Initiative for Computational Economics

Numerical Methods for Solving Auctions II

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Acknowledgements

This presentation builds on publsihed and on going work with **Harry Paarsch** and draws from research we have completed with **René Kirkegaard** and we are continuing with **Ken Judd**.

Remember our asymmetric first-price auction

Bidder n maximizes

$$\mathbb{E}(U_n|s_n) = (V_n - s_n) \prod_{m \neq n} F_m[\varphi_m(s_n)]$$

which, after playing with the FOCs, led to

$$\frac{1}{\varphi_n(s) - s} = \sum_{m \neq n} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi'_m(s),$$

or, equivalently (after some algebra)

$$\varphi'_n(s) = \frac{F_n[\varphi_n(s)]}{f_n[\varphi_n(s)]} \left\{ \left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{\varphi_m(s) - s} \right] - \frac{1}{\varphi_n(s) - s} \right\}.$$

In addition, we had two sets of boundary conditions

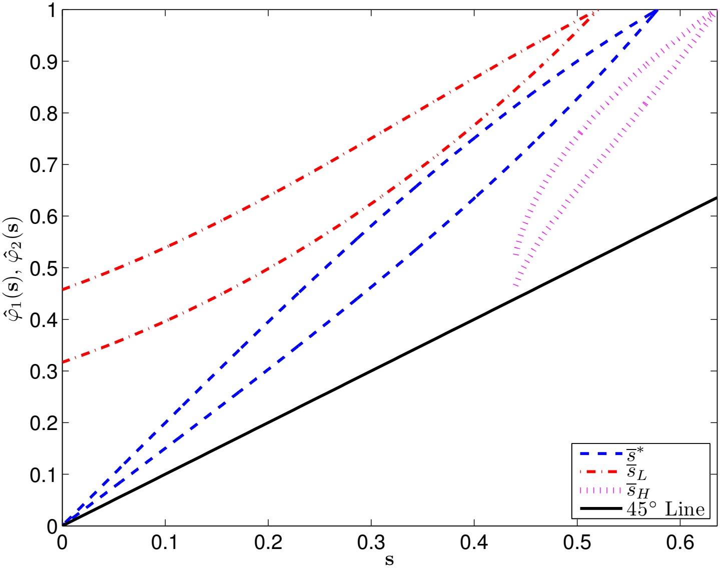
Left-Boundary Condition on Inverse-Bid Functions:
$\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$.

Right-Boundary Condition on Inverse-Bid Functions:
$\varphi_n(\bar{s}) = \bar{v}$ for all $n = 1, 2, \ldots, N$.

Why this is interesting

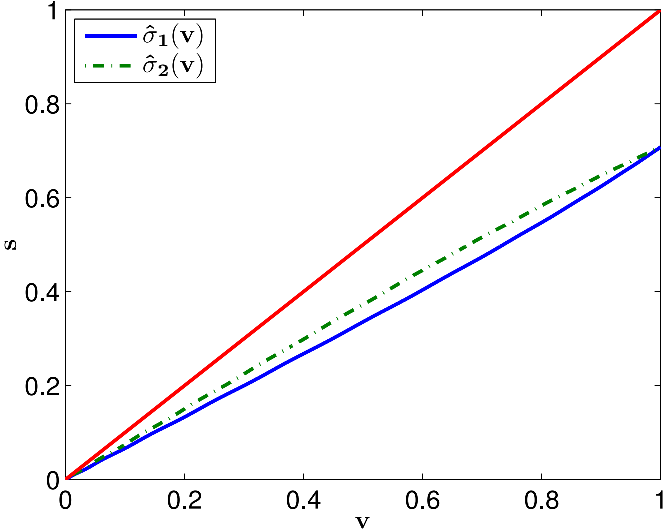
- a system of nonlinear differential equations obtain;
- no longer an initial value problem (as in symmetric case), but now a two-point boundary value problem;
- $\bar{s}$ is unkown *a priori* and determines domain of solutions;
- boundary value problem is *overidentified*;
- we know some characteristics that the solutions must respect (rationality and monotonicity);
- Lipschitz condition does not hold at $\underline{v}$!

