Solving Equations

Suppose $f : R^n \to R^n$. Consider the equation

$f(x) = 0$

Typical algorithms

Newtons' method:

- fast, but
- can diverge
- may try to evaluate f(x) where it does not exist

Homotopy method:

- reliable, but
- slow

Alternative: Reformulate as an optimization problem

Min_x	1
s.t.	$f(x) = 0$

Advantages

Variety of solvers

Can use KNITRO, CONOPT, Filter, SNOPT, NPSOL, etc

Can impose constraints

Impose domain conditions

Suppose f(x) is not defined for x<= 0. Then solve

$$\begin{array}{ll} \text{Min}_x & 1 \\ \text{s.t.} & f(x) = 0 \\ & x \geq \epsilon \end{array}$$

for some small $\epsilon > 0$. Can’t use x=>0 because then solver may consider some $x_i = 0$.

Use auxiliary information about solution

Suppose you know that the solution to f(x)=0 satisfies a < x < b (a and b are vectors). We can use that information in

$$\begin{array}{ll} \text{Min}_x & 1 \\ \text{s.t.} & f(x) = 0 \\ & a \leq x \leq b \end{array}$$

More generally, if we know that $a \leq g(x) \leq b$, for some a, b, and g, then solve

$$\begin{array}{ll} \text{Min}_x & 1 \\ \text{s.t.} & f(x) = 0 \\ & a \leq g(x) \leq b \end{array}$$

Stabilize algorithm with L_2 penalty

Suppose that the Jacobian of f is nearly singular near the solution. Then, the following quadratic penalty formulation stabilizes the algorithm:

$\text{Min}_x \quad \|x\|_2^{\,2}$

s.t. $\quad f(x) = 0$

where P is some penalty parameter, preferably small.

Stabilize algorithm with L_1 penalty

An L_1 penalty might also help

Min$_x$ $\|x\|_1$
s.t. f(x) =0

where P is some penalty parameter, preferably small. (More later about how to do this.)

Stabilize via relaxation

We can instead try to find something that nearly solves the equations:

$$
\begin{aligned}
&\text{Min}_{x,\lambda} && \|\lambda\|_1 \\
&\text{s.t.} && -\lambda_i \le f(x) \le \lambda_i \\
& && \lambda_i \ge 0
\end{aligned}
$$

This will give you something instead of just saying “Can’t find a solution”

You could use the max norm

$$
\begin{aligned}
&\text{Min}_{x,\lambda} && \|\lambda\|_\infty \\
&\text{s.t.} && -\lambda_i \le f(x) \le \lambda_i \\
& && \lambda_i \ge 0
\end{aligned}
$$

Find multiple solutions

If there are many solutions, one could resolve the following problem several times for different π parameters:

$$\begin{array}{ll} \text{Min}_x & \pi . x \\ \text{s.t.} & f(x) = 0 \end{array}$$

This will go after solutions on the boundary of the set of solutions.

The following will go after the solution closest to some chosen x_0:

$$\begin{array}{ll} \text{Min}_x & \| x_0 - x \|_2^{\,2} \\ \text{s.t.} & f(x) = 0 \end{array}$$