Asymptotic Methods for Asset Market Equilibrium Analysis

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Abstract. General equilibrium analysis is difficult when asset markets are incomplete. We make the simplifying assumption that uncertainty is small and use bifurcation methods to compute Taylor series approximations for asset demand and asset market equilibrium. A computer must be used to derive these approximations since they involve large amounts of algebraic manipulation. To illustrate this method, we apply it to analyzing the allocative, price, and welfare effects of introducing a new derivative security.

1. Introduction

Precise analysis of equilibrium in asset markets is difficult since few cases can be solved exactly for equilibrium prices and volume. Many analyses assume that markets are complete, implying that equilibrium is efficient and equivalent to some social planner's problem. That approach is limited since it ignores transaction costs, taxes, and incompleteness in asset markets. This paper develops bifurcation methods to approximate asset market equilibrium without assuming complete asset markets. We begin from a trivial deterministic case where all assets have the same safe return and use local approximation methods to compute asset market equilibrium when assets have small risk. We compute Taylor series expressing equilibrium asset prices and holdings as a function of preference parameters such as absolute risk aversion, and

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asset return statistics such as mean, variance, and skewness. The formulas completely characterize equilibrium for small risks.

Implementing this approach is straightforward, but involves an enormous amount of algebraic manipulation far beyond the capacity of human hands. Fortunately, desktop computers using symbolic software can execute the necessary algebraic manipulation and compute the series expansions in reasonable time. We use Mathematica, but the computation could be executed by other symbolic languages such as Macsyma and Maple. The asymptotic expansions tell us about the qualitative properties of equilibrium and can be used to compute a numerical approximation to equilibrium of particular problems with a specified nonzero risk. Therefore, the bifurcation approach is computational in two ways: the formulas are qualitative asymptotic approximations derived by computer algebra, and can be used to produce numerical approximations to specific problems. This paper focuses on the qualitative asymptotic results and leaves the numerical applications for future study.

The result is essentially a mean-variance-skewness-etc. theory of asset demand and equilibrium pricing, similar to Samuelson's [22] analysis of asset demand. This approach is also more intuitive than the standard contingent state approach to equilibrium. The incomplete markets paradigm focuses on the difference between the number of contingent states and the number of assets. For example, welfare results in Hart [11], Cass and Citanna [3], and Elul [7] depend on how many assets are missing and the number of agents. It is difficult to interpret such indices of incompleteness since we can count neither the number of contingent states nor the number of different kinds of agents in a real economy. Furthermore, one expects that the impact of asset incompleteness on economic performance is related more to the statistical character of riskiness and the diversity of investor objectives than to the number of states and the number of agents. For example, the number of different agents is a poor measure of agent diversity since an economy with 100 types of investors with different risk aversions close to the mean risk aversion is less diverse than an economy with 10 types of investors with substantially different risk aversions. Similarly, the number of contingent states is at best a poor indicator of the magnitude and character of riskiness. This paper's analysis produces asymptotic formulas depending solely on the moments of asset returns and the differences in utility indices, showing that they, not the number of states, govern the asymptotic properties of equilibrium. Since moments are more easily observed in real markets than the number of contingent states the result is a more practical and intuitive approach to equilibrium analysis of asset markets.

Our approach is intuitive and similar in spirit to standard linearization and comparative static methods from mathematical economics. If fact, the analysis resembles Jones [12] classic analysis of international trade. Linearization methods based on the Implicit Function Theorem (IFT) are important computational tools that allow us to approximate nonlinear relationships with tractable, asymptotically valid approximations. We begin with the no-risk case where we know the equilibrium. We then use that information to compute equilibria for nearby cases of risky economies. However, the IFT does not apply here because the critical Jacobian is singular. In particular, when risk disappears all assets must become perfect substitutes and the portfolios of individuals are indeterminate when risk is zero. We cannot use the IFT if we do not know the equilibrium portfolio in the case of zero risk. Instead, we must apply tools from bifurcation theory to solve our problem. These tools are natural since they are essentially generalizations of L'Hospital's rule. Furthermore, because of the singularity at zero risk, we will need to compute higher-order approximations, not just the familiar first-order terms from linear approximation methods.