We considered shooting methods



Coalition of 3 vs. Coalition of 2

We considered the MMRS (1994) example of coalitions



Raining on the shooting parade

I was very careful about the example I chose—with uniform $F_0(\cdot)$ the maximum valuations from each coalition imply asymmetric power distributions (one of the only cases with closed-form solutions)

Nearly all researchers who used shooting methods noted that the algorithm was very sensitive and instable

Recently, Fibich and Gavish (forthcoming, GEB) have proven analytically that the inherent instability is not a technical issue, but rather an analytic property of backward integration in this setting

Furthermore, shooting methods are very costly (time wise), require more advanced programming techniques, and typically involve a lot of “fiddling”

Projection methods: an alternative

A projection method is a general strategy to approximating a true, but unknown, function by a finite number of approximating functions.

Idea: the true solution is approximated by a finite combination of simple, known functions.

In our setting this means researchers would first choose a basis to approximate the solutions to each inverse-bid function.

Basis-related choices

The full basis for the space of candidate solutions should be rich (flexible) enough to approximate any function relevant to the problem (which will be represented and approximated as a linear combination of basis functions).

The researcher would then fix the flexibility of the approximation by deciding how many basis elements to include: in short, the researcher must select the degree of the approximation.

Fixing the degree transforms an infinite-dimensional problem into a finite-dimensional one, where only the coefficients of the basis functions need then to be found; if the basis is a good choice, larger degrees should yield better approximations.

Other choices: residual function and norm

The researcher must also decide on an appropriate residual function to evaluate how closely the approximation represents the true solution.

The goal of projection methods is to find a set of coefficients which make some norm of the residual function as close to zero as possible.

After solving, the researcher can verify the quality of the candidate solution and choose either to increase the degree of the approximation or, if that fails, to begin with a different basis.

Spectral methods

Spectral methods use bases where each element is nonzero almost everywhere, as with trigonometric bases and orthogonal polynomials.

In the case of an asymmetric first-price auction problem, consider approximating each inverse-bid function by a truncated series expansion

$$\hat{\varphi}_n(s) = \sum_{k=0}^{K} \alpha_{n,k} \mathbb{P}_k(s), \ s \in [\bar{v}, \bar{s}], \ n = 1, 2, \ldots, N$$

where $\mathbb{P}_k(s)$ is some basis functions (which are typically chosen to be polynomials) and the $\alpha_{n,k}$s are referred to as the *spectral coefficients*.

Least-squares problem

For economists, perhaps the most intuitive spectral method is that of least-squares. It is compelling to us: we have reduced the problem of solving a functional equation to solving a nonlinear minimization problem, a problem with which we have considerable experience.

Consider selecting a large number T of grid points from the interval $[\underline{v}, \bar{s}]$.

Remember our asymmetric IPVP problem

Define

$$G_n(s; \bar{s}, \boldsymbol{\alpha}) \equiv 1 - [\hat{\varphi}_n(s) - s] \sum_{m \neq n} \frac{f_m[\hat{\varphi}_m(s)]}{F_m[\hat{\varphi}_m(s)]} \hat{\varphi}'_m(s)$$

where $\boldsymbol{\alpha}$ denotes a vector that collects the $N \times (K+1)$ coefficients of the polynomials.

Note that once the basis function has been decided, $\hat{\varphi}'_m(\cdot)$ are implied by the choice (literally just take the derivative of each basis element).

Ideally...

In an exact solution, $G_n(s; \bar{s}, \alpha)$ should equal zero for all bidders and at any bid $s \in [\underline{v}, \bar{s}]$ and our boundary constraints

$$\varphi_n(\underline{v}) = \underline{v}$$

and

$$\varphi_n(\bar{s}) = \bar{v}$$

will be satisfied for all $n = 1, \ldots, N$

Bajari (2001, ET)

The problem is to estimate $\bar{s}$ as well as the $\alpha_{n,k}$s for all $n = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, K$

The system can be evaluated at each grid point and the parameters can be chosen to minimize the following criterion function:

$$H(\bar{s}, \boldsymbol{\alpha}) \equiv \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \boldsymbol{\alpha})]^2 + \sum_{n=1}^{N} [\hat{\varphi}_n(\underline{v}) - \underline{v}]^2 + \sum_{n=1}^{N} [\hat{\varphi}_n(\bar{s}) - \bar{v}]^2$$

Bajari (2001, ET)

In practice, Bajari chose

- $K = 5$
- uniformly-spaced grid
- ordinary polynomials
- used a nonlinear least-squares algorithm to select $\bar{s}$ and α by minimizing a modified version of the previous objective

$$\tilde{H}(\bar{s}, \alpha) \equiv \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \alpha)]^2 + T \sum_{n=1}^{N} [\hat{\varphi}_n(\underline{v}) - \underline{v}]^2 + T \sum_{n=1}^{N} [\hat{\varphi}_n(\bar{s}) - \bar{v}]^2$$

which adds weight to the boundary conditions.

