3. Phase diagram for slow-growth case away from steady state.

follow the stable manifold in Fig. 2 to the steady state of that system, with w being determined by the price equation (14), since $p = c_f + rJ^{-1}$ along this path.

Finally, we show that there are equilibrium paths for arbitrary w.

LEMMA 3. *There exists a convergent equilibrium for arbitrary* $w_0 > 0$.

Proof. Suppose η^1 is such that $(1, \eta^1)$ is on the stable manifold of Fig. 2 and w^1 is that value of w consistent with $u = 1$ and $\eta = \eta^1$, derived by setting the price equal to p^{SS} in (14). We examine the phase diagram for η and w when $u = 1$ in Fig. 3, given by Eqs. (7) and (8) with u set equal to one. Since the steady-state w in this phase diagram is associated with a price of $c_f + \gamma J^{-1}$ and $u = 1$, it exceeds the true steady state, w^{SS}. Since $u = 1$ and $p = c_f + rJ^{-1}$, w is increasing at (η^1, w^1), and (η^1, w^1) is above the $\dot{w} = 0$ locus in Fig. 3. Therefore, we can run time back from (η^1, w^1) and always remain above the $\dot{w} = 0$ locus. This implies from our pricing formula that price is above $c_f + rJ^{-1}$ and rising as we run time backwards. Hence, if λ is J^{-1} when we are at (η^1, w^1), then λ rises as we move back in time, proving that $u = 1$ is consistent with the evolution of λ during that time. Hence, we have constructed an equilibrium for small initial w.

If w is large, initially u is zero and $\lambda < J^{-1}$, but λ increases and hits J^{-1} exactly when η^0 and w^0 are hit, where these are the values of η and w consistent with $u=0$ and η being on the stable manifold of Fig. 3. This case is even more straightforward since it is just the solution to the piecewise linear ordinary differential equations:

$$\dot{\eta} = r\eta + \max(P - c_d, 0)$$
$$\dot{\lambda} = r\lambda + \max(P - c_f, 0)$$
$$\dot{w} = -\gamma w$$

where

$$P = \arg\max_p \pi(p)$$

$$\pi(p) = \begin{cases} (p-c_d)(a-bp-w), & p > c_f, \\ (p-c_d)(a-bp), & p \leqslant c_f, \end{cases}$$

and we impose the boundary conditions

$$\eta(T_0) = \eta^0$$
$$w(0) = w_0$$
$$w(T_0) = w^0$$
$$\lambda(T_0) = J^{-1}$$

where w_0 is the initial value of w. Note that T_0, the length of time that $u=0$, is endogenous, being determined from the $\dot{w}$ equation. Q.E.D.

Theorem 3 summarizes the conclusions of the foregoing analysis.

THEOREM 3. *If $\gamma < r$, $F = \infty$, and Assumptions 1 and 2 hold, then there is an equilibrium where:*

I. *If w_0 is sufficiently small,*

(i) *price is initially above $c_f + rJ^{-1}$;*

(ii) *at some finite time, t_1, price is $c_f + rJ^{-1}$;*

(iii) *price is falling to $c_f + rJ^{-1}$ and $u = 1$ during $t \leqslant t_1$;*

(iv) *price equals $c_f + rJ^{-1}$ for $t \geqslant t_1$;*

(v) *u drops smoothly from 1 to its steady-state value for $t \geqslant t_1$.*

II. *If w_0 is sufficiently large, then*

(i) *price is initially below $c_f + rJ^{-1}$;*

(ii) *at some finite time* t_1, *price is* $c_f + rJ^{-1}$;

(iii) *price rises to* $c_f + rJ^{-1}$ *and* $u = 0$ *for* $t < t_1$;

(iv) *price equals* $c_f + rJ^{-1}$ *for* $t \geqslant t_1$;

(v) u *rises smoothly from* 0 *to its steady-state value after* t_1.

The equilibrium that we have constructed has several interesting features. First, the dominant firm does use its price-setting power to restrain fringe firm expansion, since $\eta \neq 0$. However, its "limit pricing" behavior decays over time and does not have any long-run impact on performance since the steady-state price is fringe long-run marginal cost. The dominant firm prices high initially, but not as high as its static monopoly price. Its price is reduced as the fringe grows in size. At some finite time price equals fringe long-run marginal cost and remains there forever. This does not stop fringe expansion, but after this time, the fringe reinvestment rate drops monotonically from 1 to the steady-state rate and earnings are distributed to investors.

