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ON THE PERFORMANCE OF PATENTS

BY KENNETH L. JUDD

A tractable dynamic general equilibrium model of continuous product innovation is developed. Patents, or any imitation lag, of infinite duration may achieve too much, too little, or the socially optimum level of innovation. Most surprising, finite-life patents may induce undamped oscillations in innovation.

1. INTRODUCTION

IT IS UNIVERSALLY RECOGNIZED that innovation is an important source of economic growth. There is a general presumption that patents are useful to encourage innovation despite the market distortions of the monopolies they create. This paper examines the impact that various patent rules have on the positive and normative features of general equilibrium when there is a stream of product innovation determined by profit-maximizing innovators. For reasons of tractability, we model product differentiation in the manner developed by Dixit and Stiglitz [3] and Spence [16] and limit preferences to a particular family. However, we develop the dynamic evolution of innovation, substantially generalizing the static framework of Dixit–Stiglitz and Spence. Several interesting propositions are true in this model. First, if *all* goods are patented forever, the first-best allocation may be achieved, but generally there may be too little or too much innovation in equilibrium. Second, and most surprising, patents of finite life may contribute to cyclic behavior of equilibrium innovative effort. We determine elements of taste, technology, and patent policy which tend to give rise to this instability and argue that it is not a perversity unique to the preferences studied here. Third, patents may be necessary to achieve efficiency even if perfect trade secrecy is available to innovators, since patents also prevent socially wasteful reinvention of a good which may occur if imitation is possible.

While we discuss only patents, our analysis also applies to imperfect trade secrecy. In viewing this model as one of trade secrecy, we implicitly assume that imitation lags are largely fixed by technical and institutional factors, not substantially affected by profit incentives.

The major contribution of this paper is the presentation of a model with a continuing stream of product innovation by rational agents. Most previous work examined just a single innovation [2, 7, 8, 11, 12, 13] or a collection of simultaneous innovations [3, 16]. However, innovation actually is realized as a stream with the existence and price of earlier innovations affecting the expected profitability of later innovations. In such a world one is concerned about the speed and pattern of innovation as well as the total amount. In this model, we find interesting and unexpected interactions between the market structure and the evolution of the market. While the model studied in this paper is special, it does show us one case in which general equilibrium growth with endogenous technical change can be analyzed. Our analysis is of substantive economic interest for two reasons. This model and natural extensions can serve as a tractable case where one may evaluate and examine various policies designed to enhance technological progress in a dynamic equilibrium context, as in Krugman [**9**] and Feenstra and Judd [**5**]. Our finding that finite-life patents may cause unstable development is just one example of this. Second, some of the dynamic outcomes of our model correspond to and possibly explain observed periodic behavior in innovation (see Sahal [**14**]).

2. OPTIMAL INNOVATION WITH CES UTILITY

In this section we shall examine the optimal path of innovation when the common utility function is the symmetric CES utility function

$$U = \int_0^\infty e^{-\beta t}\left(\int_0^\infty x(v, t)^c\, dv\right) dt \tag{1}$$

where $x(v, t)$ is the rate of consuming good v at time t, β is the rate of time preference, and $0<c<1$. The elasticity of substitution between any two goods and the elasticity of demand for any one good at any time are both $(1-c)^{-1}$ and will be denoted σ. Since this will be a model of monopolistic competition, the restriction on c is needed to keep the demand faced by any one monopolist elastic. CES utility is a valuable case to examine for two reasons. First, the symmetry allows us to concentrate on the amount of variety and abstract from asymmetries in preferences. Second, an important determinant of the desired level of variety is the substitutability of goods. Assuming a constant elasticity of substitution allows us to isolate the role of substitutability.

Since we want to concentrate on variety and substitutability, we assume that k is the common labor cost of inventing a new good and that the marginal cost of an invented good is one unit of labor. This model examines only product innovation. This is an appropriate focus since the majority of R&D expenditure is devoted to product development (see Scherer [**14**]). We normalize the initial inelastic labor supply to be unity, and assume it to grow at a constant rate, $\lambda \geqslant 0$.

Given these symmetry assumptions, it is easy to formulate the problem of the optimal path of innovation and consumption. Let $[0, V(t)]$ represent the interval of goods in existence at time t. $x(v, t)$ must be zero if v is outside this interval. Since all goods enter symmetrically into utility and there is diminishing marginal utility of consumption for each good, the optimal consumption of each good v at time t, $x(v, t)$, is the same for all v. Let $y(t)$ be that common level of consumption. Since labor will be fully employed, and $k\dot{V}$ is the amount of labor allocated to innovation, $y(t) = (e^{\lambda t} - k\dot{V})/V$. The problem facing our representative individual reduces to

$$\max_{0\leqslant y\leqslant e^{\lambda t}V^{-1}} \int_0^\infty e^{-\beta t} y^c V\, dt \tag{2}$$

$$\text{subject to} \quad k\dot{V} = e^{\lambda t} - yV.$$

Since the marginal utility of consumption is infinite at 0, we can ignore the constraint that $y \geq 0$. $y \leq e^{\lambda t}V^{-1}$ since innovation is irreversible, implying that V cannot decrease.

We can derive the equations describing the optimal path of V by economic reasoning.[1] The crucial trade-off at any moment is between present consumption and innovation activity which yields greater future variety. Along an optimal path, if innovation effort, $k\dot{V}$, is increased by ΔE at any time t for a period of length Δt, then the loss of utility due to less current consumption, $ce^{-\beta t}y^{c-1}\,\Delta E\,\Delta t$, must not exceed the increase in future utility due to increased variety, $\int_{t+\Delta t}^{\infty} e^{-\beta\tau}y^c(1-c)(\Delta E\Delta t/k)\,d\tau$, and equality holds if $\dot{V}(t)>0$. Letting ΔE and Δt be infinitesimally small, we have the optimality conditions

$$(3) \qquad e^{-\beta t}cy^{c-1}k \geq \int_t^{\infty} e^{-\beta\tau}y^c(1-c)\,d\tau,$$

$$(4) \qquad \dot{V} \geq 0; \quad \dot{V}>0 \text{ implies equality in (3).}$$

Due to the autonomous structure of this control problem, it is clear that if $\dot{V}(t)=0$ is optimal at t then it must be optimal to set $\dot{V}(t+\Delta t)=0$ also since the problem at $t+\Delta t$ will be the same as at t. Thus, if (3) ever holds with equality, it will continue to do so even after V ceases to grow.

