Dynamic Programming with Piecewise Linear Interpolation

Kenneth Judd

Hoover Institution

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Piecewise Linear Interpolation

If Lagrange data $\{(x_i, v_i) : i = 1, \ldots, m\}$ is given, then its piecewise linear interpolation is

$$\hat{V}(x; \mathbf{b}) = b_{j,0} + b_{j,1}x, \quad \text{if } \; x \in [x_j, x_{j+1}],$$

where

$$\begin{aligned} b_{j,1} &= \frac{v_{j+1} - v_j}{x_{j+1} - x_j}, \\ b_{j,0} &= v_i - b_{j,1}x_i, \end{aligned}$$

for $j = 1, \ldots, m - 1$.

In the maximization step of numerical DP algorithms, one could directly solve the maximization problem

$$v_i = \max_a \; u(x_i, a) + \beta \; \hat{V}\left(y; \mathbf{b}^+\right)$$

where

$$y = g(x_i, a)$$

Problem: $\hat{V}(x; \mathbf{b}^{t+1})$ is not differentiable, making it difficult to solve the optimization problem for a.

Min-Function Approach for Concave V

The differentiability problem is solved as follows:

$$
\begin{aligned}
v_i \quad = \quad & \max_{a,w,y} \; u(x_i, a) + \beta \; w \\
& \text{s.t.} \qquad y = g(x_i, a) \\
& \qquad\qquad w \leq b^+_{j,0} + b^+_{j,1} y, \qquad 1 \leq j < m
\end{aligned}
$$

Optimization solvers can still solve the new model quickly

- The objective function is smooth
    - inequality constraints are linear and sparse
    - we can apply fast Newton-type optimization algorithms to solve this problem if g is also smooth.
    - although this new model adds $(m-1)$ linear inequality constraints, few of them will be active at any iteration
- This approach does not need to find the interval containing y, while the spline approximation of value function must

Convex-Set Approach for Concave V

Both previous methods need to calculate coefficients; this is very complicated for multi-dimensional piecewise linear interpolation. Define a convex set Y

- $x_j, j = 1, ..., m$, vertices
- $\mu_j, j = 1, ..., m$, weights
- Y is the set of points y such that for some $\mu_j \geqslant 0$

$$\begin{aligned} y &= \sum_{j=1}^{m} \mu_j x_j \\ \sum_{j=1}^{m} \mu_j &= 1 \end{aligned}$$

Define Concave Approximation

Define concave function

- Suppose we have data (x_j, y_j) for a concave function on convex set.
- Concave interpolant at x is y

$$
\begin{aligned}
y = \quad & \max_{\mu_j \geq 0, w} \quad w \\
& s.t. \qquad x = \sum_{j=1}^{m} \mu_j x_j \\
& \qquad\qquad w \leq \sum_{j=1}^{m} \mu_j y_j \\
& \qquad\qquad \sum_{j=1}^{m} \mu_j = 1
\end{aligned}
$$

VFI for Concave V

- Old value function is defined by the set of $\left(v_j^+, x_j^+\right)$
- New value function is defined by v_i at nodes x_i:

$$
\begin{aligned}
v_i = \quad \max_{\mu_j \geq 0, a, w, y} \quad & u(x_i, a) + \beta \ w, \\
s.t. \quad & y = g(x_i, a), \\
& y = \sum_{j=1}^{m} \mu_j x_j^+ \\
& w \leq \sum_{j=1}^{m} \mu_j v_j^+ \\
& \sum_{j=1}^{m} \mu_j = 1
\end{aligned}
$$