Comparisons with Gaskins' Model and Comparative Exercises

We are now in a position to contrast our dynamic limit pricing results with those of Gaskins. We first discuss the comparative statics of the two models and then the comparative dynamics.

The greatest differences in the results show up in the steady-state prices. We find that eventually price drops to the fringe long-run marginal cost, at which point full reinvestment ceases and price remains constant. In contrast, in Gaskins' model, as long as growth is positive, the steady-state price always exceeds long-run fringe marginal cost. Gaskins finds this to be a "disturbing result" [14, p. 317] and provides numerical examples which show rather large deviations of price over average cost.

Comparative statics for the steady-state price are very straightforward in our model since p^{SS} always equals fringe long-run marginal cost. Any change in a component of fringe cost, r, J^{-1}, or c_f, is reflected in an equal change in p^{SS}. In contrast, because Gaskins' steady-state price exceeds fringe cost, changes in fringe cost are not fully reflected in changes in p^{SS}. Furthermore, in Gaskins' model, $dp^{\mathrm{SS}}/d\gamma > 0$, while we have already noted that $dp^{\mathrm{SS}}/d\gamma$ is zero in our formulation.

Our comparative statics for w^{SS} are, however, similar to those reported by Gaskins. (See Theorem 1 for the w^{SS} expression.) In particular, $dw^{\mathrm{SS}}/dc_f < 0$ and $dw^{\mathrm{SS}}/dc_d > 0$, implying that an increase in the cost advantage of the dominant firm increases the steady-state market share of the dominant firm. Another intuitive result is that $dw^{\mathrm{SS}}/d\gamma < 0$, or an increase in the rate of market growth raises the steady-state market share of the dominant firm.

The dependence of the steady-state fringe capacity on the other parameters is not as straightforward. For example, as J rises, the steady-state w will rise if and only if c_d exceeds c_f, whereas intuition would have suggested that as fringe firms' costs are reduced their market share should increase. The key element is the dependence of the steady-state marginal value of current fringe revenue, ηJ, on J. The steady-state equations imply that it increases in magnitude as J increases if c_f sufficiently exceeds c_d. Such an increase in the cost to the dominant firm of fringe revenue would lead the dominant firm to be more aggressive in pricing. In fact, examination of (11) shows that if c_f exceeds c_d marginal revenue will become negative if the steady state w were unchanged as J is increased, implying that the dominant firm would have incentive to lower price and push down w. Therefore, stability (this is an intuitive argument since we have not proven stability) would argue that the new steady state has a lower w. On the other hand, if c_f were less than c_d, than the more intuitive prediction of steady state w rising with greater J holds. These results are similar to Gaskins where the effect of an increase in k_0 was also ambiguous, however the analysis there did not make any distinction between fringe marginal and average costs, the crucial element here.

The impact of the interest rate is also different between the two models. Whereas an increase in the rate of interest increased the fringe firms' share in Gaskins, the effect here is ambiguous. This arises because Gaskins' r was only relevant to the dominant firm and a higher interest rate reduced only the value of limit pricing to the dominant firm, whereas here the opportunity cost of funds is relevant to the fringe firms as well, implying that a higher interest rate will make investment less attractive to the fringe firms. Since these forces push fringe share in opposite directions, the net effect is ambiguous. If the firms faced different opportunity costs for invested funds, then a straightforward generalization shows that fringe firms' share rises as fringe firms' interest rate falls and as the dominant firm's interest rate rises.

We next turn to the comparative dynamics of the two models. We can partially compare the optimal trajectories found by Gaskins with our results. The present-value Hamiltonian for Gaskins' formulation is

$$H^G(w, p, z) = (p(t) - c_d)(f(p) - w(t))\, e^{-rt} + z(t)[k_0(p(t) - \bar{p}) - \gamma w(t)]$$

and the present-value Hamiltonian for our formulation when $u = 1$ is

$$\begin{aligned} H(w, p, \eta) = {} & (p(t) - c_d)(f(p) - w(t))\, e^{-rt} + \eta(t) \\ & \times e^{-rt}[(p(t) - c_f)\, w(t)\, J - \gamma w(t)]. \end{aligned}$$

The first term in either Hamiltonian is the present value accruing from current sales while the second term reflects the effect of current entry on future profits. Differences between the two Hamiltonians occur only in the second terms and arise because of the different expansion equations. Note in particular that the value of the second, dynamic, term is proportional to the value of $w(t)$ in our Hamiltonian but not in Gaskins' because our rate of expansion, $\dot{w}(t)$, is proportional to $w(t)$. This implies that our game equilibrium analysis yields higher initial prices when fringe shares are sufficiently small. This is expected since maximizing the Hamiltonian with respect to price involves a balancing of the first and second terms, and the value of our second term becomes small as fringe output becomes small.