Hubbard and Paarsch (2009, IJIO)

Modified the approach we just discussed by

- using Chebyshev points;
- using Chebyshev basis;
- imposing boundary conditions as constraints;
- imposing shape constraints.

Thus, the problem becomes...

Hubbard and Paarsch (2009, IJIO)

$$\min_{\{\bar{s},\alpha\}} \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \alpha)]^2 ,$$

subject to each of these (for all $n = 1, 2, \ldots, N$)

1. $\underline{v} \leqslant \bar{s} \leqslant \bar{v}$
2. $\varphi_n(\underline{v}) = \underline{v}$
3. $\varphi_n(\bar{s}) = \bar{v}$
4. $\varphi_n(s_{j-1}) \leqslant \varphi_n(s_j)$ for uniform grid $j = 2, \ldots, J$

Hubbard, Kirkegaard, and Paarsch (2011)

HKP take this improved approach and push it farther by leveraging other information at the critical boundary points

Fibich, Gavious, and Sela (2002) proved the following properties concerning the high and low types, the first of which follows directly from the first-order conditions:

1. $(\bar{v} - \bar{s}) \sum_{m \neq n} f_m(\bar{v}) \varphi'_m(\bar{s}) = 1$ for all $n = 1, 2, \ldots, N$.
2. If $f_n(\underline{v}) \in \mathbb{R}_{++}$ and $\varphi_n(s)$ is differentiable at $s = \underline{v}$ for all $n = 1, 2, \ldots, N$, then $\varphi'_n(\underline{v}) = [N/(N-1)]$.

Take HP (2009, IJIO) approach and impose these as well to get...

Hubbard, Kirkegaard, and Paarsch (2011)

$$\min_{\{\bar{s},\alpha\}} \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \alpha)]^2,$$

subject to each of these (for all $n = 1, 2, \ldots, N$)

1. $\underline{v} \leqslant \bar{s} \leqslant \bar{v}$
2. $\varphi_n(\underline{v}) = \underline{v}$
3. $\varphi_n(\bar{s}) = \bar{v}$
4. $\sum_{m \neq n} (\bar{v} - \bar{s}) f_m(\bar{v}) \varphi'_m(\bar{s}) = 1$
5. $\varphi'_n(\underline{v}) = [N/(N-1)]$
6. $\varphi_n(s_{j-1}) \leqslant \varphi_n(s_j)$ for uniform grid $j = 2, \ldots, J$

HKP (2011): sidenote on collocation

Under this approach there are $4N$ conditions (constraints) in total and TN points that enter the objective function.

By comparison, there are $N(K+1)+1$ parameters to be estimated—the parameters in $\boldsymbol{\alpha}$ plus $\bar{s}$.

For the number of conditions (boundary and first-order together) to equal the number of unknowns

$$N(T+4) = N(K+1) + 1$$

or

$$(T+4) = (K+1) + \frac{1}{N}.$$

Since at auctions, N weakly exceeds two, and T and K are integers, this equality cannot hold for any (T, K) choice.

HKP (2011): initial guess

However, when comparing the $N(K+1)+1$ parameters with the $4N$ conditions, note that, if K equals three and all the conditions are satisfied, then only one degree of freedom remains.

One criticism of the polynomial approach (and projection-based methods in general) is that it works well, if the practitioner has a good initial guess.

When K equals three, the researcher obtains an initial guess that already satisfies some theoretical properties at essentially no cost because there is only one free parameter, $\bar{s}$, to minimize the nonlinear least-squares objective.

Hubbard and Paarsch (2009)

HP (2009, IJIO) makes for a nice example of projection methods as, even if shooting methods could work reliably, they would not work on this problem

This example involve bid preference programs which is now a well studied topic, especially among structural economists

- Marion (2007, JPubE)
- Krasnokutskaya and Seim (forthcoming, AER)

In these programs, bids of preferred firms are typically scaled by some discount factor which is one plus a preference rate denoted ρ. Suppose there are N_1 preferred bidders and N_2 typical (nonpreferred) bidders, where $(N_1 + N_2)$ equals N.

