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# The Law of Large Numbers with a Continuum of IID Random Variables*

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There are two problems with the common argument that a continuum of independent and identically distributed random variables sum to a nonrandom quantity in "large economies." First, it may be unintelligible in that it may call for the measure of a nonmeasurable set. However, there is a probability measure, consistent with the finite-dimensional distributions, which assigns zero measure to the set of realizations having that difficulty. A second difficulty is that the "law of large numbers" may not hold even when there is no measurability problem. *Journal of Economic Literature* Classification Number: 213. 

## 1. INTRODUCTION

In many models we find the following assertion: "Suppose that there is a continuum of agents each making a draw from a distribution $F$, such draws being independent; then the distribution of realized draws equals $F$." This appears to be a law of large numbers, but is there such a law of large numbers for a continuum of random variables? This paper demonstrates two difficulties. The first is the well-known fact that the "law of large numbers" may be unintelligible for a continuum of random variables in that the realized distribution may be nonmeasurable. However, we show that one can find a probability measure, consistent with the finite-dimensional distributions, which assigns a measure of zero to all realizations which have nonmeasurable distributions. A second difficulty is that the "law of large numbers" may not hold even when there is no measurability problem.

## 2. THE MEASURABILITY PROBLEM

First we must construct a probability space which represents a continuum of independent and identically distributed draws. For specificity, let $X_t$,

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$t \in [0, 1] \equiv I$, be such a collection of real-valued independent random variables, each with distribution function $F$. Let $m$ denote the measure on $\mathbb{R}$, the real line, induced by $F$. By the Kolmogorov construction (see, for example, [2]), we can construct a probability space in which all possible sequences of draws may be simultaneously represented. Let $\Omega = \mathbb{R}^I$ be the elements of the space, that is, $\omega \in \Omega$ represents the realization of a continuum of draws and is function from $I$ to $\mathbb{R}$. Sets in $\Omega$ of the form

$$\{\omega \mid \omega(t) \in A\} \equiv A^t$$

where $t \in I$ and $A$ is a Lebesque measurable subset of $\mathbb{R}$, generate the Borel measurable sets of $\Omega$, $\mathscr{F}$. The measure on $\Omega$, $\mu$, must be consistent with the finite-dimensional distributions of the i.i.d. random variables. This implies that

(i) $\mu(A^t) = m(A)$, and

(ii) $\mu(A^{t_1} \cap A^{t_2} \cap \cdots \cap A^{t_n}) = \mu(A^{t_1}) \times \mu(A^{t_2}) \times \cdots \times \mu(A^{t_n})$, $t_i \in I$, $t_i \neq t_j$ for $i \neq j$, $n = 1, 2, \ldots$.

In fact, these expressions define the measure $\mu$. Hence $(\Omega, \mathscr{F}, \mu)$ is the probability space generated by the Kolmogorov construction.

In this framework, we can pose the appropriate question for our problem. Let $F_\omega(\cdot)$ be the sample distribution function, i.e.,

$$F_\omega(c) = l(\{t \mid X_t(\omega) \equiv \omega(t) \leqslant c\}) \tag{1}$$

where $l$ denotes Lebesgue measure. $F_\omega(\cdot)$ is supposedly the "proportion" of agents whose draw does not exceed $c$. We desire that almost all $\omega$ have their sample distributions equal to the sampled distribution, i.e.,

$$\mu(\{\omega \mid F_\omega(\cdot) = F(\cdot)\}) = 1. \tag{2}$$

In the case of a countable collection of random variables, $\{X_n\colon n = 1, 2, \ldots\}$, one actually defines

$$F_\omega(c) = \lim_{N \to \infty} \frac{\mathrm{Card}(\{n \leqslant N \mid X_n(\omega) \leqslant c\})}{N} \tag{3}$$

and proves a statement like (2). However, we have an uncountable set of random variables. Bewley and Radner [1] recognized these difficulties and suggest that one redefines the integrals so that $F_\omega(c)$ is treated as being equal to $F(c)$ if for each countable subset of $I$ (3) holds for almost all $\omega$. In this paper we will see what exactly holds when we stay with the more direct approach explicit in (1) and (2).

The first difficulty which confronts us is the definition of $F_\omega(c)$. $F_\omega(c)$ may be undefined since the set

$$\omega_c \equiv \{t \mid X_t(\omega) \leqslant c\}$$

may be nonmeasurable. If we assume the axiom of choice, then there exist nonmeasurable sets and it is straightforward to construct $\omega$ such that $\omega_c$ is not measurable (see [5]). Moving to a set theory without choice would be unappealing to many. Therefore, we assume the usual Zermelo–Frankel set theory with choice. We then ask: "Do most realizations, $\omega$, fail to have a distribution function, $F_\omega$?" To answer this, let $\mu^*$ be the outer measure on $\Omega$ generated by $\mu$ and let $\mu_*$ be the corresponding inner measure.