The purpose of this paper is to present the key mathematical ideas and illustrate them with basic economic applications. We first apply bifurcation methods to derive approximations of asset demand, refining the similar Samuelson [22] method. We then use these approximations of asset demand to compute approximations of asset market equilibrium. We compute asymptotically valid expressions for equilibrium with different asset combinations, and use them to show how changes in asset availability affects equilibrium.

The bifurcation approach is particularly interesting since it handles the complete and incomplete asset market cases in the same way. This contrasts sharply with the conventional approach where the incomplete asset market case is far more complex than the complete market case (see Magill and Quinzii [21] for a more complete discussion). We can do this because we focus on small risks. Since our analysis makes no assumptions about the span of assets, it is also a method for computing equilibrium in some economies with incomplete asset markets. This is generally a difficult problem because the excess demand function is not continuous. Brown et al. [2] and Schmedders [23] have formulated algorithms for computing equilibria when asset markets are incomplete. Their methods aim to compute equilibrium for any such model. Our method is only valid locally but is much faster since it relies on relatively simple and direct formulas.

The applications presented in this paper are just a small sampling of the possibilities. Guu and Judd [15] applies the results of this paper to compute the optimal derivative asset. Leisen and Judd [19] uses similar methods to price options and determine equilibrium trade in options when they are not priced by arbitrage. We stay with the single good model in this paper so that we can focus on the key mathematical problems. The methods do generalize to the multicommodity models examined in Hart and others, but space limitations force us to leave that for future studies.

Section 2 reviews local approximation theory and previous small noise analyses. Section 3 presents the bifurcation to theorems that generalize the IFT. Section 4 applies the bifurcation theorems to asset demand. Section 5 presents a small noise analysis of an asset market with one risky asset and Section 6 examines a market with one fundamental risky asset plus a derivative asset. Comparisons of these cases allows us to analyze the effects of introducing a derivative asset. Section 7 discusses some computational considerations. Section 8 outlines the approach to more general models. Section 9 concludes.

2. Local Approximation Methods at Nonsingular Points

Local approximation methods are based on a few basic theorems. They begin with Taylor's theorem and the IFT for $\mathbb{R}^n$. We first state the basic theorems in this section, and then present the bifurcation theorems in the next section.

2.1. Taylor Series Approximation. The most basic local approximation is presented in Taylor's Theorem.

Theorem 1. *(Taylor's Theorem for $\mathbb{R}^n$) Let $X \subseteq \mathbb{R}^n$ and p be an interior point of X.*

Suppose $f : X \to \mathbb{R}$ is C^{k+1} in an open neighborhood $\mathcal{N}$ of p. Then, for all $x \in \mathcal{N}$

$$\begin{aligned} f(x) &= f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\ (x_i - p_i) \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\ (x_i - p_i)\ (x_j - p_j) \\ &\vdots \\ &+ \frac{1}{k!} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(p)\, (x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k}) \\ &+ \mathcal{O}\left(\|x-p\|\right)^{k+1} \end{aligned}$$

The Taylor series approximation of $f(x)$ based at p uses derivative information at p to construct a polynomial approximation. The theory only guarantees that this approximation is good near p. While the accuracy of the approximation decays as x moves away from p, this decay is often slow, implying that a finite Taylor series can be a good approximation for x in a large neighborhood of p.

2.2. The Meaning of "Approximation". We often use the phrase "$f(x)$ approximates $g(x)$ for x near p", but the meaning of this phrase is seldom made clear. One trivial sense of the term is that $f(p) = g(p)$. While this is certainly a necessary condition, it is generally too weak to be a useful concept. Approximation usually means at least that $f'(p) = g'(p)$ as well. In this case, we say that "f is a first-order (or linear) approximation to g at $x = p$". In general, "f is an n'th order approximation of g at $x = p$" if and only if

$$\lim_{x \to p} \frac{\| f(x) - g(x) \|}{\| x - p \|^n} = 0$$

This definition says that the error $\|f(x) - g(x)\|$ of the approximation $f(x)$ is asymptotically bounded above by $c\,\|x-p\|^n$ for any constant $c > 0$. Therefore, for any x near p, the approximating function $f(x)$ is very close to $g(x)$. In particular, the degree k Taylor series of a C^{k+1} function is a k'th order approximation since its error is $\mathcal{O}\left(\|x-p\|\right)^{k+1}$. This may seem trivial but this is not always the definition of n'th order approximation used in economics. We state it here for the purpose of precision.