This observation permits us to make a comparison of the average rate of decline in price. Since our terminal price is lower and price attains this lower price at some finite time, the average rate of decline in price when the initial w is small must be greater in our game analysis than in Gaskins' model. This is intuitive since the reduced long-run effectiveness of limit pricing in the game analysis encourages the dominant firm to be more aggressive in acquiring profits through high prices in the initial stages when it has a greater market share.

The necessary conditions for a maximum value for the dominant firm's problem for either formulation can be written as a system of differential equations in $p(t)$ and $w(t)$. Gaskins' system of equilibrium equations, (4, 5), was given in Section 3. The comparable $\dot{w}$ and $\dot{p}$ equations for our model are determined by differentiating the price equation, (14), in which we set $u=1$ and assume an interior price choice, valid choices when price is greater than the steady-state price, and are given by[25]

$$\dot{w}(t)=(p(t)-c_f)\,w(t)\,J-\gamma w(t)$$

$$\dot{p}(t)=\frac{(c_f-c_d)\,w(t)\,J-(r-\gamma)[f(p)-w(t)+f'(p)(p(t)-c_d)]+\gamma w(t)}{-2f'(p)-f''(p)(p(t)-c_d)}.$$

A comparison of Gaskins' $\dot{p}(t)$ equation with ours indicates that only the numerators differ. The differences arise only because our response coefficient is endogenous and depends linearly on $w(t)$. Therefore Gaskins' analysis and comparative dynamics is very similar to our $u=1$ phase.

At the outset of this study, it was not a priori clear what effect a rational fringe would have on the analysis. On the one hand a smarter fringe may find ways to circumvent the dominant firm's attempts to limit the fringe, while on the other hand a smarter fringe may realize that, since limit pricing was rational for the dominant firm given the capital market constraint, it should resign itself to this aggressive behavior and retreat or possibly

[25] The necessary conditions for a maximum value of the dominant firm's problem generates the simultaneous differential equations: (i) $\dot{w}^*(t)=(p^*(t)-c_f)\,w^*(t)\,u^*(t)\,J-\gamma w^*(t)$; (ii) $\dot{\eta}^*(t)=(p^*(t)-c_d)\,e^{-rt}+\eta^*(t)(p(t)-c_f)\,Ju(t)$. This system of differential equations can be converted into the autonomous system in the paper by eliminating $\eta(t)$.

even surrender. The mechanical expansion rule of Gaskins' model implies that fringe firms are both limited in their perception of their options, but also implies that they are stubborn. We find that when the growth rate is small, the dominant firm is less successful in keeping out rational fringe investors. We next examine the case when the growth rate is rapid, finding substantially different implications for industry performance under limit pricing with a rational but constrained fringe.

6. Case II: $\gamma > r$, $F < \infty$

Next we examine equilibrium when growth is rapid, i.e., $\gamma > r$, a case ignored previously in the literature. To keep payoffs bounded, we must assume that the game ends at some finite time. We have in mind two types of situations. First, one could think of the good as being faddish in nature with demand growing rapidly, but then dropping to zero at some time F. Second, and more realistically, this analysis will be directly useful in examining our third case where demand initially grows rapidly, but then slows down.

To focus on the interesting situations, some auxiliary assumptions are needed. In order to assure survival of the dominant firm, we need Assumption 4.

Assumption 4. $c_d < c_f + \gamma J^{-1}$.

Otherwise, even if the dominant form charged only its breakeven price, the fringe would want to fully invest (since γ exceeds r) until nearly F, and the fringe would grow more rapidly than the market, squeezing the dominant firm out.

To avoid the trivial case of natural monopoly where the fringe shrinks relative to the market even if price were p^*, we need Assumption 5.