THEOREM 1. *Let*

$$N = \{\omega \in \Omega \mid F_\omega \textit{ fails to exist}\}.$$

*Then*

(i) $\mu^*(N) = 1$,

(ii) $\mu_*(N) = 0$,

(iii) *$N$ is $\mu$-nonmeasurable,*

(iv) *for any $r \in [0, 1]$, there is an extension of $\mu$, $\mu_r$, such that $\mu_r(N) = r$.*

*Proof.* To show (i), it is sufficient to show that all nonempty Borel sets, the $\mu$-measurable sets in $(\Omega, \mathscr{F}, \mu)$, intersect $N$. Choose $B \in \mathscr{F}$ and $\omega \in B$. We shall construct an $\omega' \in B$ such that $F_{\omega'}$ does not exist. Since the Borel sets are constructed from the sets of the form $A^t$ by countable unions and intersections, any Borel set is restricted on at most a countable number of indices. That is,

$$R_B = I - \{t \mid \forall\, \omega \in B (\forall\, x \in \mathbb{R} [\omega/x, t] \in B)\}$$

is countable where we define $[\omega/x, t]: I \to \mathbb{R}$

$$[\omega/x, t](s) = \begin{cases} \omega(s), & s \neq t \\ x, & s = t \end{cases}$$

to be the function which is the same as $\omega$ except that at $t$ it takes the value $x$. The easiest way to see this is to note that these countably restricted cylinder sets do form a sigma-algebra, and hence form the Borel field of the Kolmogorov extension. $R_B$ then denotes the countable set of indices which are restricted by $B$.

We can now construct $\omega'$. If $F_\omega$ does not exist, then let $\omega' = \omega$. If $F_\omega$ exists, choose a $c$ such that

$$l(\{t \mid \omega(t) = c\}) = 0.$$

Let $U \subset R$ be Lebesgue nonmeasurable and disjoint from $R_B$. Since $R_B$ is countable, $U$ can be constructed by the usual approach, as in Royden [5]. Then define $\omega'$ by

$$\omega'(x) = \begin{cases} \omega(x), & x \in R/U \\ c, & x \in U \end{cases}$$

$\omega' \in B$ since $\omega'$ and $\omega$ are equal for $x$ in $R_B$. However, $F_{\omega'}$ does not exist since $\{t \mid \omega'(t) = c\}$ is nonmeasurable, being the union of a set of zero Lebesgue measure and a nonmeasurable set. Hence $\omega' \in B \cap N$ and (i) follows.

Since $R_B$ is countable, each $B \neq \varnothing$ contains an $\omega$ which is constant outside of $R_B$, and hence measurable. Therefore, $N$ cannot contain any nonempty $B$ and (ii) follows.

Parts (iii) and (iv) are direct consequences of (i) and (ii). Part (iii) follows because measurable sets necessarily have equal inner and outer measures, and (iv) holds since the only restictions on extending a measure to a nonmeasurable set are that the inner measure not be decreased and the outer measure not be increased. Q.E.D.

Theorem 1 shows that it is not true that "most" realizations fail to have distribution functions and that there are extensions of $\mu$ where the set of such "bad" realizations, $N$, has measure zero. Let $(\Omega, \bar{\mathscr{F}}, \bar{\mu})$ be the minimal such extension where the Borel sets in $\bar{\mathscr{F}}$ are generated by adding $N$ to $\mathscr{F}$. Any extension of $\mu$ must be consistent with the finite-dimensional distributions. Hence, we have constructed a measure on $\mathbb{R}^I$ consistent with the finite-dimensional properties of a continuum of i.i.d. random variables where almost all realizations of the continuum of random variables have distribution functions.

It appears that we may have discovered a solution to the problem: assign measure zero to all bad realizations. This effectively erases them from consideration without violating the basic i.i.d. assumptions. Since this procedure would be arbitrary, there may be some debate over the legitimacy of this move even if it solved the problem. The next result tells us, however, that our problems are not over.

## 3. The Absence of a Law of Large Numbers

Remember what the usual assertion is: first, the law of large numbers can be stated, and second, it is true. The second step of the assertion remains unproved. In fact, the following claim shows that the law of large numbers does not generally hold, even when it can be stated for all $\omega$. Suppose $\bar{\mu}^*$ and $\bar{\mu}_*$ are the outer and inner measures, respectively, corresponding to $\bar{\mu}$.

