NLCEQ: Nonlinear Certainty Equivalent Approximation Method

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Deterministic DP model

- Infinite or finite horizon deterministic dynamic programming problem:

$$V_0(x_0) = \max_{a_t \in \mathcal{D}(x_t)} \sum_{t=0}^{T-1} \beta^t u_t(x_t, a_t) + \beta^T V_T(x_T), \quad \text{s.t.} \quad x_{t+1} = g_t(x_t, a_t), \tag{1}$$

- Standard method to compute value/policy function: value function iteration or time iteration
  - how to choose appropriate approximation domains
  - the stability problem
- New method: NLCEQ (Nonlinear Certainty Equivalent Approximation method)
  - Cai, Y., K.L. Judd, and J. Steinbuks. A nonlinear certainty equivalent approximation method for dynamic stochastic problems. *Quantitative Economics*.

Basic NLCEQ

Basic NLCEQ

- Choose approximation method and a corresponding set of approximation nodes $\{x_0^i\}$
  - Smolyak approximation
  - Complete polynomials with nodes from monomial rules
  - Sparse grids, fixed or adaptive
  - Epsilon-distinguishable sets with L1 approximation methods
- Data collection step
  - For each node x_0^i, solve the large-scale optimal control problem to find $v_i = V_0\left(x_0^i\right)$ and action a_0^i
  - Economies of scale
    - Determine sparsity and use it repeatedly
    - Compute efficient differentiation code, and use it repeatedly
    - Collect nodes into clusters, create sequences that warm and hot starts
- Fitting Step.
  - Fit $\{(x_0^i, v_i) : 1 \leq i \leq N\}$ to get value function approximation $\hat{V}(x_0; \mathbf{b}_v)$
  - Fit $\{(x_0^i, a_0^i) : 1 \leq i \leq N\}$ for action a_0 to get policy function approximation $\hat{P}(x_0; \mathbf{b}_a)$
- Natural Parallelism everywhere

Infinite-Horizon Deterministic DP

- Infinite-horizon deterministic DP problem:

$$V_0(x_0) = \max_{a_t \in \mathcal{D}(x_t)} \sum_{t=0}^{\infty} \beta^t u_t(x_t, a_t),$$
$$\text{s.t.} \quad x_{t+1} = g_t(x_t, a_t),$$

- Basic NLCEQ method for infinite-horizon problems
  - Transform it to a finite-horizon problem
    - choose a terminal value function which assumes that $x_{t+1} = x_t$ with $t \geq T$
    - Apply the basic NLCEQ method for the finite-horizon problem

A simple example

- Solve optimal growth problem:

$$
\begin{aligned}
V_0(k_0) &= \max_{c} \sum_{t=0}^{\infty} \beta^t u(c_t), \\
\text{s.t.} \quad k_{t+1} &= (1-\delta)k_t + Ak_t^{\alpha} - c_t,
\end{aligned}
$$

- Use the basic NLCEQ method: see growth_NJLCEQ.gms

NLCEQ Method

- Stochastic DP problem:

$$V_0(x_0) = \max_{a_t \in \mathcal{D}(x_t)} \mathbb{E}\left\{\sum_{t=0}^{T-1} \beta^t u_t(x_t, a_t) + \beta^T V_T(x_T)\right\},$$
$$\text{s.t.} \quad x_{t+1} = g_t(x_t, a_t, \varepsilon_t),$$

- NLCEQ method:
  - Transformation Step: transform stochastic problems to deterministic (and finite-horizon) problems:

$$\widetilde{V}_0(x_0) = \max_{a_t \in \mathcal{D}(x_t)} \sum_{t=0}^{T-1} \beta^t u_t(x_t, a_t) + \beta^T \widetilde{V}_T(x_T),$$
$$\text{s.t.} \quad x_{t+1} = g_t(x_t, a_t)$$

  - Apply the basic NLCEQ method for the finite-horizon deterministic problem

Comparison with Other Methods

- Perturbation methods
  - only valid for non-evolving infinite-horizon models
  - local approximation
  - does not work for problems with occasionally binding inequality constraints
- Projection methods
  - only valid for non-evolving infinite-horizon models
  - Challenging for high-dimensional problems
  - Challenging for problems with occasionally binding inequality constraints
- Value function iteration and/or time iteration
  - Challenging for high-dimensional problems
  - Challenging for problems with occasionally binding inequality constraints
  - iteration may be unstable (approximation domains, shape-preservation)
  - accumulated errors from numerical approximation, integration and optimization could be large