Differentiation of equality in (3) yields a differential equation in y:

$$(5) \qquad \dot{y}=y\left(\frac{y}{ck}-\frac{\beta}{1-c}\right).$$

Using the method of partial fractions, one finds three kinds of solutions to (5). If $y(0)<\beta ck/(c-1)$, $y\to 0$ monotonically as $t\to\infty$, implying that $y^{c-1}\to\infty$ and $y^c\to 0$, violating (3) for large values of t. If $y(0)>\beta ck/(c-1)$, y becomes infinite at some finite t, an absurdity. Thus, if equality in (3) ever holds, the desired solution to (5) is the stationary solution,

$$(6) \qquad y(t)=\frac{\beta ck}{1-c}=\beta k(\sigma-1)\equiv\bar{y}.$$

Since $y=(e^{\lambda t}-k\dot{V})/V$, (6) yields the linear differential equation $k\dot{V}+\bar{y}V=e^{\lambda t}$, which has the solution

$$(7) \qquad V(t)=(V_0-\bar{v})\,e^{-\beta(\sigma-1)t}+\bar{v}\,e^{\lambda t}, \quad \bar{v}=(\bar{y}+\lambda k)^{-1},$$

where V_0 is the initial level of variety and $\bar{v}$ is the asymptotic level of variety per capita. If $V_0>(\bar{y})^{-1}$, there is no innovation until $V_0=e^{\lambda t}\bar{y}$. We can relabel time so that innovation begins at $t=0$. Hence, we conclude:

THEOREM 1: *The optimal innovation path for CES preferences is* (7) *for* $t\geq 0$ *if* $t=0$ *is the first moment when* $V_0\leq(\bar{y})^{-1}$, *and* $V(t)=V_0$ *for* $t<0$. *Since the objective*

[1] (3) can be solved in a straightforward fashion by standard optimal control techniques. We use an equivalent intuitive derivation here since this approach will be used in the more difficult finite patent life analysis later.

is concave in state and control in the feasible region, this is the unique optimal program.

Some aspects of this optimal program should be noted. First, the asymptotic per capita level of variety is bounded. This is expected since the marginal cost of innovation is constant and positive, whereas the marginal benefit of variety decreases to zero as the amount of variety increases. The optimal consumption of any good is constant once it is invented, a property which we shall see is due to the symmetry of the utility function. The marginal utility of income is $\bar{y}^c$ always; therefore, the implicit interest rate is constant at β. The levels of innovative effort and variety are decreasing functions of the elasticity of substitution, since innovation is less important when goods are more substitutable. Variety is a decreasing function of the cost of innovation, k, and the rate of time preference, β, which is the intertemporal opportunity cost of resources used for invention. The rate of convergence to the steady-state path is greater as the discount rate and the elasticity of substitution are greater, also expected since these factors reduce variety. Also, variety per capita is less for faster growing economies, clearly because a high rate of population growth increases the cost of sustaining any particular level of per capita variety. We therefore see that intuitive conjectures concerning innovation hold in this model.

3. MARKET PROVISION OF VARIETY WITH PATENTS

Next we study a decentralized market equilibrium for the economy of Section 2. We examine a representative agent model. The agent receives wage income and patent rents. He allocates his income between present consumption and investment, which takes the form of inventing a new good for which he receives a patent of duration $T \leqslant \infty$. Since we are interested in the decentralized market solution, we assume that he faces prices and availability of goods, costs of innovation, and profit streams parametrically. As the producer of any particular patented good, he assumes that the prices of all other goods are fixed, but recognizes his monopoly position in that good due to his patent. As an innovator, he does not consider the fact that his innovation will have an impact on his consumption set and the profitability of other patented goods. As a consumer, he takes prices as fixed. Alternatively one may view this model as a continuum of identical agents and profit-maximizing firms producing existing goods and inventing new ones. For the purposes of this study, the choice of interpretation is a matter of taste. Labor is chosen as the numeraire.

Let V_n, V_p be the amount of variety of nonpatented and patented goods, respectively. V_n changes with time as patents run out, in particular

$$V_n(t) = V_n(t-T) + V_p(t-T) = V(t-T). \tag{8}$$

Nonpatented goods sell competitively at their cost, 1. Since the elasticity of demand is $(1-c)^{-1}$, price will be c^{-1} for all monopolized goods. Therefore, nonpatented goods will have demand of $C/(V_n + d^c V_p)$ per good and patented goods will have demand of $dC/(V_n+d^cV_p)$ per good where $d\equiv c^\sigma$ and C is total consumption expenditure. Profits from a patented good, $\pi(t)$, will be $(1-c)dC/(V_n+d^cV_p)$. Let A be the number of patents an agent holds.

Suppose the agent faces $\pi(t)$, $V_n(t)$, and $V_p(t)$, and is considering an innovation effort plan, $E(t)$. Then $1+\pi A-E$ is his rate of consumption expenditure, yielding a contribution to instantaneous utility equal to $(1-\pi A-E)^c(V_n+V_pd^c)^{1-c}$. The optimality conditions for E when he maximizes discounted utility[2] are

$$\text{(9)}\qquad e^{-\beta t}c(1+\pi A-E)^{c-1}k(V_n+d^cV_p)^{1-c} \geqslant \int_t^{t+T} e^{-\beta\tau}(1+\pi A-E)^{c-1}\pi(V_n+d^cV_p)^{1-c}\,d\tau,$$

(10) $E\neq 0$ implies that (9) holds with equality.

$$\text{(11)}\qquad \dot{A}(t)=(E(t)-E(t-T))/k.$$

We are assuming no trade in patents. This restriction is not substantive since the volume of trade in patents will be zero at the equilibrium prices of this Hicksian economy. Again, the concavity of the objective guarantees a unique optimum. Note that the upper limit of integration in (9) is $t+T$ instead of infinity, representing the possibly finite life of the patent.

