Numerical Methods in Economics

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Chapter 13 Notes: Regular Perturbations of Simple Systems

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Taylor Series

- Suppose that $f : R^n \to R^m$.
  - Linear approximation at $x = x_0$ is
$$f(x) \doteq f(x_0) + f_x(x_0)(x - x_0)$$
  - Taylor series approximation of $f(x)$ based at x_0
$$f(x) \doteq f(x_0) + f_x(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^\top f_{xx}(x_0)(x - x_0) + ...$$
    and computed via repeated differentiation.
  - A function f and its Taylor series at x_0 have equal low-order derivatives at x_0.
  - Taylor series is a numerical approximation technique, useful when f and its derivatives are easily computed at some special point x_0.
- Analytic Functions
  - A function $f : R^n \to R$ is *analytic* at x iff f equals a power series on some open neighborhood of x.
  - If analytic, $f(x)$ equals its infinite-term Taylor series expansion at x
  - We approximate $f(x)$ for x near x_0 with first n terms of Taylor series expansion of f at x_0

Implicit Function Theorem

- Suppose $h : R^n \to R^m$ is defined in $H(x, h(x)) = 0$, $H : R^n \times R^m \to R^m$, and $h(x_0) = y_0$.
  - Implicit differentiation shows
  $$H_x(x, h(x)) + H_y(x, h(x))h_x(x) = 0$$
  - At $x = x_0$, this implies
  $$h_x(x_0) = -H_y(x_0, y_0)^{-1} H_x(x_0, y_0)$$
  if $H_y(x_0, y_0)$ is nonsingular. More simply, we express this as
  $$h_x^0 = -\left(H_y^0\right)^{-1} H_x^0$$
  - Linear approximation for $h(x)$ is
  $$h^L(x) \doteq h(x_0) + h_x(x_0)(x - x_0)$$
- To check on quality, we compute
$$E = \hat{H}(x, h^L(x))$$
where $\hat{H}$ is a unit free equivalent of H. If $E < \varepsilon$, then we have an ε-solution.

- If $h^L(y)$ is not satisfactory, compute higher-order terms by repeated differentiation.
  - $D_{xx}H(x, h(x)) = 0$ implies

$$H_{xx} + 2H_{xy}h_x + H_{yy}h_xh_x + H_yh_{xx} = 0$$

  - At $x = x_0$, this implies

$$h^0_{xx} = -\left(H^0_y\right)^{-1}\left(H^0_{xx} + 2H^0_{xy}h^0_x + H^0_{yy}h^0_xh^0_x\right)$$

  - Construct the quadratic approximation

$$h^Q(x) \doteq h(x_0) + h^0_x(x - x_0) + \frac{1}{2}(x - x_0)^\top h^0_{xx}(x - x_0)$$

    and check its quality by computing $E = H(x, h^Q(x))$.

Regular Perturbation: The Basic Idea

- Suppose x is an endogenous variable, ε a parameter
  - Want to find $x(\varepsilon)$ such that $f(x(\varepsilon), \varepsilon) = 0$
  - Suppose $x(0)$ known.
- Use Implicit Function Theorem
  - Apply implicit differentiation:

$$f_x(x(\varepsilon), \varepsilon)x'(\varepsilon) + f_\varepsilon(x(\varepsilon), \varepsilon) = 0 \tag{13.1.5}$$

  - At $\varepsilon = 0$, $x(0)$ is known and (13.1.5) is linear in $x'(0)$ with solution

$$x'(0) = -f_x(x(0), 0)^{-1} f_\varepsilon(x(0), 0)$$

  - Well-defined only if $f_x \neq 0$, a condition which can be checked at $x = x(0)$.
  - The linear approximation of $x(\varepsilon)$ for ε near zero is

$$x(\varepsilon) \doteq x^L(\varepsilon) \equiv x(0) - f_x(x(0), 0)^{-1} f_\varepsilon(x(0), 0)\varepsilon \tag{13.1.6}$$

- Can continue for higher-order derivatives of $x(\varepsilon)$.
  - Differentiate (13.1.5) w.r.t. ε

$$f_x x'' + f_{xx}(x')^2 + 2f_{x\varepsilon}x' + f_{\varepsilon\varepsilon} = 0 \tag{13.1.7}$$

  - At $\varepsilon = 0$, (13.1.7) implies that

$$\begin{aligned} x''(0) = &-f_x(x(0),0)^{-1}\left(f_{xx}(x(0),0)\ (x'(0))^2\right.\\ &\left.+2f_{x\varepsilon}(x(0),0)\ x'(0) + f_{\varepsilon\varepsilon}(x(0),0)\right) \end{aligned}$$

  - Quadratic approximation is

$$x(\varepsilon) \doteq x^Q(\varepsilon) \equiv x(0) + \varepsilon x'(0) + \frac{1}{2}\varepsilon^2 x''(0) \tag{13.1.8}$$