HP (2009) example

Under this program, probability of winning for a class 1 bidder,

$$\Pr(\mathsf{win}|b_1) = (1 - F_1[\varphi_1(b_1)])^{N_1-1}\left(1 - F_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]\right)^{N_2},$$

while for a class 2 bidder probability of winning is

$$\Pr(\mathsf{win}|b_2) = [1 - F_1\left(\varphi_1\left[(1+\rho)b_2\right]\right)]^{N_1}(1 - F_2[\varphi_2(b_2)])^{N_2-1}$$

HP (2009) example

Each firm then chooses its bid b to maximize its

$$\mathbb{E}(U_i|b_i) = (b_i - C_i)\Pr(\text{win}|b_i)$$

which yields two FOCs

$$\frac{\partial \mathbb{E}(U_1|b_1)}{\partial b_1} = 1 - [b_1 - \varphi_1(b_1)]\left[\frac{(N_1-1)f_1[\varphi_1(b_1)]\,\varphi_1'(b_1)}{1-F_1[\varphi_1(b_1)]} + \frac{N_2 f_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]\frac{1}{1+\rho}\varphi_2'\left(\frac{b_1}{1+\rho}\right)}{1-F_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]}\right] = 0$$

and

$$\frac{\partial \mathbb{E}(U_2|b_2)}{\partial b_2} = 1 - [b_2 - \varphi_2(b_2)]\left[\frac{N_1 f_1\left(\varphi_1[(1+\rho)b_2]\right)(1+\rho)\varphi_1'[(1+\rho)b_2]}{1-F_1\left(\varphi_1[(1+\rho)b_2]\right)} + \frac{(N_2-1)f_2[\varphi_2(b_2)]\,\varphi_2'(b_2)}{1-F_2[\varphi_2(b_2)]}\right] = 0$$

HP (2009) example

Right-Boundary Conditions (on Inverse-Bid Functions):

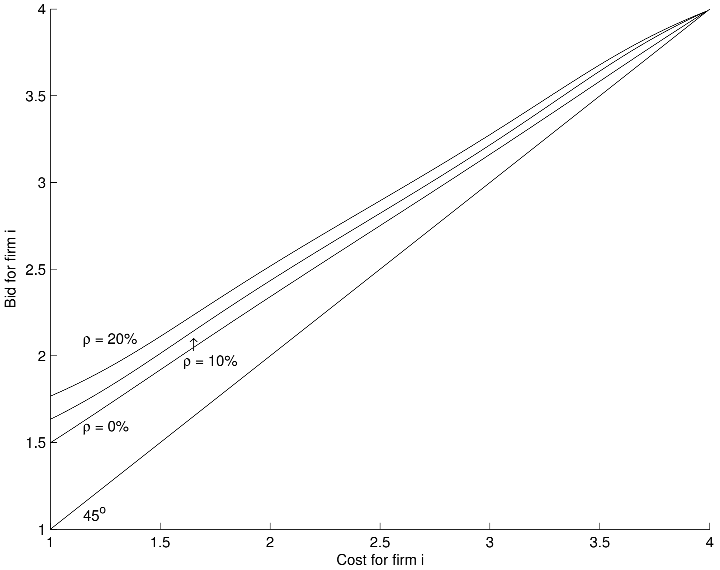
- for all nonpreferred bidders of class 2, $\varphi_2(\bar{c}) = \bar{c}$;
- for all preferred bidders of class 1, $\varphi_1(\bar{b}) = \bar{c}$, where $\bar{b} = \bar{c}$ if $N_1 > 1$, but when $N_1 = 1$, then $\bar{b}$ is determined by

$$\bar{b} = \underset{b}{\operatorname{argmax}} \left[(b - \bar{c}) \left(1 - F_2 \left[\varphi_2 \left(\frac{b}{1+\rho} \right) \right] \right)^{N_2} \right].$$