Assumption 5. $c_f + \gamma J^{-1} < p^*$.

For technical convenience we need Assumption 6.

Assumption 6. $c_f < c_d$.

Assumption 6 is needed to assure that the dominant firm will not set price at fringe short-run marginal cost in the last moments of the game. This case is the more interesting since it is the case where the fringe has some chance to compete. Again, we assume demand is linear. In this case, Lemma 4 provides the crucial facts.

LEMMA 4. *Under Assumptions* 2, 3, 4, 5, *and* 6, *if* $\gamma > r$, *p rises whenever* $u = 1$ *and* $p > c_f$, *and* λ *is concave whenever it is falling and u is* 0 *or* 1.

Proof. This follows from

$$\dot{p} = \frac{\dot{w}}{2b}(\eta J - 1) + \frac{w}{2b}\dot{\eta}J$$

$$= \frac{w}{2b}((c_f - c_d)J + \gamma + (r - \gamma)\eta J)$$

which is positive since $r < \gamma$ and $c_d < c_f + \gamma J^{-1}$. If $u = 0$, then $p = p^* - w/2b$ and p rises since w falls with no reinvestment.

From this, we may further conclude that $\ddot{\lambda} < 0$ also when $u = 0$ or 1 and λ is falling since

$$\ddot{\lambda} = r\dot{\lambda} - \dot{p}, \qquad u = 0$$

$$= r\dot{\lambda} - \dot{p}\lambda, \qquad u = 1,$$

is negative if λ is falling and p is rising. Q.E.D.

Next we establish that in equilibrium, the fringe will go through possibly three basic stages, first either initially reinvesting less than all earnings or being indifferent concerning the reinvestment rate, then reinvesting at a 100% rate until some time F_0, at which time investment ceases forever.

LEMMA 5. *In the fast growth equilibrium under Assumptions* 2, 3, 4, *and* 5, *there exist* t_1 *and* F_0 *such that* $t_1 < F_0$, $u(t) < 1$ *or* $\lambda = J^{-1}$ *for* $0 < t < t_1$, $u(t) = 1$ *and* $\lambda > J^{-1}$ *for* $t_1 < t < F_0$, *and* $u(t) = 0$ *for* $t > F_0$. *Furthermore,* $F - F_0$ *is bounded above independent of F and the initial conditions.*

Proof. First, we show that if λ approaches J^{-1} from above, then λ must pass through J^{-1} immediately. Since $\dot{\lambda} < 0$ during such an approach to J^{-1}, price must exceed $c_f + rJ^{-1}$ from (12). Since $u = 1$, price is rising and cannot approach $c_f + rJ^{-1}$ from above as λ converges to J^{-1}. When λ hits J^{-1}, u cannot rise since it equals 1, implying that price cannot fall. Since λ is concave in time, price must exceed $c_f + rJ^{-1}$ when λ is J^{-1} and $\dot{\lambda}$ must be negative. Once λ falls from J^{-1}, then $u = 0$ and λ continues to decline, implying that $u = 0$ thereafter. Also, when $u = 0$, $\ddot{\lambda}$ is bounded above by $-r(p^* - c_f - rJ^{-1})$ implying that λ moves from J^{-1} to 0 in an amount of time bounded above independent of F and w. However, $\lambda(F) = 0$ is the fringe firm's transversality condition at the end of the game, implying that the length of the final stage is bounded above independent of F.

Second, this argument implies that once λ exceeds J^{-1}, it must never equal or fall below J^{-1} again until the terminal stage of the game is reached. Therefore, if λ does not always exceed J^{-1}, there is an initial stage where either $u=0$ or $\lambda=J^{-1}$. Q.E.D.

In summary, we have shown that once λ hits J^{-1} from above it must continue to decline. Therefore, in equilibrium the fringe goes through at most the three stages described above since once it begins reinvesting all earnings, it continues that policy until it stops all reinvestment forever. During the final stage of the game from $t=F_0$ to $t=F$, η follows

$$\dot{\eta}=r\eta+p^*-c_d-w/2b.$$

Since $\dot{w}=-\gamma w$ during this stage and since $\eta(F_1)=0$ also, it follows from solving these linear differential equations that $|\eta(F_0)|$ is also bounded above independent of F. Let $N(w)$ be the value of η at F_0 if the fringe is w at F_0 and the fringe decides to stop expanding at F_0, i.e., $\lambda(F_0)=J^{-1}$. Since η, w, and λ are governed by the linear differential equations above, there is a unique such $N(w)$. Let $W(\eta)$ be the inverse correspondence of $N(w)$. $W(\eta)$ is then the possible sizes of the fringe at F_0 if the fringe ceases to expand when the dominant firms costate is η.

