# Nonlinear Complementarity Problems and Applications

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May 13, 2020

# There is more to life than equations

We often ignore corners and kinks when perhaps we shouldn't

- Inada condition: demand for a good is always positive
- Zero marginal cost at zero output: supply will always be positive
- Armington formulation means all countries buy all goods made by each other country
- Borrowing constraints: sometimes ignored, sometimes too stringent

Life is not so simple

- There are some goods I do not want
- Sometimes marginal costs are everywhere positive, and firm produces zero
- There are some goods made in Italy that we do not import to USA
- I can borrow, but the interest rate schedule has kinks

Nonlinear complementarity problems to the rescue!

# Nonlinear complementarity problems

Problem statement: Given $F : \mathbf{R}^n \to \mathbf{R}^n$, find an $x \in \mathbf{R}^n$ such that

$$0 \leq x \perp F(x) \geq 0, \qquad \text{(NCP } (F))$$

- perp notation "$\perp$" means $x^T F(x) = 0$ .
- $x^T F(x) = \sum_{i=1}^n x_i F_i(x)$ implies NCP(F) is equivalent to

$$x \geq 0, \quad F(x) \geq 0, \quad x_i F_i(x) = 0, \quad i = 1, 2, ..., n.$$

- Complementarity implies either $x_i = 0$ or $F_i(x) = 0$ for each $i = 1, 2, ..., n$.

Generalizes concept of equations

- Equivalent to nonsmooth system of equations

$$\min[x_i, F_i(x)] = 0, \quad i = 1, 2, ..., n$$

- If all $x_i > 0$, then same as system of equations

$$F_i(x) = 0, \quad i = 1, 2, ..., n$$

# Example: Constrained Optimization

General problem:

$$\begin{array}{ll} \min_x & f(x) \\ s.t. & g(x) = 0 \\ & h(x) \leq 0 \end{array} \tag{1}$$

- $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$: objective function with $n$ choices
    - $g : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$: $m$ equality constraints
    - $h : X \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$: $\ell$ inequality constraints
    - $f, g$, and $h$ are $C^2$ on $X$

- Karush-Kuhn-Tucker (KKT): if $x^*$ is local minimum, then exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^\ell$ such that $x^*$ is a *stationary*, or *critical*, point of $\mathcal{L}$, the *Lagrangian*,

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top g(x) + \mu^\top h(x) \tag{2}$$

- First-order conditions, $\mathcal{L}_x(x^*, \lambda^*, \mu^*) = 0$, imply that $(\lambda^*, \mu^*, x^*)$ solves

$$\begin{array}{rl} f_x + \lambda^\top g_x & +\mu^\top h_x = 0 \\ g(x) & = 0 \\ 0 \leq -h(x) & \perp \mu \geq 0 \end{array} \tag{3}$$

# Application: Arrow-Debreu Model

Model Formulation

- Economy with $n$ agents and $m$ commodities
  - $e \in \Re^{n \times m}$ are the endowments
  - $\alpha \in \Re^{n \times m}$ and $\beta \in \Re^{n \times m}$ are the utility parameters
  - $p \in \Re^{m}$ are the commodity prices
- Agent $i$ maximizes utility with budget constraint

$$
\begin{array}{ll}
\max\limits_{x_{i,*} \geq 0} & \displaystyle\sum_{k=1}^{m} \frac{\alpha_{i,k}(1 + x_{i,k})^{1-\beta_{i,k}}}{1 - \beta_{i,k}} \\
\text{subject to} & \displaystyle\sum_{k=1}^{m} p_k \left(x_{i,k} - e_{i,k}\right) \leq 0
\end{array}
$$

- Market $k$ sets price for the commodity

$$0 \leq p_k \quad \perp \quad \sum_{i=1}^{n} (e_{i,k} - x_{i,k}) \geq 0$$

asserting that either price is zero or supply equals demand.

- Equilibrium is consumer focs plus the nonlinear complementarity for the market.

## Application: Nash Equilibrium

- Characterization of two player equilibrium $(x^*, y^*)$

$$x^* \in \left\{ \begin{array}{ll} \arg\min\limits_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x) \leq 0 \end{array} \right.$$

$$y^* \in \left\{ \begin{array}{ll} \arg\min\limits_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{array} \right.$$

- Complementarity Formulation
  - Assume each player's optimization problem is convex. Then

$$\begin{array}{ll} \min\limits_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq x & \perp & \nabla_x f_1(x, y^*) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\ 0 \leq \lambda_1 & \perp & -c_1(x) \geq 0 \end{array}$$

$$\begin{array}{ll} \min\limits_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(y) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq y & \perp & \nabla_y f_2(x^*, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\ 0 \leq \lambda_2 & \perp & -c_2(y) \geq 0 \end{array}$$

  - Nash equilibrium is solution to

$$\begin{array}{lcl} 0 \leq x & \perp & \nabla_x f_1(x, y) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\ 0 \leq y & \perp & \nabla_y f_2(x, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\ 0 \leq \lambda_1 & \perp & -c_1(y) \geq 0 \\ 0 \leq \lambda_2 & \perp & -c_2(y) \geq 0 \end{array}$$

# Application: Oligopoly Model

- Firm $f \in \mathcal{F}$ chooses output $x_f$ to maximize profit
  - $u$ is the market utility function

$$u = \left(1 + \sum_{f \in \mathcal{F}} x_f^\alpha\right)^{\frac{\eta}{\alpha}}$$

  - $\alpha$ and $\eta$ are parameters
  - $c_f$ is the unit cost for each firm
  - $p_f = \frac{\partial u}{\partial x_f}$ is the firm's price
- In particular, each firm $f \in \mathcal{F}$ maximizes its profits subject to nonnegative output

$$x_f^* \in \arg\max_{x_f \geq 0} \left(\frac{\partial u}{\partial x_f} - c_f\right) x_f$$

- Nash equilibrium is system for $f \in \mathcal{F}$

$$0 \leq x_f \quad \perp c_f - \frac{\partial u}{\partial x_f} - x_f \frac{\partial^2 u}{\partial x_f^2} \geq 0$$

# Solution Methods

- Sequential linearization methods (PATH)
  - Solve the linear complementarity problem for $z$

$$0 \leq z \quad \perp \quad F(x_k) + \nabla F(x_k)(z - x_k) \geq 0$$

  - Perform a line search along ray $x_k + (z - x_k)$
    * use a "merit" function to choose good $x_{k+1} = z$
    * merit function rewards points $z$ that look close to a solution
  - Repeat until convergence

- Semismooth Reformulation
  - Define Fischer-Burmeister function

$$\phi(a, b) := a + b - \sqrt{a^2 + b^2}$$

    or some function such that $\phi(a, b) = 0$ iff $a \geq 0$, $b \geq 0$, and $ab = 0$
  - Define the system

$$[\Phi(x)]_i = \phi(x_i, F_i(x))$$

  - $x^*$ solves complementarity problem iff $\Phi(x^*) = 0$
  - Use methods for solving nonsmooth system of equations (not discussed here)

- Some Available Software
  - PATH – sequential linearization method
    * Todd Munson's PhD thesis at U. Wisconsin
    * Not in public domain but available for free through NEOS if you use GAMS or AMPL
    * Widely used in applied general equilibrium work
  - MILES – sequential linearization method
  - SEMI – semismooth linesearch method
  - TAO – Toolkit for Advanced Optimization