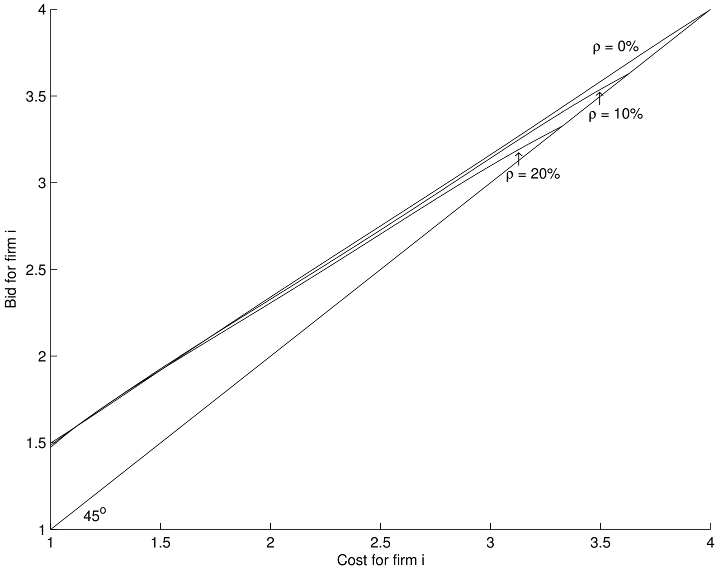
Left-Boundary Conditions (on Inverse-Bid Functions): there exists an unknown bid $\underline{b}$ such that

- for all nonpreferred bidders of class 2, $\varphi_2(\underline{b}) = \underline{c}$;
- for all preferred bidders of class 1, $\varphi_1\left[(1+\rho)\underline{b}\right] = \underline{c}$.

Endogenous changes in preferred behavior



Endogenous changes in nonpreferred behavior



Are we comfortable with less control?

While shooting methods have so many issues, a distinct advantage is that the error tolerance (at $\underline{v}$—the point we were shooting to) can be controlled explicitly. The researcher had to specify this beforehand.

Of course, whether this could be achieved and whether the shooting approach would be successful is another issue.

Nonetheless, that control is a nice feature. How do you know whether a polynomial approximation is sufficient? Should you select a higher degree polynomial?

Regardless of the approach, theory tells us more

I mentioned earlier that we had more information concerning the derivatives of the bid functions at the boundaries.

We can also use theory to inform us about some qualitative properties of the bid functions. In the projection approach this can also help inform us about whether we're capturing essential features of the solution.

This comes from Hubbard, Kirkegaard, and Paarsch (2011)—HKP

Comparing bidder behavior

Define

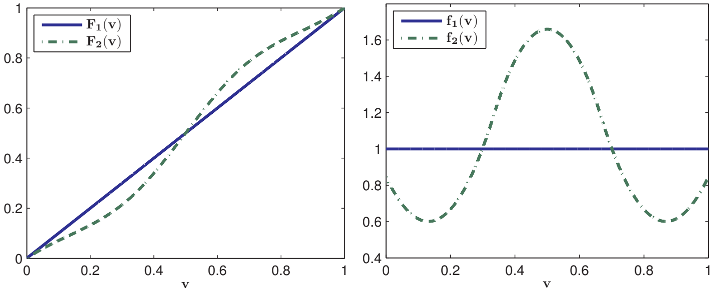
$$D_{n,m}(s) = \hat{\varphi}_n(s) - \hat{\varphi}_m(s)$$

Then if $K = 3$ this becomes

$$D_{n,m}(s) = \hat{\varphi}_n(s) - \hat{\varphi}_m(s) = \left[\frac{f_n(\bar{v}) - f_m(\bar{v})}{f_n(\bar{v})f_m(\bar{v})}\right]\left[\frac{s^2(\bar{s} - s)}{\bar{s}^2(\bar{v} - \bar{s})}\right]$$

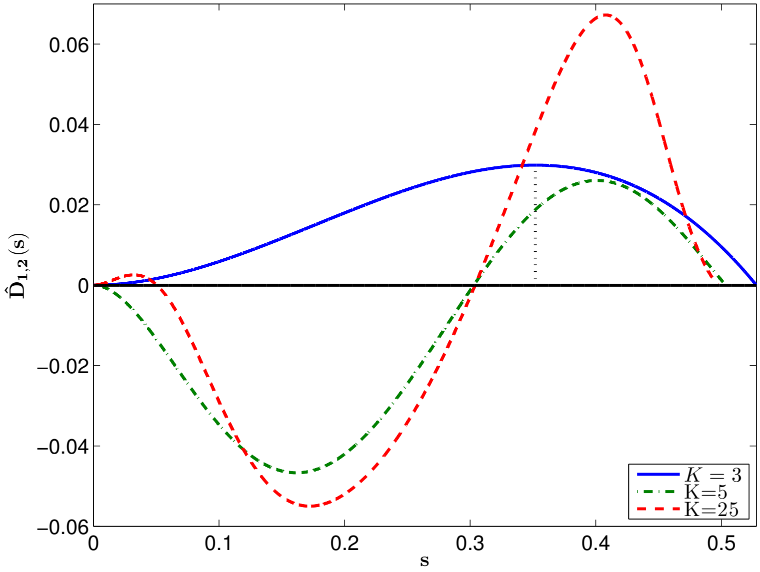
Proposition:
Assume (i) $f_n(\underline{v}) \in \mathbb{R}_{++}$ and (ii) $\varphi_n(s)$ is a polynomial of degree K, $K \geqslant 3$, with real coefficients that satisfy Conditions 1a, 1b, 2a, and 2b, for all $n = 1, 2, \ldots, N$. If $f_n(\bar{v}) \neq f_m(\bar{v})$, then $\varphi_n(s)$ and $\varphi_m(s)$ cross at most $(K - 3)$ times on $(\underline{v}, \bar{s})$, $m, n = 1, 2, \ldots, N$.