Multi-country optimal growth problem

- N-country optimal growth problem

$$\max_{c,\ell,I} \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t)\right)$$

subject to

$$K_{t+1,j} = (1-\delta)K_{t,j} + I_{t,j}$$

$$\ln(\theta_{t+1,j}) = \rho \ln(\theta_{t,j}) + \sigma(\epsilon_{t+1,j} + \varepsilon_{t+1})$$

$$\sum_{j=1}^{N} (c_{t,j} + I_{t,j} - \delta\, K_{t,j}) = \sum_{j=1}^{N} (\theta_{t,j}\, f(K_{t,j}, \ell_{t,j}) - \Gamma_{t,j})$$

- with

$$f(K_{t,j}, \ell_{t,j}) = A(K_{t,j})^{\alpha}(\ell_{t,j})^{1-\alpha}$$

$$\Gamma_{t,j} \equiv \frac{\phi}{2} K_{t,j} \left(\frac{I_{t,j}}{K_{t,j}} - \delta \right)^2$$

$$U(c_t, \ell_t) = \sum_{j=1}^{N} \tau_j u_j(c_{t,j}, \ell_{t,j})$$

$$u_j(c_{t,j}, \ell_{t,j}) = \frac{(c_{t,j})^{1-\frac{1}{\gamma_j}}}{1-\frac{1}{\gamma_j}} - B_j \frac{(\ell_{t,j})^{1+\frac{1}{\eta_j}}}{1+\frac{1}{\eta_j}}$$

Errors

- Euler Errors (unit-free):

$$E_1(K,\theta) = \max_{1\le j\le N} \left| \mathbb{E}\left\{ F_j\left(K,\theta,\theta^+\right)\right\} - 1 \right|$$

with

$$F_j\left(K,\theta,\theta^+\right) \equiv \frac{\beta \frac{\partial u_j}{\partial c}\left(c_j^+,\ell_j^+\right)}{\frac{\partial u_j}{\partial c}\left(c_j,\ell_j\right)\omega_j} \left[\pi_j^+ + \theta_j^+ \frac{\partial f}{\partial K}\left(K_j^+,\ell_j^+\right)\right]$$

- Other Errors:

$$E_2(K,\theta) = \max_{2\le j\le N} \left| \frac{\frac{\partial u_j}{\partial c}\left(c_j,\ell_j\right)\tau_j}{\frac{\partial u_1}{\partial c}\left(c_1,\ell_1\right)\tau_1} - 1 \right|$$

$$E_3(K,\theta) = \max_{1\le j\le N} \left| \frac{\frac{\partial u_j}{\partial c}\left(c_j,\ell_j\right)\theta_j \frac{\partial f}{\partial \ell}\left(K_j,\ell_j\right)}{\frac{\partial u_j}{\partial \ell}\left(c_j,\ell_j\right)} + 1 \right|$$

$$E_4(K,\theta) = \left| \frac{\sum_{j=1}^N \left(c_j + I_j - \delta K_j + \Gamma_j\right)}{\sum_{j=1}^N \left(\theta_j f(K_j,\ell_j)\right)} - 1 \right|$$

Results for Low-Dimensional Problems

- Results of NLCEQ for 2-country problems
- Use degree-n complete Chebyshev polynomials for approximation
- See growth4D_NLCEQ.gms
- Use true solutions from value function iteration to estimate the errors

β	γ	η	Global Error $\mathcal{E}$			
			degree-D Chebyshev		level-l Smolyak	
			$D=2$	$D=4$	$l=1$	$l=2$
0.99	0.25	0.1	2.4(−2)	1.7(−3)	5.3(−2)	6.7(−3)
		0.5	2.1(−2)	2.0(−3)	6.5(−2)	1.0(−2)
	0.5	0.1	2.0(−2)	1.3(−3)	6.1(−2)	5.3(−3)
		0.5	2.1(−2)	1.1(−3)	6.5(−2)	6.1(−3)
0.95	0.25	0.1	2.8(−2)	2.6(−3)	5.1(−2)	9.3(−3)
		0.5	1.8(−2)	3.7(−3)	7.0(−2)	1.3(−2)
	0.5	0.1	2.0(−2)	1.5(−3)	5.7(−2)	5.6(−3)
		0.5	1.5(−2)	1.7(−3)	6.2(−2)	8.7(−3)