- General Perturbation Strategy
  - Find special (likely degenerate, uninteresting) case where one knows solution
    * General relativity theory: begin with case of a universe with zero mass: ε is mass of universe
    * Quantum mechanics: begin with case where electrons do not repel each other: ε is force of repulsion
    * Business cycle analysis: begin with case where there are no shocks: ε is measure of exogenous shocks
  - Use local approximation theory to compute nearby cases
    * Standard implicit function may be applicable
    * Sometimes standard implicit function theorem will not apply; use appropriate bifurcation or singularity method.
  - Check to see if solution is good for problem of interest
    * Use unit-free formulation of problem
    * Go to higher-order terms until error is reduced to acceptable level
    * *Always* check solution for range of validity

Single-Sector, Deterministic Growth - canonical problem

Consider dynamic programming problem

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c) dt$$
$$\dot{k} = f(k) - c$$

Ad-Hoc Method: Convert to an LQ problem:

- Popular idea in macroeconomics
  - Compute steady state of deterministic problem
  - Convert nonlinear deterministic problem to LQ problem around steady state
  - Apply LQ methods to the approximate problem
  - Add noise to law of motion to get stochastic approximation

- McGrattan, JBES (1990)
  - Replace $u(c)$ and $f(k)$ with approximations around c^* and k^*
  - Solve linear-quadratic problem

$$\begin{array}{l} \max_c \int_0^\infty e^{-\rho t} \left(u(c^*) + u'(c^*)(c - c^*) + \frac{1}{2} u''(c^*)(c - c^*)^2 \right) \, dt \\ \text{s.t.} \quad \dot{k} = f(k^*) + f'(k^*)(k^* - k) - c \end{array}$$

  - Resulting approximate policy function is

$$C^{McG}(k) = f(k^*) + \rho(k - k^*) \neq C(k^*) + C'(k^*)(k - k^*)$$

    * Local approximate law of motion is $\dot{k} = 0$
    * Add noise to get

$$dk = 0 \cdot dt + dz$$

    * Approximation is *random walk* when theory says solution is stationary

- Recent papers by Benigno and Woodford repeat this point.

- Kydland-Prescott
  - Restate problem so that $\dot{k}$ is linear function of state and controls
  - Replace $u(c)$ with quadratic approximation
  - Note 1: such transformation may not be easy
  - Note 2: special case of Magill (JET 1977).
- Lesson
  - Kydland-Prescott, McGrattan provide no mathematical basis for method
  - Formal calculations based on appropriate IFT should be used.
  - Beware of *ad hoc* methods based on an intuitive story!

Perturbation Method for Dynamic Systems

- Formalize problem as a system of functional equations
  - Bellman equation:

$$\rho V(k) = \max_c \; u(c) + V'(k)(f(k) - c) \tag{1}$$

  - $C(k)$: policy function defined by

$$\begin{aligned} 0 &= u'(C(k)) - V'(k) \\ \rho V(k) &= u(C(k)) + V'(k)(f(k) - C(k)) \end{aligned} \tag{2}$$

  - Apply envelope theorem to (1) to get

$$\rho V'(k) = V''(k)(f(k) - C(k)) + V'(k)f'(k) \tag{1_k}$$

  - Steady-state equations

$$\begin{aligned} c^* &= f(k^*) \\ 0 &= u'(c^*) - V'(k^*) \\ \rho V(k^*) &= u(c^*) + V'(k^*)(f(k^*) - c^*) \\ \rho V'(k) &= V''(k)(f(k^*) - c^*) + V'(k)f'(k) \end{aligned}$$

– Steady State: We know k^*, $V(k^*)$, $C(k^*)$, $f'(k^*)$, $V'(k^*)$:

$$\begin{aligned} \rho &= f'(k^*) \\ C(k^*) &= f(k^*) \\ V(k^*) &= \rho^{-1} u(c^*) \\ V'(k^*) &= u'(c^*) \end{aligned}$$

– Want Taylor expansion:

$$\begin{aligned} C(k) &\doteq C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + ... \\ V(k) &\doteq V(k^*) + V'(k^*)(k - k^*) + V''(k^*)(k - k^*)^2/2 + ... \end{aligned}$$

- Linear approximation around a steady state

  – Differentiate (1_k, 2) w.r.t. k:

$$\rho V'' = V'''(f - C) + V''(f' - C') + V''f' + V'f'' \tag{1_{kk}}$$

$$0 = u''C' - V'' \tag{2_k}$$

  – At the steady state

$$0 = -V''(k^*)C'(k^*) + V''(k^*)f'(k^*) + V'(k^*)f''(k^*) \tag{1_k^*}$$

  – Substituting (2_k) into (1_k^*) yields

$$0 = -u''(C')^2 + u''C'f' + V'f''$$

  – Two solutions

$$C'(k^*) = \frac{\rho}{2}\left(1 \pm \sqrt{1 + \frac{4u'\left(C\left(k^*\right)\right)f''\left(k^*\right)}{u''\left(C\left(k^*\right)\right)f'\left(k^*\right)f'\left(k^*\right)}}\right)$$