2.3. The Implicit Function Theorem for Analytic Functions. Our analysis will rely on the IFT for analytic functions. It is useful to review some basic facts about analytic functions that will help us understand our results. The following definition for analytic functions is the most helpful of the many equivalent definitions.

Definition 2. *A function $f(x) : \mathbb{R} \to \mathbb{R}$ is analytic at x_0 if and only if there is some nonempty open set $\Omega \subset \mathbb{R}$ such that $x_0 \in \Omega$ and for all $x \in \Omega$, $f(x) = \sum_{i=0}^{\theta} a_i x^i$ and $\sum_{i=0}^{\theta} a_i |x|^i < \infty$ for all $x \in \Omega$.*

Basically, analytic functions are C^∞ and locally equal to the power series created by Taylor series expansions. The key word here is "local". For example, the power series expansion of $\log x$ around $x_0 = 1$ cannot be globally valid since $\log x$ is not defined at $x = 0$. To make this precise, we need the concept of radius of convergence. The next theorem states the key result that the domain of convergence for a power series is a disk.

Theorem 3. *Let $C = \left\{ x | \sum_{i=0}^{\theta} a_i x^i \right\} < \infty$. Then the closure of C, $\overline{C}$, is a disk, and the radius of $\overline{C}$ is called the radius of convergence of $\sum_{i=0}^{\theta} a_i x^i$.*

The focus on analytic functions is essential since some C^∞ functions are not analytic. The best example of this is e^{-1/x^2}. The function e^{-1/x^2} is defined everywhere, even at $x = 0$. Furthermore, it is C^∞ everywhere, even at $x = 0$ where each derivative equals zero. This implies that the Taylor series expansion based at $x_0 = 0$ is the zero function. However, e^{-1/x^2} equals zero just at $x = 0$, not in any neighborhood of $x = 0$. Therefore, e^{-1/x^2} does not equal its Taylor series expansion in any open neighborhood of $x = 0$ and is not analytic at $x = 0$. In general, a C^∞ function is analytic at x_0 if and only if it equals its power series in some nondegenerate neighborhood of x_0.

We have discussed just the univariate case. Analytic functions on R^n are similarly defined; see, for example, Zeidler [26]. The next important tool is the Implicit Function Theorem (IFT) for analytic functions.

Theorem 4. *(Implicit Function Theorem) Let $H(x, y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be analytic at (x_0, y_0) and assume $H(x_0, y_0) = 0$. If $H_y(x_0, y_0)$ is nonsingular, then there is a unique function $h : \mathbb{R}^n \to \mathbb{R}^m$ such that $h(x)$ is analytic at x_0 and $H(x, h(x)) = 0$ for*

(x,y) in an open neighborhood of (x_0,y_0). Furthermore, the derivatives of h at x_0 can be computed by implicit differentiation of the identity $H(x,h(x))=0$.

The IFT states that h can be uniquely defined for x near x_0 by $H(x,h(x))=0$ if $H_y(x_0,y_0)$ is not singular and allows us to implicitly compute the derivatives of h. For example, the gradient of h at x_0 is

$$\frac{\partial h}{\partial x}(x_0) = -H_y(x_0,y_0)^{-1}H_x(x_0,y_0)$$

and provides us with the first–order terms of the power series representation for $h(x)$ based at x_0. When we combine Taylor's theorem and the IFT, we have a way to compute a locally valid polynomial[1] approximation of a function $h(x)$ for x near x_0 implicitly defined by $H(x,h(x))=0$. There is an IFT for C^∞ functions, but it does not give us a positive radius of convergence for the implied power series. Therefore, we must proceed with an analytic function perspective.