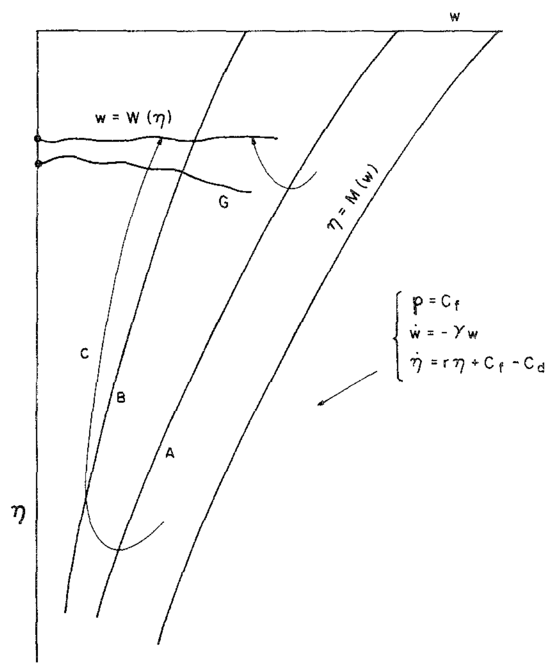

FIG. 4. Equilibrium phase diagram under fast growth.

The final piece for the fast growth case is the phase diagram of the dominant firm's behavior in the intermediate stage when $u = 1$. This is described in Fig. 4, which is the phase diagram in $\eta - w$ space of Eqs. (7), (8) with $u = 1$, $\gamma > r$, and p chosen by (14). The dominant firm must choose between setting price equal to the fringe marginal cost and making the interior choice. Since $\eta < 0$ (more w depresses the dominant firm's profits), our price equation, (14), implies that when w and/or η are large in magnitude, $p = c_f$ will be chosen. This says that if the fringe is large or if the future lost profits from expansion of the fringe is large, then the dominant firm sets price equal to fringe firm costs, causing the fringe firm's expansion to cease. Let $M(w)$ be the η such that

$$\eta < M(w) \Rightarrow p = c_f$$

$$\eta > M(w) \Rightarrow p = p^* + \frac{w}{2b}(\eta J - 1);$$

$M(w)$ exists because H evaluated at $p^* + (w/2b)(\eta J - 1)$ is strictly monotonic in η. Examination of the Hamiltonian also shows that $M(w)$ is increasing in w, that is, the larger the fringe is, the smaller is the critical η at which price is set at c_f by the dominant firm. $M(w)$ is therefore as displayed in Fig. 4. Also, as c_d decreases and as c_f increases, $M(w)$ shifts up, increasing the likelihood that $p = c_f$ is chosen.

With these pieces, we can now construct an equilibrium.

THEOREM 4. *In any equilibrium of our game under Assumptions* 2, 3, 4, *and* 5, *with* $\gamma > r$, *and* $F < \infty$, *there are up to five phases, which are, in order,*

(i) *the fringe does not want to reinvest, i.e.,* $\lambda < J^{-1}$, *and price is the static leadership price;*

(ii) *the fringe wants to invest and price is set below the static leadership price;*

(iii) *the fringe chooses* $u = 1$ *and price is set equal to* c_f;

(iv) *the fringe chooses* $u = 1$ *and price rises above* c_f;

(v) *the fringe chooses* $u = 0$, *and price is the static leadership price and continues to rise.*

Proof. The basic elements needed for construction of our phase diagram are the stationary loci for w and η when $u = 1$ and $\eta > M(w)$—that is, when the fringe wants to fully reinvest earnings and is not shut down. Straightforward calculations show that, if $\eta > M(w)$,

$$\dot{\eta}=0 \Rightarrow w=\left(\frac{p^*-c_f}{1-\eta J}\right)2b+\frac{r\eta 2b}{(1-\eta J)^2}+\frac{(c_f-c_d)\,2b}{(1-\eta J)^2}$$

$$\dot{w}=0 \Rightarrow w=\left(\frac{p^*-c_f}{1-\eta J}\right)2b-\frac{\gamma 2b}{(1-\eta J)\,J}.$$

At this point we use the assumption that $\gamma<(p^*-c_f)\,J$ to ensure a positive w when $\dot{w}=0$.