The equilibrium conditions are $V=A$, $C=1+\pi A-E$, $E=\dot{V}$, and labor market clearing. Supply of labor to goods producers must equal demand, so

$$\text{(12)}\qquad (1-k\dot{V})=\left(\frac{V_n+dV_p}{V_n+d^cV_p}\right)(1+\pi A-k\dot{V}),$$

implying that (since $(1-c)d^c=d^c-d$)

$$\text{(13)}\qquad \pi=(1-c)d^c\left(\frac{1-k\dot{V}}{V_n+dV_p}\right).$$

It will be convenient to define "effective" variety, V_e, to be

$$\text{(14)}\qquad V_e=d^{-1}V_n+V_p.$$

The reason for the terminology "effective" will be revealed below. Define

$$\text{(15)}\qquad y=\frac{e^{\lambda t}-k\dot{V}}{V_e}.$$

If there are no unpatented goods, y is consumption per good as in Section 2. Otherwise y is consumption per "effective" good. Combining (9–11), the optimality conditions, with the equilibrium conditions yields

$$\text{(16)}\qquad e^{-\beta t}y^{c-1}\geqslant\int_t^{t+T} e^{-\beta\tau}y^c(1-c)\,d\tau$$

(17) $\dot{V}\neq 0$ implies that (16) holds with equality.

[2] This is equivalent to value maximization by a competitive firm.

When V is growing, differentiation of (16) yields a differential-difference equation in y:

$$\dot{y}(t)=y(t)\left[\frac{1-e^{-\beta T}(y(t+T)/y(t))^c}{ck}y(t)-\frac{\beta}{1-c}\right]. \tag{18}$$

The equivalent expression, (62), will prove useful:

$$\begin{aligned} y(t+T)&=\left[1-\left(\frac{\dot{y}(t)}{y(t)}+\frac{\beta}{1-c}\right)\frac{ck}{y(t)}\right]^{1/c}e^{\beta T/c}y(t) \\ &\equiv g(t)y(t). \end{aligned} \tag{19}$$

Note that the constant function $y=\bar{y}_T\equiv\beta(\sigma-1)k/(1-e^{-\beta T})$, solves (18). It will be our strategy to show that it is the only feasible solution if innovation never stops.

The first simple case is when T is infinite, i.e., new goods are protected forever, but some old goods are not monopolized. Examples of such goods would include leisure and household production goods, which cannot be patented, as well as goods that are allowed to be competitively produced. As in the proof of Theorem 1, consumption of each good must be bounded above and below. Equation (18) in y reduces to equation (5), which we saw had only the constant solution. Here that constant solution yields a linear differential equation in V_p. For example, if $\lambda=0$, the equilibrium V_p is

$$\begin{aligned} V_p(t)=&\left(V_p(0)+\left(V_n(0)d^{-1}-\frac{1}{\beta(\sigma-1)k}\right)\right)e^{-\beta(\sigma-1)t} \\ &+\left(\frac{1}{\beta(\sigma-1)k}-V_nd^{-1}\right). \end{aligned} \tag{20}$$

If $V_n(0)\geqslant d(\beta(\sigma-1)k)^{-1}$, then $V_p(t)=V_n=V(t)$. Otherwise, V_p converges to $(\bar{y})^{-1}-V_n(0)d^{-1}$ asymptotically.

Conditions (16) and (17) provide us with insight into the behavior of the decentralized economy when $T=\infty$. They are the Euler conditions for

$$\max_{V_e}\int_0^\infty e^{-\beta t}\left(\frac{e^{\lambda t}-k\dot{V}_e}{V_e}\right)^c V_e\,dt \tag{21}$$

$$\text{subject to}\quad \dot{V}_e\geqslant 0,\quad V_e(0)=V_nd^{-1}+V_p(0),$$

i.e., the market is confusing V_e for V, showing why we call V_e "effective" variety. The decentralized economy is providing variety as if unpatented variety were V_nd^{-1} instead of V_n, because unpatented goods are cheaper and reduce demand for patented goods. This misperception is also reflected in the fact that asymptotic variety for the decentralized economy provides $V_n(d^{-1}-1)$ less variety than the optimum, the amount of underprovision being exactly the level of misperception of unpatented variety implicit in (16) and (17). As expected, the solution for variety, (20), shows that variety is decreasing in σ, β, k. More surprising, when $V_n(0)=0$, we find:

THEOREM 2:[3] *The optimal innovation path is realized as an equilibrium of the economy with infinite life patents on all goods when the utility functional is the symmetric CES utility function and all goods have identical costs of innovation and constant marginal cost production functions.*

Initially it is surprising that an equilibrium with a continuum of monopolies could achieve the first-best allocation. However, since all monopolies charge the same price, the marginal rate of substitution equals the marginal rate of transformation, and the allocation of consumption across goods will be efficient at any instant given the allocation of labor between innovation and production. The surprising aspect of the efficiency of equilibrium is that the dynamic allocation of resources is also efficient. Later we shall see that with non-CES utility infinite-life patents may lead to excessive or deficient innovation in equilibrium.

We have seen that as long as the patents are of infinite duration, innovation will continue indefinitely if the initial level were sufficiently small. We now examine whether this property still holds when patent lives, or trade secrecy durations, are finite. We shall find that lags in the diffusion of information may cause rather different dynamic behavior.

We first examine the case of no growth, $\lambda = 0$. If innovation were to continue forever, the solution of each innovator's problem would always be an interior solution, implying that (16) would always hold with equality. The constant solution $\bar{y}_T$ solves (16). If y becomes arbitrarily large, then V must become arbitrarily close to zero, implying that if $V(0) > 0$, then $\dot{V}(t) < 0$ at some t, violating the $\dot{V} \geqslant 0$ constraint. Since (16) cannot hold with equality for sufficiently small y, there are positive bounds, B and b, such that

$$0 < b < y(t) < B, \quad \forall t. \tag{22}$$

To complete the demonstration that $\bar{y}_T$ is the desired solution to (16) we show that it is the only solution to (18) satisfying (22).