  – However, we know $C'(k^*) > 0$; hence, take positive solution

- Higher-Order Expansions
  - Conventional perception in macroeconomics: "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..." – Marcet (1994, p. 111)
  - Mathematics literature: No problem (See, e.g., Bensoussan, Fleming, Souganides, etc.)
- Compute $C''(k^*)$ and $V'''(k^*)$.
  - Differentiate $(1_{kk}, 2_k)$:

$$\begin{aligned} \rho V''' &= V''''(f-C) + 2V'''(f'-C') + V''(f''-C'') \\ &\quad + V'''f' + 2V''f'' + V'f''' \end{aligned} \tag{1_{kkk}}$$

$$0 = u'''(C')^2 + u''C'' - V''' \tag{2_{kk}}$$

  - At k^*, (1_{kkk}) reduces to

$$0 = 2V'''(f'-C') + 3V''f'' - V''C'' + V'f''' \tag{1^*_{kkk}}$$

  - Equations $(1^*_{kkk}, 2^*_{kk})$ are *LINEAR* in unknowns $C''(k^*)$ and $V'''(k^*)$:

$$\begin{pmatrix} u'' & -1 \\ V'' & -2(f'-C') \end{pmatrix} \begin{pmatrix} C'' \\ V''' \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Unique solution since determinant $-2u''(f'-C') + V'' < 0$.

- Compute $C^{(n)}(k^*)$ and $V^{(n+1)}(k^*)$.
  - Linear system for order n is, for some A_1 and A_2,

$$\begin{pmatrix} u'' & -1 \\ V'' & -n(f' - C') \end{pmatrix} \begin{pmatrix} C^{(n)} \\ V^{(n+1)} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Higher-order terms are produced by solving linear systems
  - The linear system is always determinate since $-nu''(f' - C') + V'' < 0$
- Conclusion:
  - Computing first-order terms involves solving quadratic equations
  - Computing higher-order terms involves solving linear equations
  - Computing higher-order terms is easier than computing the linear term.

Accuracy Measure

Consider the one-period relative Euler equation error:

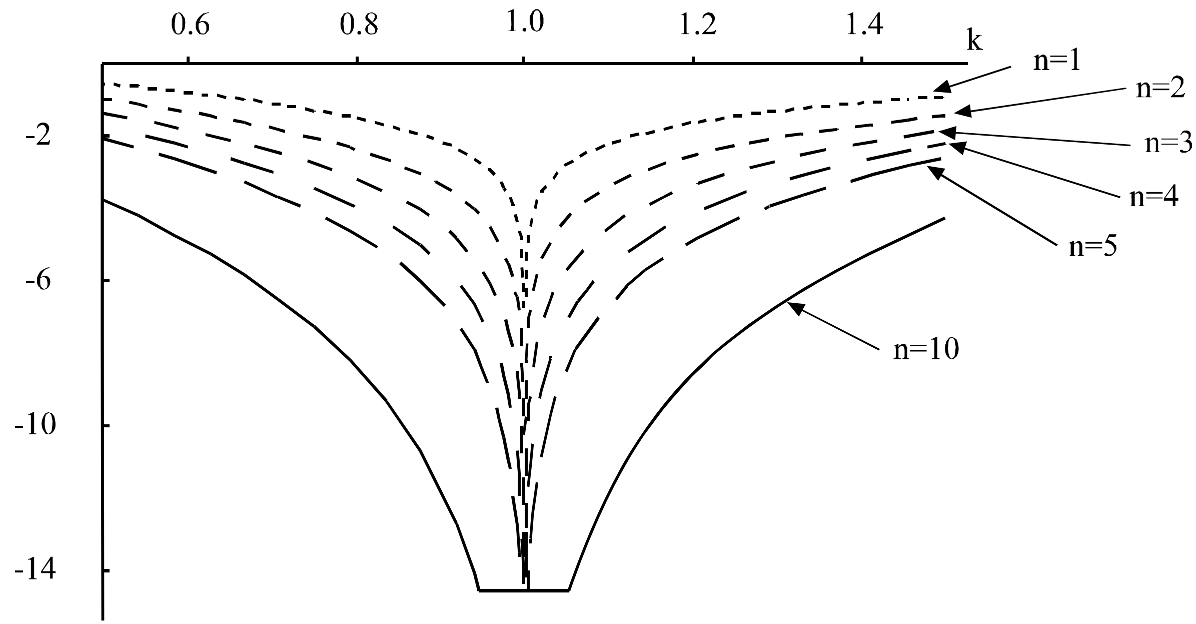
$$E(k) = 1 - \frac{V'(k)}{u'(C(k))}$$

- Equilibrium requires it to be zero.
- $E(k)$ is measure of optimization error
  - 1 is unacceptably large
  - Values such as .00001 is a limit for people.
  - $E(k)$ is unit-free.
- Define the L^p, $1 \le p < \infty$, *bounded rationality accuracy* to be

$$\log_{10} \parallel E(k) \parallel_p$$

- The L^∞ error is the maximum value of $E(k)$.

Global Quality of Asymptotic Approximations



Graph of $\log_{10} |E(k|$

- Linear approximation is very poor even for k close to steady state
- Order 2 is better but still not acceptable for even $k = .9, 1.1$
- Order 10 is excellent for $k \in [.6, 1.4]$