It is straightforward to calculate that the $\dot{\eta}=0$ and $\dot{w}=0$ loci never intersect for $\eta \leqslant 0 \leqslant w$. Also, the w intercept of the $\dot{u}=0$ locus exceeds the w intercept of the $\dot{w}=0$ locus. Since $c_f<c_d$, at the w intercept of the $\dot{\eta}$ locus p exceeds c_f. That is, when there is no dynamic consideration because η is zero and w is at the value where $\dot{\eta}=0$, the incumbent does not choose to shut down the fringe. In fact, for $c_f \leqslant c_d$, the $\dot{\eta}=0$ locus lies to the left of the $\eta=M(w)$ locus.

To piece together the analyses of $u=1$ and $t>F_0$, note that just prior to $\lambda=J^{-1}$, u is 1. Since λ is falling, $p>c_f+rJ^{-1}$. Hence

$$\dot{\eta}=(c_f+rJ^{-1}-p)\,\eta J+p-c_d>0,$$

proving that η is rising just before the moment the fringe shuts down.

Putting these pieces together, we get the phase diagram presented in Fig. 4. Since $u=1$ in Fig. 4, it represents the possible paths of the game when the fringe is fully reinvesting.

We now can determine that the game goes through possibly five phases. First, the fringe may not want to expand and the dominant firm sets price equal to the static leadership price. This will be the case if w is initially large, making the static leadership price small and fringe investment unattractive. Eventually w will be sufficiently small and the static leadership price sufficiently large that the fringe will want to expand. At first, λ may stick at J^{-1}. This is a possibility which we cannot rule out. However, if this occurs then (12) implies that $p=c_f+rJ^{-1}$ and (15) implies that u increases from 0 to 1. At this point, the game begins to be described by Fig. 4. The game may be below and to the right of the $\eta=M(w)$ locus, where $p=c_f$ and the fringe capacity decreases relative to market size. In this case, the dominant firm decides to limit price and prevent any fringe growth. η also decreases, that is, the current value of the marginal cost to the dominant firm of fringe expansion increases. Eventually, the $\eta=M(w)$ locus is hit. For a while, the game may move along $\eta=M(w)$. This would be accomplished by the dominant firm using a "mixed" (using a generalized curve sense of mixing) strategy, alternating between limit pricing, $p=c_f$, and the alternative, $p=p^*-(w/2b)(\eta J-1)$. Eventually, however, the game moves to a fourth phase (or is in this fourth phase when the fringe begins to want to expand) where price exceeds c_f and is increasing, but is being kept low by the dominant firm to slow expansion of the fringe. During this phase the game proceeds through regions A, B, and C. In region A, the price is low enough that w and fringe market share are decreasing. However, η is also decreasing, meaning that marginal fringe capacity is increasingly more costly to the dominant firm. This decline in η is ended at some point where the $\dot{\eta}$ locus is crossed and the game moves from A to region B. In B, w is still falling, but η is rising. Since the marginal cost of fringe capacity is declining the dominant firm eases up on the price. This continues until the marginal cost of fringe capacity is so small that the dominant firm will allow it to grow. This happens when the stationary w locus is crossed and the game moves to region C. In C fringe market share is rising until the $w = W(\eta)$ locus is reached, after which the fringe ceases to reinvest and the dominant firm engages in static leadership pricing. It is straightforward to check that p rises all through regions A, B, and C and jumps when the fringe ceases to expand. Q.E.D.

Comparative dynamics are again difficult to determine. However, one very interesting feature is clear from Fig. 4. As the horizon, T, becomes large, the path followed by the game must move closer to the $w = 0$ axis in Fig. 4. Lemma 5 showed that the terminal period when $u = 0$ was bounded above independent of F. Furthermore, if w is small, $\lambda < J^{-1}$, and $u = 0$, price is set at the monopoly price and λ is falling by (12), implying that the game cannot languish in any $u = 0$ phase for an arbitrarily long period of time. Therefore, for games of arbitrarily large T, $u = 1$ in equilibrium for an arbitrarily long time. If w is large or η is small in magnitude, then it will take only a short time for the game to reach G, implying that w must be close to zero for a substantial period of time in long games. This is a type of turnpike theorem since $w = 0$ is a collection of steady-state points.