If y satisfies (16) with equality but not (22), there are four cases: (A) $\lim_{t\to\infty} y(t) = B'$ and $\dot{y}(t) > 0$, for all sufficiently large t; (B) $\lim_{t\to\infty} y(t) = b'$ and $\dot{y}(t) < 0$, for all sufficiently large t; (C) $\dot{y}(t) = 0$ and $y(t) > \bar{y}_T$, for an unbounded set of t's; (D) $\dot{y}(t) = 0$ and $y(t) < \bar{y}_T$, for an unbounded set of t's.

In case (A), $\dot{y}(t) \to 0$ and $y(t+T) \approx y(t) \to B'$ as t becomes infinite. Using this fact, (18) shows that $y(t) \to \bar{y}_T$. Hence $B' = \bar{y}_T$. (A similar analysis in (B) shows $b' = \bar{y}_T$.) Thus $y(t)$ increases to $\bar{y}_T$ (in (B), $y(t)$ decreases to $\bar{y}_T$). But $y(t) < y(t+T) < \bar{y}_T$ implying via (18) that $\dot{y}(t) < 0$, a contradiction. (Similarly, (B) can be ruled out.)

In case (C), let t_0 be such that $\dot{y}(t_0) = 0$, $y(t_0) \equiv y_0 > \bar{y}$. From (19) $y(t_0+T) = g(t_0)y(t_0) \equiv g_0 y_0$. Since $\dot{y}(t_0) = 0$ and $y_0 > \bar{y}$, $g_0 > 1$, and $y(t_0+T) > y_0$. Since y is not monotonically increasing after t_0, there is a local maximum $y_1 = y(t_1) > y(t_0+T)$ for some $t_1 > t_0$. Then $y_1 > y(t_0+T) = g_0 y_0 > y_0$ and $y(t_1+T) = g(t_1)y(t_1) \equiv g_1 y_1$. Since $y_1 > y_0$, $g_1 > g_0$. So $y(t_1+T) > g_0^2 y_0$. Again, find a local

[3] The author has also established that Theorem 2 also holds if the instantaneous utility function is $(\int_0^\infty x(v,t)^c \, dv)^\gamma$. Since later analysis does not continue to hold if $\gamma \neq 1$, we examine only the case $\gamma = 1$.

maximum $y_2 = y(t_2) > y(t_1 + T)$, for some $t_2 > t_1$. Continuing this process yields a sequence of times, $\{t_k\}_{k=1}^{\infty}$, such that $y(t_k) > g_0^k y_0$, where $g_0 > 1$, showing that $y(t)$ is unbounded, violating (22). Similarly, (D) implies that $y(t) \to 0$, which violates (22). From these arguments, we can conclude that if the equilibrium never lies on the corner of the representative innovator's problem, then the equilibrium y is the constant solution $\bar{y}_T$, implying that the variety variables satisfy

$$\frac{(1 - k\dot{V})d}{V_n + dV_p} = \bar{y}_T. \tag{23}$$

By (8), (23) reduces to

$$k\dot{V}_n(t) + \bar{y}_T V_n(t) + \frac{(1-d)}{d}\bar{y}_T V_n(t-T) = 1. \tag{24}$$

The characteristic equation for this differential-difference equation is

$$sk + \bar{y}_T + \frac{(1-d)}{d}\bar{y}_T e^{-sT} = 0. \tag{25}$$

(See [**1** and **4**] for the theory of differential-difference equations.)

The complete analysis of (24) is beyond the scope of this paper and unnecessary for our purposes. To determine whether innovation would continue forever we examine the Laplace transform of (24). Define, for complex s with sufficiently large positive real part,

$$\psi(s) = \int_0^{\infty} e^{-st} V_n(t)\, dt,$$

i.e., ψ is the Laplace transform of V_n. By Laplace transform techniques, we know that for all s in some right half plane of the complex plane

$$\psi(s) = \frac{V_n(0) + \dfrac{1}{sk} - e^{-sT} I(s) \dfrac{(1-d)}{d}\dfrac{\bar{y}_T}{k}}{s + \bar{y}_T/k + e^{-sT}\dfrac{(1-d)}{d}\dfrac{\bar{y}_T}{k}}$$

where $I(s)$ is the Laplace transform of the initial condition,

$$I(s) = \int_{-T}^{0} e^{-st} V_n(t)\, dt.$$

Let μ be a root to the characteristic equation with positive real part. Then the denominator in $\psi(\mu)$ is zero and $\psi(\mu)$ could be defined only if the numerator were also zero. That would happen only accidentally since it depends on the underlying parameters and the history of V_n before $t = 0$, which cannot be reasonably restricted so that $\psi(\mu)$ would be finite. Hence, it is generically true that if there are roots of (25) with positive real parts, the Laplace transform of V_n will not be defined at such values, implying that V_n must grow asymptotically at a positive rate. This contradicts the boundedness of V which was demonstrated above. Hence we have proved Lemma 1.

LEMMA 1: *If there are roots to* (25) *with positive real parts, then it is generically true that innovation must cease at some finite time if* $\lambda = 0$.

Since the dynamic behavior of the solution to (24) is governed by the sign of the real part of the roots of (25), we adapt the usual tests to yield the following conditions:

(Necessity) If all roots of (24) have negative real parts, then

$$\left(\frac{(1-d)\beta T(\sigma-1)}{d(1-e^{-\beta T})}\right)^2 < \pi^2 + \left(\frac{\beta T(\sigma-1)}{1-e^{-\beta T}}\right)^2.$$

(Sufficiency) All roots of (24) have negative real parts if

$$\left(\frac{(1-d)\beta T(\sigma-1)}{d(1-e^{-\beta T})}\right)^2 < \frac{\pi}{4} + \left(\frac{\beta T(\sigma-1)}{1-e^{-\beta T}}\right)^2$$

where $\pi = 3.1459 \ldots$. Since $d < e^{-1}$, $(1-d)/d < 1$. Therefore, for large values of T, unstable roots appear. Other conditions guaranteeing roots with positive real part can be derived. For example, if σ is sufficiently large, then for any $T > 0$, some root has positive real part. This indicates that a bias toward unstable solutions to (24) is induced by the finite life patent, and therefore innovation must cease at some finite time in equilibrium. The necessity condition for roots with negative real part yields Theorem 3.