The focus on analytic functions is not restrictive since most functions economists use are locally analytic at points of economic relevance. For example, $\log c$ is a common utility function and is analytic at each positive value of c. Similarly for Cobb-Douglas production functions $k^\alpha \ell^{1-\alpha}$. However, these functions are only locally analytic, implying that different power series representations are valid over different finite intervals. For example, suppose we construct a power series for $u(c)=\log c$ based at $c_0=1$. Since $\log c$ is undefined at $c=0$, the radius of convergence for that power series is at most 1, which in turn implies that that power series is not valid for any $c>2$. However, the power series based at $c_0=2$ is valid for $c\in(0,4)$. When we use the IFT for analytic functions, we need to be aware of the radii of convergence of the power series we implicitly use and be sure that they are consistent with our application of the IFT.

The power series constructed in the IFT for analytic functions will have a positive radius of convergence, but we know anything about its magnitude in general. This is a drawback in some contexts. This issue is not important in this paper since we

[1] The derivative information could also be used to compute a Padé approximant, or other nonlinear approximation schemes. Judd and Guu (1993) and Judd (1998) examine both approaches. In this paper, we will stay with the conventional Taylor expansions.

examine only the asymptotic properties of models. We will return later to the issue of the range of validity for our formulas.

2.4. Previous Small Noise Analyses. The small noise approach is not new to the economics literature, but the approach we take differs in substance and formalism from previous efforts. One line of previous work is taken by Fleming [8], which was elaborated on by Judd and Guu [14]. Fleming showed how to go from the solution of a deterministic control problem to one with small noise added to the law of motion. Specifically, consider the problem

$$\begin{aligned} \max \quad & E\left\{\int_0^T e^{-\rho t}\pi(x,u)dt\right\} \\ & dx = f(u,x)dt + \epsilon\sigma(u,x)dz \end{aligned} \tag{1}$$

Fleming approximated the problem in (1) for small ϵ by finding the control law $u = U(x,t)$ of the $\epsilon = 0$ problem and then apply the IFT to Bellman's equation. A key detail was that the control law needed to be unique in the $\epsilon = 0$ case. Judd and Guu implement this approach for infinite horizon problems, and show that the Fleming procedure produces good approximations.

The problem discussed in Fleming, and Judd and Guu was easy since it could be handled by the standard IFT. A less trivial problem was examined in Samuelson [22]. He examined the problem of asset demand when riskiness was small. We will return to that problem below.

A third example of the small noise analysis is Magill's [20] analysis of what is now called real business cycles. Magill showed how to compute linear approximations to (1), use these approximations to compute spectra of the resulting linear model, and proposed that the spectra of these models be compared to empirical data on spectra. Kydland and Prescott [18] focussed on the special case of Magill's method where the law of motion $f(u,x)$ is linear in (u,x), and partially implemented Magill's spectral comparison ideas by comparing variances and covariances of these linear approximations of deterministic models to the business cycle data. This special case of Magill's approach to stochastic dynamic general equilibrium has been important in the Real Business Cycle literature. Gaspar and Judd [10] shows how to compute higher-order expansions around deterministic steady states. Also, the methods in Magill, and Kydland and Prescott were "certainty equivalent approximations", that is, they compute a linear approximation to the deterministic problem, $\epsilon = 0$, and apply it to problems where $\epsilon \neq 0$, whereas Gaspar and Judd [10] computes approximations which includes the effect of ϵ. Similarly, we will compute high-order expansions where ϵ is allowed to vary.

A fourth example that particularly illustrates the importance of using bifurcation theory is Tesar [25]. Tesar used a linear-quadratic approach to evaluate the welfare impact on countries from opening up trade in assets. Some of her numerical examples showed that moving to complete markets would result in a Pareto inferior allocation, a finding that contradicts the first welfare theorem of general equilibrium. Kim and Kim [16] have shown that this approach will often produce incorrect results. These examples illustrate the need for using methods from the mathematical literature instead of relying on *ad hoc* approximation procedures based loosely on "economic intuition."

