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AN ALTERNATIVE TO STEADY-STATE COMPARISONS IN PERFECT FORESIGHT MODELS

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This note examines a technique for the computation of the welfare impact of a perturbation of the steady state in a typical perfect foresight model. The major innovation is the ability to analyze non-stationary perturbations.

The rational expectations hypothesis and its deterministic counterpart, the perfect foresight hypothesis, have had a tremendous influence on economic theory in the past decade. However, their influence in applied economic analysis has been limited by the technical difficulties inherent in computing such equilibria and their associated comparative dynamics. It is the intent of this note to illustrate a technique which demonstrates that comparative dynamics in a perfect foresight model is not impractical. In fact, this technique is able to handle non-stationary as well as stationary perturbations of the steady state. We will thereby have a tractable alternative to steady-state analysis and a generalization of linearization approximations of adjustment processes.

We shall examine an economy described by the differential equations,

$$\dot{p} = g^1(p, k, \gamma h(t)), \tag{1a}$$

$$\dot{k} = g^2(p, k, \gamma h(t)) \tag{1b}$$

and the boundary conditions,

$$\lim_{t \to \infty} = k(t) < \infty, \qquad k(0) = k_0, \tag{2}$$

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where p, k are economic variables, γ is a scalar parameter, initially equal to zero, and $h(t)$ is bounded and eventually constant. One example of this is the simple one-good optimal growth model where k is the capital stock, $u(c)$ is the instantaneous flow of utility if c is the rate of consumption, ρ is the instantaneous rate of discount, $p=u'(\rho)$ is the instantaneous marginal utility of instantaneous rate of discount, $p=u'(c)$ is the instantaneous marginal utility of consumption, and the production function is $(1+\gamma h(t))f(k)$, with $h(t)=1$, where we can interpret γ as a productivity parameter. A change in γ away from zero would represent the perturbation of the economy away from the steady state due to an unanticipated output-augmenting change in the production function. In this case $g^1(p,k,\gamma)=p(\rho-f'(k)(1+\gamma))$ and $g^2(p,k,\gamma)=(1+\gamma)f(k)-c(p)$, where $c(\cdot)$ is defined by $u'(c(p))=p$.

Suppose we are initially in a steady state of the system, i.e., k_0, p_0 are such that $g^1(p_0,k_0,0)=g^2(p_0,k_0,0)=0$, and that there is an unanticipated change in γ from its initial value of zero. We usually are interested in the induced change of a dynamic evaluation function:

$$W=\int_0^\infty \mathrm{e}^{-\rho t}v(p(t),k(t))\mathrm{d}t,$$

i.e., we want to know $\mathrm{d}W/\mathrm{d}\gamma$ given the initial condition, k_0. If the perturbation of the system were autonomous, the direct way to do this is to linearize the system (1) around $(p_0+\mathrm{d}p, k_0+\mathrm{d}k)$, where $\mathrm{d}p$ and $\mathrm{d}k$ are the steady-state changes in p and k, and solve the resulting linear vector differential equation. However, for non-autonomous perturbations, the linearization technique may involve the solution of many intermediate systems or be impossible; for these more complex problems the following method is a tractable solution.

Before continuing, we should note two things. Since the perturbation $h(t)$ is assumed to be eventually constant, after some point the system is autonomous and the stability conditions assure us that the solution converges to the new steady state, around which the system behaves according to the aforementioned linear approximation. This tells us how the system behaves asymptotically. Also note that the problem discussed here is a boundary value problem as opposed to an initial value problem – we are specifying the initial value and asymptotic behavior of one variable, k, as opposed to specifying the initial values of both variables. In order for this problem to be well-defined, we may need some information on the nature of the linearized system; in particular, in order for the solution to exist and be unique, we are assuming that the eigenvalues of the linearization are distinct and of opposite signs. This is the case in the anticipated economic applications, such as the optimal growth example cited above. Let μ be the positive eigenvalue and λ the negative eigenvalue.