Results for High-Dimensional Problems

- Results of NLCEQ for high-dimensional N-country problems ($2N$ dimensional)
- Use level-n Smolyak grid and Chebyshev-Smolyak polynomial approximation (degree 2^n)
- Parallelization in optimization step
- Errors and running times:

N	Level l	Num of Points	Num of Cores	T	Max Euler Error	Global Error	Time (minutes)
50	1	201	201	20	2.3(−3)	1.8(−2)	0.8
	2	20,201	2,048	20	1.5(−3)	2.7(−3)	8.3
			20,201	50	3.5(−4)	2.6(−3)	8.6
100	1	401	401	20	1.9(−3)	1.8(−2)	2.2
200	1	801	801	20	1.6(−3)	1.8(−2)	8.0

Note: $a(-n)$ represents $a \times 10^{-n}$.

RBC with investment constraint

- The RBC model

$$\max_{c} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t U(c_t)\right\}$$

subject to

$$c_t + I_t = A_t k_t^{\alpha},$$
$$k_{t+1} = (1-\delta)k_t + I_t,$$
$$I_t \geq \phi I_{ss},$$

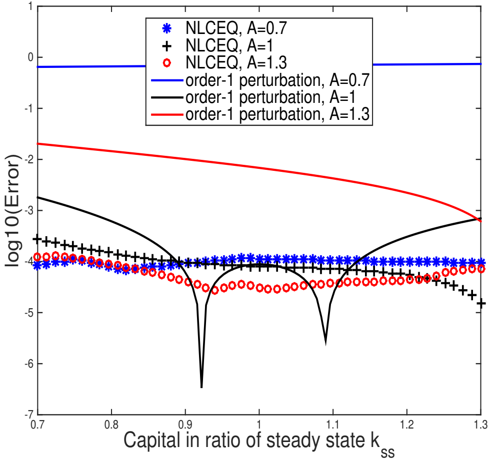
- Technology A_t:

$$\ln(A_{t+1}) = \rho \ln(A_t) + \sigma \epsilon_{t+1},$$

- See growth_bind_NLCEQ.gms

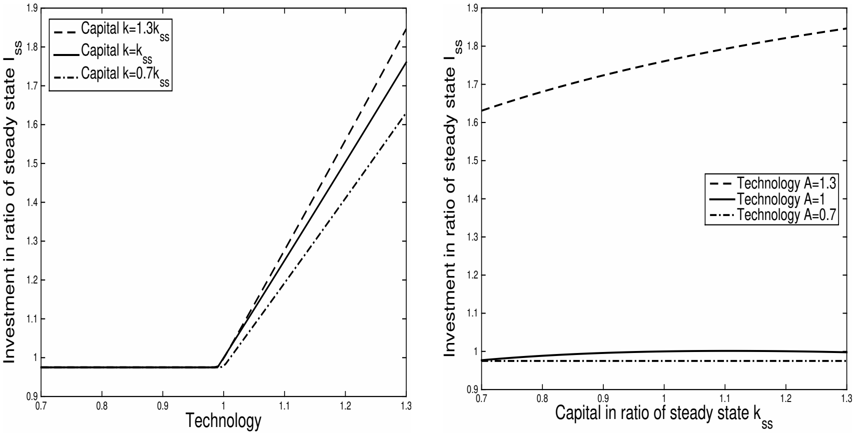
Results

Figure: Errors of the solutions from NLCEQ or log-linearization for the RBC model with a constraint on investment



Results

Figure: Investment policy function for the RBC model with a constraint on investment



NLCEQ with Discrete Stochastic States

- Discrete stochastic state θ_t
  - an exogenous Markov chain with k possible values, $\{\vartheta_1, ..., \vartheta_k\}$
  - transition matrix P
- If the initial-time state $\theta_0 = \vartheta_i$, then unconditional probability vector at time t is: $p_{t,i} = P^t \mathbf{e}_i$
- In the transformation step of NLCEQ, replace θ_t by $\sum_{j=1}^{k} p_{t,i,j} \vartheta_j$ for an initial $\theta_0 = \vartheta_i$