This paper illustrates the critical mathematical structure of asset market problems with small risks, and develops the relevant mathematical tools. While the model analyzed below is simple, the basic approach is generally applicable.

3. Bifurcation Methods

Our asset market analysis requires us to approximate an implicitly defined function at a point where the conditions of the IFT do not hold. Fortunately, we will be able to exploit additional structure and arrive at a solution using bifurcation methods. We first present the general theorems and then apply them to some asset problems.

3.1. Bifurcation in $\mathbb{R}^1$.

Suppose that $H(x, \epsilon)$ is C^2 and $x(\epsilon)$ is implicitly defined by $H(x(\epsilon), \epsilon) = 0$. One way to view the equation $H(x, \epsilon) = 0$ is that for each ϵ it defines a collection of x that solves $H(x, \epsilon) = 0$. The number of such x may change as we change ϵ. We next define the concept of a *bifurcation point.*

Definition 5. *(x_0, ϵ_0) is a bifurcation point of H iff the number of solutions x to $H(x, \epsilon) = 0$ changes as ϵ passes through ϵ_0, and there are two distinct parametric paths, $(X_i(s), E_i(s)),\ i = 1, 2$, such that $H(X_i(s), E_i(s)) = 0$, and $\lim_{s\to 0}(X_i(s), E_i(s)) = (x_0, \epsilon_0),\ i = 1, 2$.*

A trivial example of a bifurcation is $H(x, \epsilon) = \epsilon(x - \epsilon)$ at $(x, \epsilon) = (0, 0)$. If $\epsilon \neq 0$, the unique solution to $H = 0$ is $x(\epsilon) = \epsilon$, but at $\epsilon = 0$ any x solves $H = 0$. There is a bifurcation point at $(x, \epsilon) = (0, 0)$, and the two branches of solutions to $H = 0$ are $X_1(s) = E_1(s) = s$ and $X_1(s) = s,\ E_1(s) = 0$. We cannot apply the IFT to $H(x(\epsilon), \epsilon) = 0$ at $(0, 0)$ directly since the Jacobian of H_x is singular at $(0, 0)$. Suppose that we are interested in the branch $x(\epsilon) = \epsilon$, and not the trivial branch where $\epsilon = 0$ and x is arbitrary. This is natural since we want to know how x changes as ϵ changes, not just the situation at $\epsilon = 0$. Bifurcation theorems help us accomplish this. The case for $x \in \mathbb{R}$ is summarized in the Theorem 6.

Theorem 6. *(Bifurcation Theorem for $\mathbb{R}$) Suppose $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, H is analytic for (x, ϵ) in a neighborhood of $(x_0, 0)$, and $H(x, 0) = 0$ for all $x \in \mathbb{R}$. Furthermore, suppose that*

$$H_x(x_0, 0) = 0 = H_\epsilon(x_0, 0),\ \ H_{x\epsilon}(x_0, 0) \neq 0.$$

Then $(x_0, 0)$ is a bifurcation point and there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and a function $h(\epsilon)$, $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that h is analytic and $H(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$.

Proof. The strategy to prove this theorem follows the trick of "solving a singularity through division by ϵ" (see Zeidler, 1998, Chapter 8). Define

$$F(x, \epsilon) = \begin{cases} \frac{H(x,\epsilon)}{\epsilon}, & \epsilon \neq 0 \\ \frac{\partial H(x,0)}{\partial \epsilon}, & \epsilon = 0 \end{cases} . \tag{2}$$

Since H is analytic and $H(x, 0) = 0$ for all x, $H(x, \epsilon) = \epsilon F(x, \epsilon)$ and F is analytic in (x, ϵ). Since $0 = H_\epsilon(x_0, 0)$, $F(x_0, 0) = 0$. Direct computation shows $F_x(x, \epsilon) + \epsilon F_{x\epsilon}(x, \epsilon) = H_{x\epsilon}(x, \epsilon)$, which implies $F_x(x_0, 0) = H_{x\epsilon}(x_0, 0) \neq 0$. Since $F_x(x_0, 0) \neq 0$, we can apply the IFT to F at $(x_0, 0)$. Therefore, there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and an analytic function $h(\epsilon)$, $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that $F(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$, which in turn implies $H(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$. ■