The change in W due to an infinitesimal change in γ is

$$\frac{\mathrm{d}W}{\mathrm{d}\gamma}=\int_0^\infty \mathrm{e}^{-\rho t}\big(v_p p_\gamma(t)+v_k k_\gamma(t)\big)\mathrm{d}t$$

$$=\begin{pmatrix} v_p(p_0,k_0) \\ v_k(p_0,k_0)\end{pmatrix}^T \int_0^\infty \mathrm{e}^{-\rho t}\begin{pmatrix} p_\gamma(t) \\ k_\gamma(t)\end{pmatrix}\mathrm{d}t, \tag{3}$$

where $p_\gamma(t)$, $k_\gamma(t)$ are the partial derivatives of $p(t)$, $k(t)$ with respect to the change in γ which occurs at $t=0$. [I am following the standard practice of suppressing the dependence of p and k on the parameter γ in writing $p(t)$ and $k(t)$.] Note that the integral above is actually the Laplace transform of $(p_\gamma, k_\gamma)^T$, which we denote by $(P_\gamma(s), K_\gamma(s))^T$, evaluated at ρ. [In general, if $f(t)$: $R^1 \to R^n$ is a function of exponential order, then the Laplace transform of $f(t)$ is $L\{f\}$: $R^1 \to R^n$, where $\int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t \equiv L\{f\}(s)$ defines $L\{f\}(s)$.]

Differentiation of the system (1) with respect to γ yields

$$\begin{pmatrix} \dot{p}_\gamma \\ \dot{k}_\gamma\end{pmatrix}=J\begin{pmatrix} p_\gamma \\ k_\gamma\end{pmatrix}+\begin{pmatrix} g_3^1(p_0,k_0,0)h(t) \\ g_3^2(p_0,k_0,0)h(t)\end{pmatrix}, \tag{4}$$

where J is the Jacobian of the vector function G: $R^2 \to R^2$ evaluated at (p_0, k_0) where $G(p,k)=(g^1(p,k,0), g^2(p,k,0))^T$. The Laplace transform of (4) yields an algebraic equation in the transforms K_γ, P_γ:

$$s\begin{pmatrix} P_\gamma(s) \\ K_\gamma(s)\end{pmatrix}=J\begin{pmatrix} P_\gamma(s) \\ K_\gamma(s)\end{pmatrix}+\begin{pmatrix} p_\gamma(0)+g_3^1(p_0,k_0,0)H(s) \\ g_3^2(p_0,k_0,0)H(s)\end{pmatrix}, \tag{5}$$

where $p_\gamma(0)$ is the change in p at $t=0$ induced by γ and $H(s)$ is the Laplace transform of $h(t)$. $p_\gamma(0)$ is an unknown at this point since the initial value of only k is fixed, and fixed at k_0. This fact does however yield an initial condition for (5), $k_\gamma(0)=0$. The basic equation in the transforms solves easily since it is linear in the unknown functions $P_\gamma(s)$,

$K_\gamma(s)$. Hence

$$\begin{pmatrix} P_\gamma(s) \\ K_\gamma(s) \end{pmatrix} = (sI - J)^{-1} \begin{pmatrix} p_\gamma(0) + g_3^1 H(s) \\ g_3^2 H(s) \end{pmatrix} \tag{6}$$

gives the solution for $P_\gamma(s)$ and $K_\gamma(s)$ in terms of $p_\gamma(0)$.

To pin down $p_\gamma(0)$, we need another boundary condition for (5). To that end we assume that $k_\gamma(t)$ must be bounded as well as $k(t)$. This is expected to hold in the envisaged applications since capital is necessarily a continuous function of time, h is eventually constant, we expect $k_\gamma(t)$ to converge to the derivative of the steady-state value of k with respect to γ, and usually the steady state is continuously differentiable in the parameters. Under this assumption, $K_\gamma(s)$ must be finite for all $s > 0$, since $K_\gamma(s)$ is $\int_0^\infty \mathrm{e}^{-st} k_\gamma(t) \mathrm{d}t$. In particular, $K_\gamma(\mu)$ must be finite. Straightforward manipulations show that this implies

$$p_\gamma(0) = \frac{-(\mu - J_{11}) g_3^2 H(\mu)}{J_{12}} - g_3^1 H(\mu). \tag{7}$$

Substituting (7) into (6), and substituting the results into (3) yields

$$\frac{\mathrm{d}W}{\mathrm{d}\gamma} = \begin{pmatrix} v_p \\ v_k \end{pmatrix}^T (\rho I - J)^{-1} \begin{pmatrix} g_3^1 (H(\rho) - H(\mu)) - \dfrac{\mu - J_{11}}{J_{12}} g_3^2 H(\mu) \\ g_3^2 H(\rho) \end{pmatrix}. \tag{8}$$

The example discussed in this note is only one application of Laplace transforms to general equilibrium dynamics that is possible. The generalization to higher dimensions is straightforward. For example, the author has used this technique to analyze the problem of capital crowding out effects of government financing decisions in a perfect foresight model of capital accumulation. Other applications will include a large variety of dynamic taxation questions. Given the relative ease of these manipulations, it is clear that many questions in dynamic welfare economics can be analyzed taking into account the dynamic adjustment paths, yielding answers which will be more sound than comparative steady-state analysis.

References

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