THEOREM 3: (*i*) *For any* σ, (24) *has positive real roots for sufficiently large* βT. (*ii*) *For sufficiently large* σ, *some root of* (24) *will have positive real part for all* $\beta T > 0$. *Therefore, if* $\lambda = 0$ *innovation will cease at a finite time if the patent life is long, discount rate is high, or if goods are highly substitutable.*

These biases towards cessation of innovation have an intuitive explanation. If innovation ends at t_f, then all goods are unpatented at $t_f + T$. If the net profits to innovation are small at t_f, then they are likely to be negative at $t_f + T$ since all goods which were patented at t_f, hence selling at c^{-1}, are unpatented at $t_f + T$, selling at 1. This decline in price for a nonnegligible number of substitutes reduces the demand curve facing any potential innovator, making innovation unprofitable and causing it to cease prematurely at a finite time.

We have focused on finding conditions which cause innovation to cease at some finite time. Next, suppose innovation never ends. Therefore, equation (24) holds forever. A complete solution is not feasible; however, we can determine the initial rate of innovation. Rearranging (24) shows that

$$\dot{V}(0) = \dot{V}_n(T) = k^{-1} - \bar{y}_T(-V_n(0) + V_n(T) + d^{-1} V_n(0)).$$

Since $V_n(0)$ and $V_n(T)$ are given by the initial condition, we can directly compute the initial amount of innovation from the initial condition. This expression shows that the initial level of innovation is greater as (i) the cost of innovation is less; (ii) as the number of currently patented goods, $V_n(T) - V_n(0)$, and the number of currently unpatented goods, $V_n(T)$, is less; and, (iii) the "steady-state" number of goods, $(\bar{y}_T)^{-1}$, increases. If there are no outstanding patents, then $V_n(0) = V_n(T)$ and the initial innovative effort increases as T is greater and substitutability is less.

Straightforward manipulation of the equilibrium equations shows that if there are too many unpatented goods initially, in particular, if $V_n(0)$ exceeds $(1-e^{-\beta T})(1-c)d/\beta ck$, then no innovation occurs. Note that this critical initial value is below the level which makes innovation socially desirable. If $V_n(0)$ is not too big, the asymptotic level of variety is bounded above by $(1-e^{-\beta T})(1-c)/\beta ck - (d^{-1}-1)V_n(0)$, which is exact at $T=\infty$.

At this point, it is useful to summarize our investigation of innovation in a dynamic model where utility is CES over all possible goods. First, if all goods are patented and the patents have infinite life, then the first-best is achieved. Second, if some goods are not patented then there will be an underprovision of variety in equilibrium with infinite patents. The amount of the deficiency in variety is proportional to the number of unpatented goods with the proportionality constant increasing as goods are less substitutable. Exact solutions for the path of innovation are not given, but an upper bound for variety is found. This upper bound decreases as patent life decreases and is a tight upper bound since it is exact at $T=\infty$. Third, if patents have finite lives, there is a tendency for innovation to cease at some finite time, in contrast to the continuous (though diminishing) innovation which occurs in the optimal program. This will happen in particular when the patent life is long or if goods are highly substitutable. Next we show that this dynamic has interesting effects in a growing economy.

Suppose $\lambda > 0$. We saw that no nonstationary solution to equality in (16) is dominated by any multiple of $e^{\alpha t}$ if $\alpha < \beta/c$. In particular, such a solution would not be not dominated by any multiple of $e^{\lambda t}$ if $\lambda < \beta/c$, since that would cause $V_n(t)/e^{\lambda t}$ to be unbounded, an impossibility since the cost of innovation implies that $V_n(t)/e^{\lambda t}$ is bounded. Since $c<1$, a sufficient condition to rule out nonstationary solutions to (16) is $\lambda < \beta$, i.e., the growth rate is less than the discount rate, a natural condition often needed to ensure efficiency of competitive equilibrium in growth models. So, if the patent life is sufficiently long or the goods are sufficiently substitutable, the only interior solution to the equilibrium conditions is again described by $y(t) = \bar{y}_T$, and the definition of y implies a differential-difference equation for V_n.

$$(26) \qquad k\dot{V}_n(t) + \bar{y}_T V_n(t) + \left(\frac{1-d}{d}\right)\bar{y}_T V_n(t-T) = e^{\lambda(t-T)}.$$

The particular solution to (32) for V_n, V_n^p, is

$$(27) \qquad V_n^p(t) = \frac{e^{\lambda(t-T)}}{k\lambda + \bar{y}\left(1 + \left(\frac{1-d}{d}\right)e^{-\lambda T}\right)\Big/(1-e^{-\beta T})}$$

which is always feasible. This particular solution is the logical candidate for the steady-state solution to which the equilibrium tends, since we expect product variety to grow at the same rate as population. From the particular solution V_n^p we also may compute the particular solutions for patented variety, V_p^p, and total variety, V^p, finding that

$$(28) \qquad V^p(t) = e^{\lambda T} V_n^p(t), \quad V_p^p(t) = (e^{\lambda T} - 1) V_n^p(t).$$

Comparing (27) and (28) with (7) shows the extent to which variety is undersupplied when patent life is finite. Equations (27) and (28) isolate the separate effects of finite life patents and the monopoly distortions. As goods are less substitutable, d is smaller, $(1-d)/d$ is larger, and the monopoly price mark-up is greater causing more demand to go to unpatented goods and reducing profitability of innovation. This distortion is less when the proportion of unpatented goods, $e^{-\lambda T}$, is small. Also, there is a loss of goods due to the reduced incentive to innovate, reflected as before in the $1 - e^{-\beta T}$ term.

It is doubtful, however, that the equilibrium will be this steady state. We saw above that (26) often has unstable roots, which will exceed λ for small λ and that these unstable solutions will contribute to any equilibrium in which innovation never ceases. This appears also to be the case here. It seems highly unlikely that when innovation begins, the initial condition is just right to assign a zero coefficient to all unstable roots. These considerations yield Lemma 2.

LEMMA 2: *If λ is less than the real part of some root of* (25), *innovation generically cannot continue indefinitely; hence, innovation will proceed for a while, cease, and then resume when demand is sufficiently large.*

Recall that Theorem 3 gives conditions for some root of (26) having a positive real part. This observation yields Theorem 4.