In general, Theorem 6 tells us we can compute derivatives through implicit differentiation. In particular, $h'(0)$ and $h''(0)$ are defined by

$$\begin{aligned} h'(0) &= -[F_x(x_0, 0)]^{-1} F_\epsilon(x_0, 0) = -\frac{1}{2}[H_{x\epsilon}(x_0, 0)]^{-1} H_{\epsilon\epsilon}(x_0, 0) \\ 3H_{x\epsilon}(x_0, 0)h''(0) &= -[3h'(0)H_{xx\epsilon}(x_0, 0)h'(0) + 3H_{x\epsilon\epsilon}(x_0, 0)h'(0) + H_{\epsilon\epsilon\epsilon}(x_0, 0)] \end{aligned}$$

which implies a unique value for $h'(0)$ and $h''(0)$ as long as $H_{x\epsilon}(x_0, 0) \neq 0$. Notice the sequentially linear character of the problem. One only needs linear operations to compute $h'(0)$, and once we have computed $h'(0)$ the problem of computing $h''(0)$ is also a linear problem. The existence of $h'(0)$, $h^{''}(0)$, and all higher derivatives of h relies solely on the solvability condition $H_{x\epsilon}(x_{0,}0) \neq 0$ and the existence of the higher-order derivatives of H at the bifurcation point.

Theorem 6 resolves the problem when $H(x, \epsilon) = \epsilon(x - \epsilon) = 0$. In this case, $H(x, 0) = 0$ for all x, $H_x(0, 0) = 0 = H_\epsilon(0, 0)$, but $H_{x\epsilon}(x_0, 0) = 1 \neq 0$. Implicit differentiation shows that $h'(0) = 1$, and that every other derivative of h at $x = 0$ is zero. The example of $H = \epsilon(x - \epsilon)$ seems quite trivial, but our problems will have a similar form and Theorem 6 gives us conditions under which the general problem is really no more complex than this simple example.

Implicit differentiation of $H(x(\epsilon), \epsilon) = 0$ will produce a power series expansion for $x(\epsilon)$ around $\epsilon = 0$, but we know nothing about the radius of convergence of that power series. For example, $H(x, \epsilon) = \epsilon \left(x - (\epsilon + 1)^{1/2}\right) = 0$ has the obvious global solution $x(\epsilon) = (\epsilon + 1)^{1/2}$ but the power series for $(\epsilon + 1)^{1/2}$ around $\epsilon = 0$ is valid only when $-1 < \epsilon < 1$ because there is a singularity at $\epsilon = -1$.[2] Also, in practice, we will only be able to use finite-order Taylor series approximations, which are just the initial segments of the full power series. In general, any such Taylor series approximation will do well for ϵ close to zero, but the quality of the approximation will degrade as ϵ moves away from zero.

We assumed $H_{x\epsilon}(x_0, 0) \neq 0$ in Theorem 6. The division-by-zero trick can be applied to problems with higher-order degeneracies. If $H_{x\epsilon}(x_{0,}0) = 0$ then $F_x(x_{0,}0) = 0$, and we cannot apply the IFT to F in the proof. But if $F_\epsilon(x_{0,}0) = 0$ and $F_{x\epsilon}(x_{0,}0) \neq 0$ we can apply the bifurcation theorem to F.

3.2. Bifurcation in $\mathbb{R}^n$: The Zero Jacobian Case. The foregoing focussed on one-dimensional functions h. We can also apply these ideas for functions over $\mathbb{R}^n$. The same trick used in Theorem 6 works for Theorem 7; therefore, its proof is

[2] The difficulty in this case could be fixed by a nonlinear change of variables. Appropriate and clever nonlinear change of variables can help with this problem, but we do not pursue that strategy in this paper.

omitted.