Example 2



(a) CDFs (b) PDFs

Example 2: $\hat{D}_{1,2}(s)$ for various K



Exogenous and endogenous ratios

Let

$$P_{n,m}(v) = \frac{F_m(v)}{F_n(v)}, \; v \in (\underline{v}, \bar{v}]$$

measure bidder n's strength relative to bidder m at a given v

Let

$$R_{n,m}(v) = \frac{U_n(v)}{U_m(v)}, \; v \in [\underline{v}, \bar{v}]$$

denote bidder n's equilibrium pay-off relative to bidder j's equilibrium pay-off at a given v; i.e.,

$$U_n(v) = (v_n - s) \prod_{m \neq n} F_m\left[\varphi_m(s)\right]$$

Properties of the ratios

- Ratios vs. Bids

$$R_{n,m}(v) \gtreqless P_{n,m}(v) \iff \sigma_n(v) \gtreqless \sigma_m(v), \text{ for } v \in (\underline{v}, \bar{v}].$$

- Endogenous vs. Exogenous Ratios

$$R'_{n,m}(v) \gtreqless 0 \iff R_{n,m}(v) \gtreqless P_{n,m}(v), \text{ for } v \in (\underline{v}, \bar{v}].$$

- Right-Boundary Condition

$$R_{n,m}(\bar{v}) = P_{n,m}(\bar{v}) = 1,$$

- Left-Boundary Condition

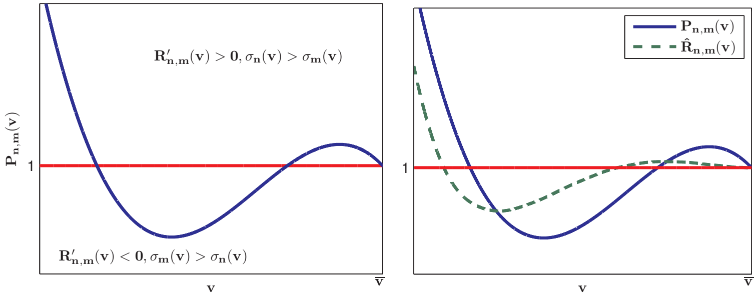
$$\lim_{v \to \underline{v}} R_{n,m}(v) = \frac{f_m(\underline{v})}{f_n(\underline{v})} = \lim_{v \to \underline{v}} P_{n,m}(v).$$

Implications for approximations

1. Slope:
   If $P_{n,m}(v) = \hat{R}_{n,m}(v)$ (i.e., $\hat{\sigma}_n(v) = \hat{\sigma}_m(v)$), $\hat{R}_{n,m}$ should be *flat*. This is true anytime bids coincide (for any $v > \underline{v}$, including $\bar{v}$).

2. Location:
   $P_{n,m}(v) = R_{n,m}(v)$ *at most* once between any two peaks of $P_{n,m}$. With diminishing wave property they *must* cross between any two peaks (not counting v equals $\bar{v}$).