THEOREM 4: (*i*) *For any σ the necessary condition of Lemma* 2 *is satisfied for sufficiently small λ and sufficiently large βT.* (*ii*) *For sufficiently large σ the necessary condition of Lemma* 2 *is satisfied for any βT and sufficiently small λ.*

Theorem 4 shows us that as patent life is longer, goods are more substitutable, and for small rates of population growth we will not get smooth continuous innovation, but rather will have a period of no innovation between the initial and later periods of innovation. The intuition is the same as discussed after Theorem 3.

Since population is growing, at some point after innovation ceases it will begin again. A reasonable conjecture is that the initial condition again will not be just right to prevent explosive innovation. What would be desirable to find are conditions under which these periods of no innovation recur infinitely often. This is very difficult to do in this continuous-time model. Therefore, we next examine the issue of stable equilibrium innovation in the context of a discrete-time model.

4. A DISCRETE-TIME MODEL

Suppose we have a discrete-time model with population at t, L_t, obeying $L_{t+1}=(1+\delta)L_t=(1+\delta)^{t+1}$, $\delta>0$. Suppose that a good is invented at the beginning of a period and is granted a patent which lasts only for that same period. Again $V_n(t)$ and $V_p(t)$ will be unpatented and patented variety in period t, respectively, with $V(t)$ being their sum. Let $C(t)$ be the consumption expenditure, which in equilibrium must equal net income; but net income must be equal to labor endowment, L_t, since equilibrium net profit[4] must be zero if there is innovation. Therefore,

$$(29)\qquad k=(1-c)dL_t/[V_n(t)+d^cV_p(t)]$$

if $V_p(t)>0$. If (29) implies a negative $V_p(t)$ given $V_n(t)$, then net profits from any innovation must be negative, and $V_p(t)$ is zero. Therefore, if $P_t=V_n(t)/(1+\delta)^t$ is "per capita" unpatented variety,

$$(30)\qquad (1+\delta)P_{t+1}=P_t+\max\left(0,\frac{c(1-c)}{k}-\frac{P_t}{d^c}\right)\equiv h(P_t)(1+\delta)$$

where $P_t=V_n(t)/L_t$ is unpatented variety per capita. The steady state value of P in (30) is $c(1-c)(\delta+d^{-c})^{-1}k^{-1}\equiv\bar{P}$. Let μ_t be the deviation from this steady state, i.e., $\mu_t\equiv P_t-\bar{P}$. If the equilibrium has continuous innovation, then $V_p(t)$ is always positive and equilibrium obeys

$$(31)\qquad \mu_{t+1}=\mu_t\left(\frac{1-d^{-c}}{1+\delta}\right).$$

$1-d^{-c}$ is always negative since $1-d^{-c}\leq 1-e<0$. Thus (31) oscillates.

If $\delta<d^{-c}-2$, these oscillations are explosive, hence eventually infeasible. $\mu_t=0$; there is a first t, t_0, such that $\mu_{t_0}=0$. Following the difference equation back to time 0 shows that only a finite number of initial conditions could cause $\mu_{t_0}=0$. Therefore, only a countable set of initial conditions could result in the equilibrium achieving the steady state. Outside of this set of initial conditions, the equilibrium μ sequence is a sequence of explosive oscillations followed by periods of $\mu_{t+1}=\mu_t(1+\delta)^{-1}$, where no innovation occurs.

The sufficient condition for undamped cycling of innovation is satisfied for c close to one if $\delta<e-2=0.7128$, a reasonable assumption on the time unit if protection from imitation is not long. If c is close to zero, however, undamped cycling will not occur. Hence, this persistent fluctuation will arise only if growth is not too rapid and if goods are moderately or highly substitutable. These are the same biases we discovered in the continuous time case. The important feature of this analysis is the demonstration of the possibility of continued alternation between no innovation and much innovation. Even if the roots to (30) are stable,

[4] Since the patent is valid only during that period in which the good was invented, we do not have any discounting.

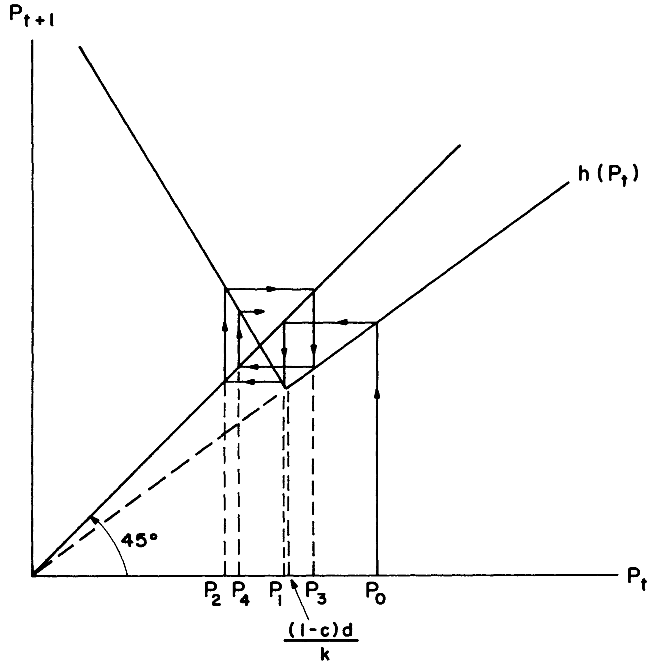

FIGURE 1.—Difference equation for one-period patents in the discrete-time model.

we will generically have damped fluctuations converging to the steady state. Hence, the finite duration of the monopoly power induces an undesirable fluctuation in innovative effort.

The difference equation in (30) can be graphically analyzed, showing clearly this oscillatory behavior. The graph of $h(P)$ is illustrated in Figure 1. $h(\cdot)$ consists of two linear segments. If $P_t \in [0,(1-c)d/k]$, innovation occurs and $P_{t+1} = P_t(1-d^{-c})+(1-c)d/k$. However, as P_t increases, innovation drops even faster, in fact d^{-c} times faster, reducing P_{t+1}. This continues until $P_t \geq (1-c)d/k$. In this region, $P_{t+1} = (1+\delta)^{-1}P_t$, since no innovation occurs, yielding the segment with slope $(1+\delta)^{-1}$. The steady-state value of P is the intersection of $h(\cdot)$ with the 45° line, at $P_t = \bar{P}$. If $P < (1-c)d/k$, innovation occurs but we are in the unstable region where $h(\cdot)$ has greater absolute slope than the 45° line. Unless $\bar{P}$ is hit, an explosive cobweb appears until $P_t > (1-c)d/k$. Then P falls until innovation begins again.