Theorem 7. *(Bifurcation Theorem for $\mathbb{R}^n$) Suppose $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is analytic near $(x_0, 0)$, and $H(x, 0) = 0$ for all $x \in \mathbb{R}^n$. Furthermore, suppose that*

$$H_x(x_0, 0) = 0_{n\times n} \tag{3}$$

$$H_\epsilon(x_0, 0) = 0_n \tag{4}$$

$$\det(H_{x\epsilon}(x_0, 0)) \neq 0 \tag{5}$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and an analytic function $h(\epsilon) : \mathbb{R} \to \mathbb{R}^n$ such that $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, and $H(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$.

Since Theorem 7 shows that h is analytic, it can be approximated by a multivariate Taylor series. In particular, the first-order derivatives are defined by

$$h'(0) = -\frac{1}{2} H_{x\epsilon}^{-1}(x_0, 0)\ H_{\epsilon\epsilon}(x_0, 0) \tag{6}$$

Theorem 7 assumes $H_x(x_0, 0)$ is a zero matrix. There are generalizations that only assume that $H_x(x_0, 0)$ is singular. We do not present any extensions here since they are substantially more complex to present and are not needed below. See Zeidler or Chow and Hale for more complete treatments of bifurcation problems.

4. PORTFOLIO DEMAND WITH SMALL RISKS

The key assumption we exploit is that risks are small. This is motivated not by any claim that actual risks are small, but is reasonable for three reasons. First, this assumption allows us to solve the problem without making any parametric assumptions for either tastes or returns. We derive critical formulas for allocations and welfare in a parameter-free fashion. The results tell us which moments of asset returns are important and which properties of the utility function are important for the case of small risks. Second, the results for small risks may be suggestive of general results. For example, the asymptotic results could provide counterexamples to conjectures since the asymptotic results are asymptotically explicit solutions. Furthermore, any general property of the model will be true for the case of small risks and will be revealed as general properties of our asymptotic solutions. In this paper, we pursue the implications of the small risk assumption, leaving it for later work to see how robust those results.

Third, the period of time in our model is not meant to be an entire life, but rather the period of time between trades. Given modern markets and the presence of many high-volume, low-transaction cost traders, it is reasonable to assume that only a moderate amount of risk is borne between trading periods. A dynamic model is necessary to examine the validity of this point, but we believe that our static analysis will give useful insights and leave dynamic generalizations for future work.

4.1. Demand with Two Assets. We begin by applying the bifurcation approximation methods to asset market demand. Suppose that an investor has W in wealth to invest in two assets. The safe asset, called a bond, yields one dollar per dollar invested, and the risky asset, called stocks or equity, yields Z dollars per dollar invested. There is no savings-consumption decision in this model. Therefore, this is equivalent to making bonds in the second period the numeraire. If an investors has θ shares of stock, final wealth is $Y = (W - \theta) + \theta Z$. We assume that he chooses θ to maximize $E\{u(Y)\}$ for some concave utility function $u(\cdot)$.

Economists have studied this problem by approximating u with a quadratic function and then solving the "approximate" quadratic optimization problem. The bifurcation approach allows us to examine this procedure rigorously and extend it. We first create a continuum of portfolio problems by assuming

$$Z = 1 + \epsilon z + \epsilon^2 \pi \tag{7}$$

where z is a fixed random variable. We assume $E\{z\} = 0$ since we want (7) to decompose Z into its mean, $1 + \epsilon^2 \pi$, and its risky component, ϵz. We also assume $\sigma_z^2 = 1$; this makes ϵ the standard deviation of Z and ϵ^2 its variance in the ϵ problem. Both of these assumptions are just normalizations, implying no loss of generality. At $\epsilon = 0$, Z is degenerate and equal to 1, the payoff of the bond. The scalar π represents the risk premium. More precisely, $\sigma_z^2 = 1$ implies that π is the the price of risk, that is, the risk premium per unit variance. In this demand problem we make the natural assumption that $\pi > 0$ but that is not necessary for the analysis.