This close approach to $w = 0$ in long games implies that the dominant firm is so aggressive that the fringe is reduced in size and becomes relatively small during a long game. This temporary decline in w and the aggressive pricing by the dominant firm which achieves this result are features which distinguishes the fast-growth case from the slow-growth case. This extra aggressiveness by the dominant firm under fast demand growth indicates that industry performance may be worse in our model than under slow growth, though confirmation of this conjecture awaits more precise quantitative analysis.

Note that the fast and slow-growth cases differ substantially. In the slow-growth case, the fringe size moves in a monotonic fashion whereas in the fast-growth case relative fringe size may go through phases of both expansion and contraction. Price movements also differ in the same fashion with prices being much more volatile in the fast-growth case. The dominant firm is much more aggressive in the fast-growth case, or at least is more successful in limiting fringe firm growth. This is not surprising, since the value of limiting fringe firm growth is greater relative to the current sacrifices implicit in limit pricing as the future market is larger relative to the current market.

7. Case III: Fast Growth followed by Slow Growth

Casual empiricism suggests that for many goods demand first grows rapidly then slows as the industry matures. This can be modeled in our analysis by assuming that the rate of growth initially exceeds the rate of interest, as in Case II, then at some known time, R, the rate of growth drops to a level below the rate of interest, as in Case I. The analysis of this case is accomplished by a simple union of the two preceding cases. The time R denotes the end of the fast-growth phase as did F in Case II, except that at R the shadow prices are not zero, but rather are given by the initial equilibrium relationship between the w and shadow prices in the equilibrium of a slow-growth game. All that is altered is the terminal surface of the fast-growth phase. The phase diagram for the fast-growth game in Fig. 4 continues to be the phase diagram for the fast-growth phase with fringe firm reinvestment. The curve G in Fig. 4 represents the equilibrium $\eta - w$ relationship which exists at the beginning of the slow-growth infinite horizon game of Case I. In this case, G is the terminal surface of the fast growth equilibrium system instead of the $w = W(\eta)$ locus. (Nothing is to be inferred from Fig. 4 concerning their relative position.) This terminal surface G represents the transition between the two growth phases. Unifying the analysis of the two phases in this fashion shows some interesting features of the resulting equilibrium. In the initial fast-growth phase, the dominant firm will be very aggressive, keeping price low to slow fringe expansion. As the fast-growth phase nears its end, the dominant firm cashes in by letting price rise, reaping large profits because the fringe is small but the market has grown to a large size. When the slow-growth phase begins the fringe is able to grow sufficiently rapidly to increase its market share and force price down to its long-run marginal cost, which is also the long-run market price.

8. Generalizations and Alternative Solutions

We have characterized the open-loop equilibrium of the dynamic price leadership model. First note that continuous limit pricing is never an equilibrium of our open-loop game. If the dominant firm would choose a price path $\tilde{p}(t)$ with $\tilde{p}(t) < c_f + rJ^{-1}$ always, then the fringe will never expand. However, if the fringe firms are committed to no expansion, it is not rational for the dominant firm to react with such a price path. It may be that the dominant firm would make more money charging $\tilde{p}(t)$ with no fringe expansion than it does in our equilibrium, but that outcome is not an open-loop equilibrium unless the static leadership price is less than $c_f + rJ^{-1}$. It may be the outcome if the game were a Stackelberg game where the dominant firm could not only commit itself to a fixed price path, but could also communicate such a commitment to the fringe firms before they committed themselves to any investment policy. This structure is not studied here because of the excessively strong commitment advantage enjoyed by the dominant firm compared to the symmetric commitment structure in our open-loop structure.