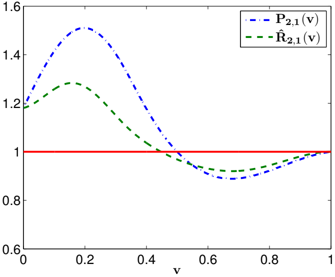
Implications for approximations



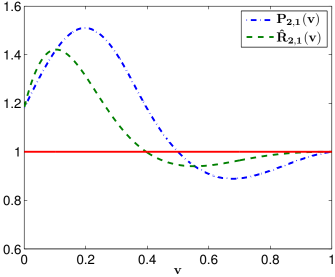
(c) Exogenous Ratio

(d) Both Ratios

Example 2 continued

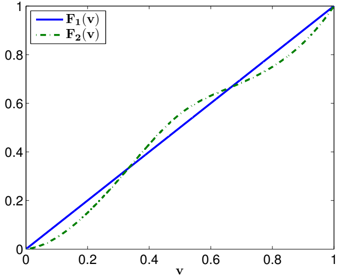


(e) $K = 5$

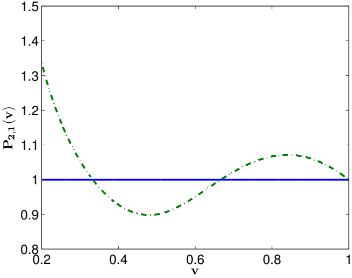


(f) $K = 25$

Example 3: diminishing wave property satisfied

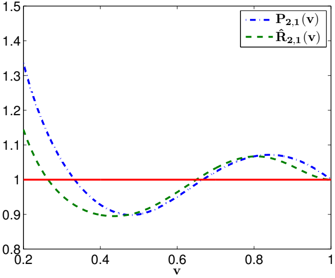


(g) CDFs

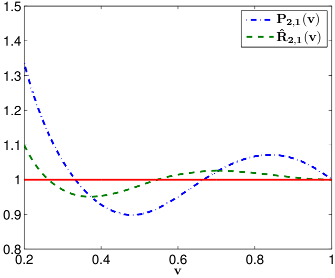


(h) Exogenous Ratio

Example 3 approximations



(i) $K = 5$



(j) $K = 25$

Are good solutions really that important?

Hopefully you see that these graphical “tests” can be used to evaluate some qualitative features of the approximation

But, really, how important are these small differences in the approximated (inverse) bid functions?

Well, let’s simulate some auctions and find out...

Results of a simulation study: one trillion auctions

	Order of $\hat{\varphi}_n$	Expected Revenue	Proportion of Inefficiencies	Prop. Wins Player 1	Prop. Wins Player 2	$E(U_1)$	$E(U_2)$
Ex. 2	$K = 3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
	$K = 4$	0.3473	0.0223	0.5209	0.4791	0.1645	0.1434
	$K = 5$	0.3458	0.0227	0.5089	0.4911	0.1634	0.1459
	$K = 10$	0.3431	0.0337	0.5028	0.4972	0.1635	0.1481
	$K = 15$	0.3432	0.0338	0.5027	0.4973	0.1634	0.1481
	$K = 20$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 25$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 30$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	SPA	0.3445	0.0000	0.5000	0.5000	0.1555	0.1555
Ex. 3	$K = 3$	0.3475	0.0721	0.5491	0.4509	0.1568	0.1568
	$K = 4$	0.3452	0.0670	0.5287	0.4713	0.1584	0.1613
	$K = 5$	0.3364	0.0557	0.5041	0.4959	0.1605	0.1678
	$K = 10$	0.3334	0.0704	0.4964	0.5036	0.1632	0.1699
	$K = 15$	0.3330	0.0719	0.4947	0.5053	0.1634	0.1702
	$K = 20$	0.3328	0.0725	0.4944	0.5056	0.1635	0.1703
	$K = 25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	$K = 30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

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	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

Poor approximation $\Rightarrow$ wrong revenue ranking

Results of a simulation study: one trillion auctions

	Order of $\hat{\varphi}_n$	Expected Revenue	Proportion of Inefficiencies	Prop. Wins Player 1	Prop. Wins Player 2	$E(U_1)$	$E(U_2)$
	$K=3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
	$K=4$	0.3473	0.0223	0.5209	0.4791	0.1645	0.1434
	$K=5$	0.3458	0.0227	0.5089	0.4911	0.1634	0.1459
	$K=10$	0.3431	0.0337	0.5028	0.4972	0.1635	0.1481
Ex. 2	$K=15$	0.3432	0.0338	0.5027	0.4973	0.1634	0.1481
	$K=20$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K=25$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K=30$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	SPA	0.3445	0.0000	0.5000	0.5000	0.1555	0.1555
	$K=3$	0.3475	0.0721	0.5491	0.4509	0.1568	0.1568
	$K=4$	0.3452	0.0670	0.5287	0.4713	0.1584	0.1613
	$K=5$	0.3364	0.0557	0.5041	0.4959	0.1605	0.1678
	$K=10$	0.3334	0.0704	0.4964	0.5036	0.1632	0.1699
Ex. 3	$K=15$	0.3330	0.0719	0.4947	0.5053	0.1634	0.1702
	$K=20$	0.3328	0.0725	0.4944	0.5056	0.1635	0.1703
	$K=25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	$K=30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
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$E(U_n^{SP}) - E(U_m^{SP}) = \mu_n - \mu_m$, but not in FPAs here.