We can compute a bound on the amplitude of these oscillations. From (30) and Figure 1, it is clear that the amount of innovation is negatively related to the amount of unpatented variety and that unpatented variety always exceeds $(1-c)d^{-1}k^{-1}$. Hence, the largest amount of innovation per capita that would ever occur is $V_p(t)/L_t$ when $V_n(t) = (1-c)d^{-1}k^{-1}$, at which time V_p/L become $(1-c)(c-d)/k$. Since $d \to 0$ as $c \to 0$, and $c \geq d$ for all $c \in [0,1]$, the maximum possible amplitude in the oscillations occurs for an intermediate value of σ, being zero when σ is one or infinite.

We can say a little more about this cyclic behavior. The average rate of growth of the goods space cannot exceed λ, since that is infeasible. In equilibrium, it cannot be less than λ, since that would eventually imply positive net profits. Therefore the goods space grows on average at rate λ.

In summary, we have demonstrated Theorem 5.

THEOREM 5: *If $0 < \delta < d^{-c} - 2$, then, for almost all initial conditions the discrete-time, symmetric CES economy with one-period patents alternates between periods of innovation and no innovation. The amplitude is greatest for intermediate values of c, being zero if $c = 0$ or $c = 1$.*

The obvious question to ask is whether there is any evidence of this cyclicity in innovation which we have demonstrated. In fact, there is such evidence in several indices of innovative behavior. For a more detailed discussion, see Sahal [**14**]. For example, analysis of the time series of patented inventions in the construction industry indicates cycles of 40, 20, and 13 years. Similar evidence exists for the farm equipment, railroad, and electric generation industries. Sahal, among others, argues that the observed temporal clustering of innovation in these industries cannot be explained by the existence of major innovations which generate several minor ones, nor by other interactions between current and past innovations. Also, the cycles are too long to be driven by business cycle fluctuations. Our models show that fluctuation in innovation is a possible equilibrium phenomenon within an otherwise constant structure.

The simple intuitive explanation for this instability is most clearly illustrated in this discrete-time model. Suppose that initially there is little variety; then the demand for any good will be high and incentives to innovate will be relatively large. However, as goods are invented, the demand for any particular good will decline, and drop precipitously when patents expire and prices for those goods drop to marginal cost. At such times profits available to a potential patent holder will be low, choking off innovation. This will not be the end of the story, however, since growth will imply that demand for the marginal good will increase, leading to a new phase of innovation. We have seen in the discrete-time model that these oscillations in innovative effort may not be damped, leading to equilibria which have an infinite number of alternations between periods of innovation and no innovation. In the continuous-time model, we saw under similar conditions that continuous innovation would generically not proceed smoothly from the beginning. We would argue for a presumption that the continuous-time model would also display the unstable innovation path under plausible conditions.

We see that the instability is driven by a simple intuition: spurts of innovation will flood the market with variety which throttles future innovation because the current patents will expire, lowering prices. Given that this simple intuition lies at the heart of Theorems 4 and 5, it is likely that they are robust to changes in the preferences, indicating that innovation with finite-life patents may often display unstable behavior.

5. THE GENERAL SYMMETRIC CASE: OPTIMAL AND MARKET DEVELOPMENT

In this section we examine the market and optimal development of innovation and consumption when the utility function is symmetric, not necessarily CES, i.e., $U=\int_0^\infty e^{-\beta t}\,(\int_0^\infty g(x(v,t))\,dv)\,dt$ for some g. We assume that $g'>0>g''$, $g'(0)=\infty$, and $g'(\infty)=0$. All other assumptions of Section 2 remain unchanged.

By symmetry, optimality reduces to maximizing

$$\int_0^\infty e^{-\beta t} g\left(\frac{e^{\lambda t}-k\dot{V}}{V}\right) V\,dt$$

subject to feasibility. The solution here is found exactly as before. If y is consumption per good, $(e^{\lambda t}-k\dot{V})/V$, optimality implies

$$\text{(32)}\qquad e^{-\beta t}kg'(y)\geqslant\int_t^\infty e^{-\beta\tau}[g(y)-g'(y)(y)]\,d\tau;$$

(33) if $\dot{V}>0$, equality holds in (32).

There is a constant function $y(t)=y_0$, which solves (32), where y_0 solves

$$\text{(34)}\qquad \beta k=\left(\frac{1}{\rho(y_0)}-1\right)y_0,$$

where $\rho(y)$ is the elasticity of utility $yg'(y)/g(y)$. Due to the assumptions placed on g, y_0 exists and is unique. Due to the concavity of the functional in the feasible region, this is the unique maximum.

Next we turn to the market solution. The elasticity of demand for a good is $\eta(x)=g'(x)/xg''(x)$ if x units of that good are consumed. Hence, the monopoly price is $(1+\eta(x)^{-1})^{-1}$ when x units are sold. The usual manipulations show that there is a constant, y_M, which solves

$$\text{(35)}\qquad \beta k=(1+\eta(y_M))^{-1}y_M$$

and is the constant equilibrium rate of consumption of each good once it is invented. Since y_M is a constant solution to the equilibrium, the corresponding price is constant for all goods at all times. This does not show that the only equilibria are ones with constant price, but only that such equilibria are described by (35). There may be many such equilibria since the behavior η has not been sufficiently restricted. If demand is less elastic as consumption increases, and $|\eta|>1$, then there is a unique y_M.

We would like to compare the market provision of variety with the optimal provision. As in Dixit–Stiglitz, the identity $x\rho'/\rho=1-(1/\eta)\rho$, together with the definitions of y_0 and y_M, implies that $y_M\gtreqless y_0$ as $\rho'(y_M)\gtreqless 0$, proving the following theorem.