NLCEQ Method for Competitive Equilibrium

- Equations for equilibrium (Euler equations, transition laws of states; market clearing conditions; other first-order conditions)

$$\mathbf{F}(\mathbf{x}_t, \mathbf{a}_t, \mathbf{x}_{t+1}, \mathbf{a}_{t+1}) = 0, , \quad t = 0, 1, 2, ...$$

- State and control variables converge to steady values

$$\mathbf{x}_\infty = \mathbf{x}_{ss}, \; \mathbf{a}_\infty = \mathbf{a}_{ss}.$$

- Optimization method for competitive equilibrium:

$$\begin{aligned} \min_{\mathbf{a}_t \in \mathcal{D}(\mathbf{x_t})} \quad & \left\| \mathbf{x}_T^{\text{Endo}} - \mathbf{x}_{ss}^{\text{Endo}} \right\| + \left\| \mathbf{a}_T - \mathbf{a}_{ss} \right\| \\ \text{s.t.} \quad & \mathbf{F}(\mathbf{x}_t, \mathbf{a}_t, \mathbf{x}_{t+1}, \mathbf{a}_{t+1}) = 0, \quad t = 0, 1, ..., T-1, \\ & \mathbf{x}_0 = \mathbf{x}_0^j, \end{aligned}$$

New Keynesian Model with Zero Lower Bound

The New Keynesian model: a representative household, a government, a final-good firm, and intermediate firms

- The representative household

$$\max_{c_t,\ell_t,b_t} \mathbb{E}\left\{\sum_{t=0}^{\infty}\left(\prod_{i=0}^{t}\beta_i\right)U(c_t,\ell_t)\right\}$$

subject to

$$p_t c_t + \frac{b_t}{1+r_t} = w_t\ell_t + b_{t-1} + T_t + \Pi_t$$

  - b_t: bond face value
  - p_t: price of consumption c_t from the production of the final-good firm
  - T_t: lump sum transfer from the government
  - Π_t: profit from all firms

Specifications

- Stochastic discount factor β_t:

$$\ln(\beta_{t+1}) = (1-\rho)\ln(\beta^*) + \rho\ln(\beta_t) + \sigma\epsilon_{t+1}$$

- Utility

$$U(c,\ell) = \ln(c) - \frac{\ell^{1+\eta}}{1+\eta}$$

- Final-good firm

$$\max_{y_{i,t}} p_t y_t - \int_0^1 p_{i,t} y_{i,t} di$$

with

$$y_t = \left(\int_0^1 y_{i,t}^{\frac{\alpha-1}{\alpha}} di\right)^{\frac{\alpha}{\alpha-1}}$$

Specifications

- Intermediate firms: production $y_{i,t+j} = \ell_{i,t+j}$
  - Calvo-type prices: a fraction $1-\theta$ of the firms have optimal prices and the remaining fraction θ of the firms keep the same price as in the previous period.
  - A re-optimizing intermediate firm $i \in [0,1]$ chooses its price $p_{i,t}$ to maximize the current value of profit over the time when the optimal $p_{i,t}$ remains effective:

$$\max_{p_{i,t}} \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \beta_{t+k} \right) \lambda_{t+j} \theta^j \left(p_{i,t} y_{i,t+j} - w_{t+j} \ell_{i,t+j} \right) \right\}$$

    - λ_t: the Lagrange multiplier of the budget constraint

Prices, labor, and government spending

- price

$$
\begin{aligned}
p_t &= \left(\int_0^1 p_{i,t}^{1-\alpha} di\right)^{\frac{1}{1-\alpha}} \\
&= \left((1-\theta)(q_t p_t)^{1-\alpha} + \theta \int_0^1 p_{i,t-1}^{1-\alpha} di\right)^{\frac{1}{1-\alpha}} \\
&= \left((1-\theta)(q_t p_t)^{1-\alpha} + \theta p_{t-1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

- labor

$$\ell_t = \int_0^1 \ell_{i,t} di,$$

- Government: spend $g_t = s_g y_t$; issue bonds and pay dividends; lump sum transfer

Equilibrium

- Euler equation

$$1 = \mathbb{E}_t \left\{ \beta_{t+1} \frac{1 + r_t}{\pi_{t+1}} \frac{c_t}{c_{t+1}} \right\}$$