Results of a simulation study: one trillion auctions

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	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

SPA $\Rightarrow$ bigger pie;
In these: for seller, SPA $\succ$ FPA, but for bidders, FPA $\succ$ SPA
(collectively, but not individually)

Results of a simulation study: one trillion auctions

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	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

SPA $\rightarrow$ FPA $\Rightarrow$ small change in probability of winning

Player 1 wins $\approx$1% more of the time than player 2 in FPA

Results of a simulation study: one trillion auctions

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Ex. 2	$K = 3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
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	$K = 30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

SPA $\rightarrow$ FPA $\Rightarrow$ bigger change payoff

$E(U_1) > E(U_2)$ by $\approx$10% in FPA

It's a small world

I think auctions are an area where there's a lot of back-and-forth between numerical work, applied work, and theoretical work

Theory informs numerical work

Numerical work motivates new theory

Empirical work relies on theory and numerical work

Theory and numerical work are guided by new empirical observations

APVP example

Consider a first-price auction with no reserve price involving two bidders.

The bidders draw valuations from a joint distribution $F_{\mathbf{V}}(v_1, v_2)$ which has compact support $[\underline{v}, \bar{v}]^2$. We employ Sklar's Theorem which states that a copula $C(F_1(v_1), F_2(v_2))$ always exists, and is a unique function linking $F_{\mathbf{V}}(v_1, v_2)$ with $F_1(v_1)$ and $F_2(v_2)$.

We assume a Frank copula with dependence parameter θ set such that Kendall's τ equals 0.5 which implies non-negligible statistical dependence between V_1 and V_2.

Under these conditions, we have a model in the APVP.

Example 4

Assume further that valuations for bidder 1 have a uniform marginal distribution

$$F_1(v) \sim F(v; 1, 1)$$

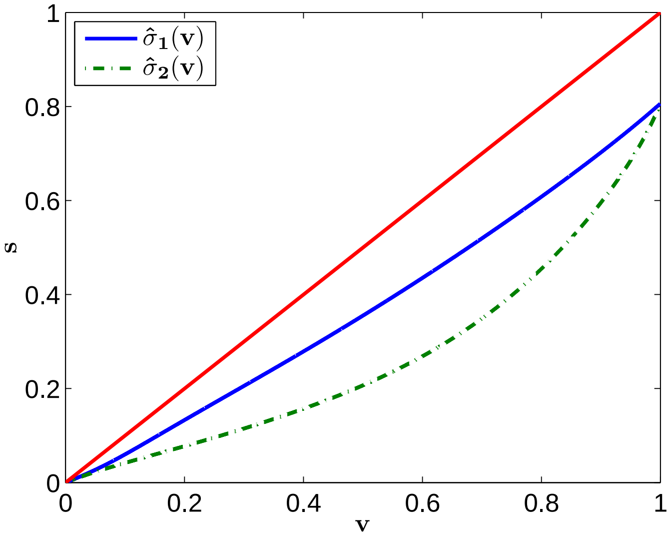
while valuations from bidder 2 are distributed via the following beta-uniform mixture marginal distribution:

$$F_2(v) \sim \gamma F(v; 1, 1) + (1 - \gamma) F(v; 2, 2)$$

with the weight γ equal to 0.1.

Note that $F_2(v)$ first-order stochastically dominates $F_1(v) \Rightarrow$ in IPVP, weakness leads aggression holds

Example 4: asymmetry and affiliation



Things to take from our discussions

When bidders are *ex ante* heterogenous at first-price auctions, closed-form solutions often do not exist and there is a role for numerical methods

We discussed two of them extensively: backwards shooting methods and perturbation methods

Theory can be used to provide some validation of the solution—regardless of the approach taken

There is room for improvements: I will circulate a current (preliminary) version of a paper I'm working on with Harry Paarsch in which we try to bring everything in the literature together and discuss some future directions