THEOREM 6: *A market equilibrium, when the utility functional is additive symmetric, has a socially excessive consumption per good (implying socially deficient innovation) if and only if the elasticity of utility is increasing at the market value of y.*

These results are similar to the results that Dixit and Stiglitz [3] have shown in the context of a static model of monopolistic competition with fixed costs of production. In fact, this model can be regarded as a dynamic version of their problem. The intuition which they give also applies here: revenue to producers is proportional to $xg'(x)$ whereas the social benefit is $g(x)$; $\rho(x)$ is the ratio and if $\rho'(x)$ is positive then the benefit to the firm from expansion of production exceeds the social benefit.

6. THE GENERAL CES CASE: OPTIMAL AND MARKET DEVELOPMENT

In this section, we move away from the symmetric CES utility functional and examine the general CES utility functional:

$$U=\int_0^\infty e^{-\beta t}\left(\int_{\Omega(t)} \alpha(v)x(v,t)^c\,dv\right)$$

where $\alpha(\cdot)$ is bounded and measurable, $c\in(0,1)$, and $\Omega(t)\subset\mathbb{R}$ is the set of goods in existence at time t.

The social problem is

$$\max \int_0^\infty e^{-\beta t}\left(\int_{\Omega(t)} \alpha(v)x(v,t)^c\,dv\right)$$

$$\text{subject to}\quad k\frac{d}{dt}(l(\Omega(t)))+\int_{\Omega(t)} x(v,t)\,dv=1$$

where $l(\cdot)$ is Lebesque measure. Comparison of equilibrium and optimality conditions again[5] yields the following theorem.

THEOREM 7: *The optimal development of innovation is an equilibrium, when we have a general CES utility function and all goods are covered by patents of infinite life.*

Therefore, even when utility is not symmetric across goods, infinite-life patents may still realize the first-best. The crucial difference between this case and the symmetric case is that the interest rate varies with time. At first, the very desirable goods are innovated and later the less desirable goods are innovated. Free entry implies that the discounted profits of all firms must be zero after accounting for the costs of innovation. This is possible with unequal demands because the initial high rate of innovation drops sufficiently that consumption per good rises and marginal utility of income falls, driving up the interest rate, and causing the future high profit flows to be discounted heavily, whereas later when less desirable goods are innovated, there will be much less innovation, marginal utility of income stabilizes, and the interest rates will be lower.

[5] Since one proceeds just as in the earlier CES case, the details are omitted. They are available in the author's thesis [6].

There has been a presumption in the literature that monopolistic competition results in excessive product diversity, since each firm produces at a point where price exceeds marginal cost. This argument is invalid because it ignores the fact that these goods are distinct and are not perfect substitutes for a consumer. One way to view the results obtained here is as an extension of this point and other similar points made by Dixit and Stiglitz to a dynamic context.

A more substantive interpretation of these results relates to the exact nature of the Dixit–Stiglitz optimality results, i.e., when utility over all goods is symmetric CES, the monopolistically competitive equilibrium realizes the social optimum. In Dixit and Stiglitz, two conceptually different interpretations may be given to k: first, k is a fixed set-up cost borne by any entrant, and second, k is the cost of the R & D necessary to invent the good. In the symmetric CES case, these two yield the same equilibrium if innovators can keep the R & D results secret. This also is the equilibrium when R & D results cannot be kept secret but infinite life patents keep all potential competitors out, whether imitators or independent innovators. This latter equilibrium equivalence holds because symmetry in demand implies that no entrant would pay k in order to enter an existing market and receive duopoly profits since with the same expenditure he can invent a new good and receive monopoly profits.

This equivalence breaks down when we have an asymmetric CES utility function. In this case it is possible that duopoly profits from producing a very desirable good may exceed the monopoly profits which would flow from inventing the most desirable good not yet in existence. If k is either a fixed set-up cost or a cost of innovation when trade secrecy is possible, then this entry into existing markets may occur. If there was an infinite life patent available, then this entry would be prohibited.

Since infinite life patents yield the social optimum in the CES case and a monopoly for each good, monopolistic competition without patents won't generally yield the optimum because that equilibrium would likely have a mixture of oligopoly and monopoly. The only case where this would not happen is where one presumes that Bertrand price competition would result from entry. Hence, the general theorem is that infinite life patents yield the social optimum with CES utility, not that monopolistic competition achieves the optimum. Since monopolistic competition with fixed costs is equivalent to R & D protected by secrecy we also see that infinite patent life in this case is a strictly superior policy to relying on secrecy.

7. CONCLUSIONS

In this paper, we have provided the basis for the study of patents in an economy with a continuum of goods. First we examined the case where the utility functional was symmetric and CES. This yielded the initially surprising result that the decentralized equilibrium with patent rights yielded the optimum dynamic allocation between consumption and innovation. In Section 5, the general symmetric utility function was postulated and a condition for determining the bias in the patent economy was derived. It depended on the elasticity of the utility function: if the elasticity of utility is rising at the market equilibrium, then there is too little variety, generalizing the static model of Dixit and Stiglitz.

Using the CES utility function, we then investigated an economy where not all goods are patented. Since infinite life patents on all goods achieve the social optimum in this case, any loss in efficiency with finite-life patents and incomplete coverage is due to these limitations. In this model, we examined how much variety is lost due to the presence of goods which are either unpatentable, e.g., leisure, or unpatented because its patent had expired. In the case of finite patent life and growth in labor endowment, we discovered a tendency towards cyclic behavior in investment in innovation and explicitly demonstrated it in a discrete-time model. Since the optimal program calls for a constant rate of innovation, finite imitation lags introduce an undesirable cyclical component to the innovation path. The intuition for this instability is quite simple. Initially, there may be little variety per capita in which case profits to innovators are high. However, as the resulting patents expire, prices will drop and profits to an innovator will be insufficient. With growth, this period of innovative inactivity will cease since growth will cause the demand curve for any potential good to increase at the rate of population growth. This new surge of innovative effort begins anew the cycle.

The utility functions used in this paper are very special. In particular, they assumed additive separability in time and across goods, ignoring any aspect of closeness of two particular goods and concentrating only on the substitutability of one good with respect to all others as a group. However, sacrificing generality is not without reward since we are able to simply and compactly formalize important ideas about the dynamics of innovation.

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