THEOREM 2. *Let*

$$L = \{\omega \mid F_\omega \text{ exists and } F_\omega \neq F\}$$

*be the set of realizations where the law of large numbers fails. Then*

(i) $\bar{\mu}^*(L) = 1$,

(ii) $\bar{\mu}_*(L) = 0$.

*Hence,*

(iii) *$L$ is not $\bar{\mu}$-measurable,*

(iv) *there are extensions of $\bar{\mu}$ where $F_\omega$ exists almost surely and the law of large numbers, $F_\omega = F$, holds with probability $\alpha$, for any $\alpha \in [0, 1]$.*

*Proof.* Again (i) is demonstrated if all Borel sets of positive measure intersect $L$. Choose $\bar{B} \in \bar{\mathscr{F}}$, $\omega \in \bar{B}$ with $\bar{\mu}(\bar{B}) > 0$. By construction of $\bar{\mathscr{F}}$, $\bar{B} = B \cap N^c$, for some $B \in \mathscr{F}$, $B \neq \varnothing$. Define $R_B$ as in the proof of Theorem 1. We will adopt the same approach: show that $L$ intersects all sets of positive measure in $(\Omega, \bar{\mathscr{F}}, \bar{\mu})$. We need to construct a $\omega' \in L \cap B \cap N^c$. If $F_\omega \neq F$, let $\omega' = \omega$. Otherwise, choose $c$ such that $F_\omega \neq 1_{[c,\infty]}$. Define

$$\omega'(s) = \begin{cases} \omega(s), & s \in R_B \\ c, & s \notin R_B \end{cases}.$$

Since $B$ only restricts realizations on the countable set $R_B$, and $\omega'$ agrees with $\omega$ on $B$, we conclude that $\omega' \in B$. $F_{\omega'} = 1_{[c,\infty]}$ since $\omega$ is $c$ at all except a countable set of points. Hence, $F_{\omega'} \neq F_\omega = F$ and $\omega' \in L$. By construction, $\omega'$ is measurable, hence, $\omega' \in N^c$. This proves (i). Part (ii) is also proved as in Theorem 1. Parts (iii) and (iv) follow immediately. Q.E.D.

The content of Theorem 2 is that the natural probability measure on realizations cannot tell us that most realizations have sample distributions equal to the sampled distribution, which is precisely what is done in the usual strong law of large numbers. For example, in search models with a continuum of agents, one cannot say that all but a few realizations reproduce the sampled distribution.

Frequently, the weaker law is invoked: the mean of the realized distribution equals the mean of the sample distribution. This law is also disproved in the proof of Fact 2 since the choice of $c$ was arbitrary.

## 4. Conclusions

Does this have any relevance for *economic* theory? Absolutely not. What has been shown is that continuum models do not provide us with good approximations to finite models in this context. We have two alternatives. First, we may just ignore the problem by assuming that the measures we work with have the desirable properties that almost all paths are measurable and that a law of large numbers holds. This paper has shown that such measures do exist. The problem with this approach is that there is no direct way to relate the continuum model with limits of large but finite models, which is desirable in economic applications. If we want a "large numbers" model which is a good approximation to finite models, we must look for other mathematical objects which yield the desired approximations. Hyperfinite discrete models from nonstandard analysis have been used in other contexts for this purpose and would presumably assist in solving the modeling problems discussed here. These models have the advantage of simultaneously approximating both the function theory of the real line and the probability theory of large discrete models, as shown in Keisler [4]. Furthermore, they automatically yield theorems about the limiting behavior of finite economies, as in Brown and Robinson [3]. Since a law of large numbers does exist for these hyperfinite models, we suggest that working economists assume that they have an extension of the Kolmogorov measure which satisfies the law of large numbers when they use these continuum models. Theorem 2 shows that such an extension exists and its similarity to hyperfinite models indicates that it is the one of economic interest.

In this paper we have examined the mathematical problems of modeling a continuum of random variables and using a law of large numbers. This problem occurs in a large number of economic models where a writer wants an individual to face uncertainty, but that there be no aggregate uncertainty. Two basic facts concerning this problem are established here. First, there are measures consistent with the implicit finite-dimensional distributions where all realizations are measurable, thereby making it possible to define the realized distribution. This lays to rest the fear of some that this would not be possible and makes it possible to state a law of large numbers. However, we also established that even if there are no problems with the measurability of the paths, the law of large numbers may fail to be true. The sole constructive result is that it is always possible to construct measures where the law of large numbers can be stated and is true. While this may appear to be a weak straw to clutch, it does show that the work which uses the law of large numbers with a continuum of random variables is not inconsistent with basic mathematics.

## REFERENCES

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