At this point it is clear that our assumptions about various methods of financing expansion are somewhat stronger than necessary. While the assumption that new share issues are not possible is quite strong, it is not necessary for equilibria to display the qualitative features displayed here. A much more reasonable assumption is that equity financing in the form of new share issues is possible but more costly than retentions. Suppose the tax laws were such that the effective tax rate on equity investment was higher than that on investment financed by retentions. See, e.g., Auerbach [1], for an exposition as to how this is likely to be true under current U.S. tax law. Furthermore, suppose that this higher tax pushes the marginal cost of equity finance above the marginal value of capital, λ, everywhere along an equilibrium path. This is possible since either the game is finite in length and λ, being continuous, has a maximum on a finite interval, or the game is infinite in length with λ converging to a finite limit. Then the equilibrium path that we construct assuming that equity investment is impossible will continue to be an equilibrium since the marginal value of capital for a *competitive* fringe firm, λ, is determined at any time solely by the future path of prices and a fringe firm will not use equity finance as long as its cost exceeds λ. Therefore, as long as λ is below the cost of new share issues, our equilibrium remains an equilibrium if equity financing is also possible since fringe firms will choose to issue no new shares. Similar comments can be made concerning debt financing. In the case of debt financing, it may be natural to assume that new debt can be issued roughly in proportion to retained earnings. This, however, is handled straightforwardly by altering J, the amount of new capacity made feasible by one dollar of retained earnings. Therefore, our analysis is more general than indicated by the initial assumptions.

The final issue we should discuss is the subgame perfection of our equilibria. While a complete closed-loop subgame perfect (also known as feedback) equilibrium analysis of this game is beyond our reach at this time, certain aspects are immediately apparent. The steady-state closed-loop equilibrium price in the slow growth case must also be $c_f + rJ^{-1}$. A larger constant steady-state price would cause the fringe to become indefinitely large relative to the market because price would always exceed long-run fringe average cost, and therefore fringe firms would want to invest all earnings, causing the fringe to grow at r, faster than demand growth. However, price would have to fall to the minimum of c_f and c_d if the fringe grew to such a size. A smaller steady-state price would imply that the fringe disappears, causing the dominant firm to charge p^* instead of the supposed steady-state price. Therefore, if a closed-loop equilibrium converges to a steady state, the long-run price must be the same as in the steady state of our open-loop solution. These same arguments also show that a cyclical closed-loop equilibrium must oscillate between being below and above $c_f + rJ^{-1}$. Therefore, the introduction of a rational fringe causes the long-run behavior of closed-loop equilibria to be closer to our open-loop equilibrium than to that of Gaskins'.

9. Conclusions

In this essay we have examined the optimal pricing strategy of a dominant firm facing expansion by a competitive fringe. This problem was first examined by Gaskins [14], who labeled the pricing strategy of the dominant firm "dynamic limit pricing." While his analysis has received widespread application, its strategic assumptions have come under telling criticism in recent years. The principle differences between our formulation and Gaskins' is that we precisely specify the constraint on fringe expansion, restriction to internal finance, and we treat the fringe as a rational, maximizing economic agent. The capital market imperfection provides a rational basis to the dominant firm's choice to keep price low today in order to limit the rate of expansion of the fringe.

In solving the noncooperative differential game between the dominant firm and the competitive fringe, we first examined the case Gaskins considered, market growth less than the discount rate, but we also examined the case of rapid growth. In equilibrium with "slow" growth, we find that: (i) if the fringe share of the market is sufficiently small, the dominant firm will set price above its cost of production; (ii) at some finite time, price will drop to the fringe long-run marginal cost; and (iii) the fringe firms will retain 100% of their earnings until price equals their long-run marginal cost. While our results resemble Gaskins' in that the dominant firm will use its power to slow fringe growth, in the equilibrium of our model, price converges to a lower level at a faster rate. Therefore, the long-run anticompetitive nature of dynamic limit pricing is less in our game equilibrium analysis compared to Gaskins'. If the firms have similar long-run marginal costs, then long-run performance as measured by social surplus is also better here. However, initially prices will be greater if the fringe starts small.

When the market initially goes through a period of rapid growth, any comparison of performance is ambiguous. In this case the dynamic incentives for reducing current fringe size during rapid growth are sufficiently large relative to the current cost of limit pricing that the dominant firm will price to reduce the fringe share and keep it small. However, when this initial phase of rapid growth draws to a close, the dominant firm cashes in by raising prices, allowing fringe firms to accumulate the necessary earnings for growth. When the market growth rate slows, the fringe grows and prices drop, converging to the slow growth steady-state values.

In conclusion, we find that when fringe firms are faced with capital market imperfections limiting the availability of external finance, dynamic limit pricing will be an important feature of dominant firm decision-making. However, in equilibrium, the importance of this behavior will depend crucially on the rate of growth of demand.

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