- Zero Lower Bound (ZLB) for nominal interest rates r_t

$$r_t = \max(z_t, 0)$$

with

$$z_t = (1 + r^*) \left(\frac{\pi_t}{\pi^*} \right)^{\phi_\pi} \left(\frac{y_t}{y^*} \right)^{\phi_y} - 1 \tag{2}$$

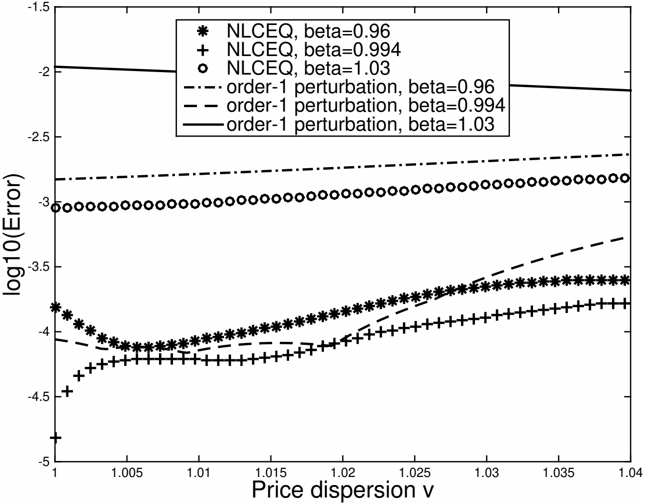
- Other equilibrium equations

$$
\begin{aligned}
1 &= \frac{1}{\chi_{t,1}}\left(y_t \ell_t^\eta + \theta \mathbb{E}_t\left\{\beta_{t+1}\pi_{t+1}^\alpha \chi_{t+1,1}\right\}\right) \\
1 &= \frac{1}{\chi_{t,2}}\left(\frac{y_t}{c_t} + \theta \mathbb{E}_t\left\{\beta_{t+1}\pi_{t+1}^{\alpha-1} \chi_{t+1,2}\right\}\right) \\
q_t &= \frac{\alpha \chi_{t,1}}{(\alpha-1)\chi_{t,2}} = \left(\frac{1-\theta\pi_t^{\alpha-1}}{1-\theta}\right)^{\frac{1}{1-\alpha}} \\
v_{t+1} &= \frac{\ell_t}{y_t} = (1-\theta) q_t^{-\alpha} + \theta \pi_t^\alpha v_t
\end{aligned}
$$

- State variables: β_t and v_t
- Find policy functions $(c_t, \chi_{t,1}, \chi_{t,2}, \pi_t, q_t, v_t, \ell_t, y_t, r_t, z_t)$: see newKeynesian_NLCEQ.gms

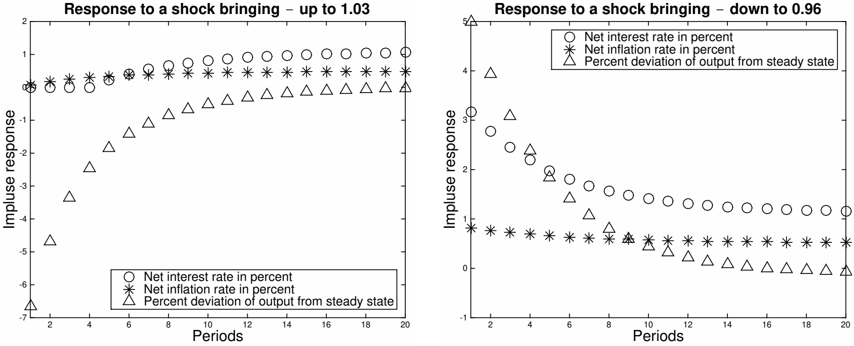
Results: Errors

Figure: Errors of the NLCEQ solution for the New Keynesian DSGE model with ZLB



Results: Impulse response to discount factor shock

Figure: Impulse responses to a shock of discount factor



Summary

- NLCEQ is a simple method
- NLCEQ is easy for coding
- NLCEQ is stable and robust even for problems with kinks
- NLCEQ is more accurate then log-linearization
- NLCEQ can deal with high-dimensional dynamic stochastic problems
  - efficient and natural parallelism
  - hundreds of dimensions (so far, thousands in the future)
  - compatible with all approximation methods
- NLCEQ can solve both social planners' problems and competitive equilibrium