Equation (7) scales its terms in a manner consistent with economic theory. We want (7) to represent a continuum of problems connecting a degenerate deterministic problem to problems with nontrivial risk. Note that (7) multiplies z by ϵ and π by ϵ^2. Since the variance of ϵz is $\epsilon^2 \sigma_z^2$, this models the intuition that risk premia are proportional to the variance. The continuum of problems parameterized in (7) all have the same price of risk π. The particular parameterization in (7) may seem to prejudge the results. That will not be a problem since the application of the bifurcation theorems will validate the assumptions implicitly made in (7).[3]

The investor chooses θ to maximize $E\{u(W + \theta(\epsilon z + \epsilon^2\pi))\}$. The first-order condition for the investor's problem is

$$\epsilon E\{u^{'}(W + \theta(\epsilon z + \epsilon^2\pi))\,(z + \epsilon\pi)\} = 0. \tag{8}$$

The condition (8) states that the future marginal utility of consumption must be orthogonal to the excess return of equity. Let μ be the probability measure for z and (a, b) the (possibly infinite) support. The choice of θ as a function of ϵ is implicitly defined by

$$0 = H(\theta(\epsilon), \epsilon) \equiv \int_a^b u'(W + \theta(\epsilon)(\epsilon z + \epsilon^2\pi))\,(z + \epsilon\pi)\,d\mu. \tag{9}$$

We want to analyze the solutions of (9) for small ϵ. However, $0 = H(\theta, 0)$ for all θ, because at $\epsilon = 0$ the assets are perfect substitutes. $\theta(0)$ is multivalued since any choice of θ satisfies the first-order condition (9) when $\epsilon = 0$. Furthermore, $0 = H(\theta, 0)$ for all θ implies $0 = H_\theta(\theta, 0)$ for all θ, violating the nonsingularity condition in the IFT. Therefore, we cannot use the IFT to compute a Taylor series for $\theta(\epsilon)$ at $\epsilon = 0$.

The situation is displayed in Figure 1. As ϵ changes, the equilibrium demand for equity, θ, follows a path like ABC or like $DEGF$. Since the asset demand problem is a concave optimization problem there is a unique path of solutions to the first-order conditions whenever $\epsilon \neq 0$. At $\epsilon = 0$, however, the entire $\epsilon = 0$ horizontal axis is also a solution to the equity demand problem. The path ABC crosses the θ axis vertically and represents a *pitchfork bifurcation*, whereas the path $DEGF$ crosses the θ axis

[3] Pages 518-519 in Judd (1998) show that alternative parameterizations of the form $Z = 1 + \epsilon z + \epsilon^\nu \pi$ for $\nu \neq 2$ lead to singularities which prevent the application of implicit function or bifurcation theorems.

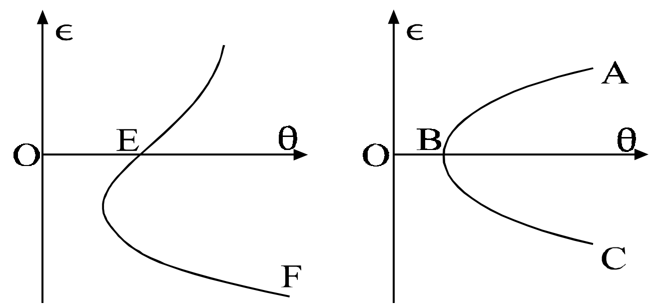


Figure 1: Bifurcation possibilities for asset demand problem

obliquely and represents a *transcritical bifurcation*. The objective is to first find the bifurcation point, B or E, where the branch of equity demand solutions crosses the trivial branch of solutions to the first-order conditions, and then compute a Taylor series that approximates $\theta(\epsilon)$ along the nontrivial branch.

Computing θ_0. We proceed intuitively to derive a solution which we validate with the Bifurcation Theorem. Since we want to solve for θ as a function of ϵ near 0, we first need to compute $\theta_0 \equiv \lim_{\epsilon\to 0}\theta(\epsilon)$. Implicit differentiation of (9) with respect to ϵ implies

$$0 = H_\theta(\theta(\epsilon),\epsilon)\theta'(\epsilon) + H_\epsilon(\theta(\epsilon),\epsilon). \tag{10}$$

Differentiating $H(\theta,\epsilon)$ with respect to θ and ϵ implies

$$\begin{aligned} H_\epsilon(\theta,\epsilon) &= \int_a^b u''(Y)\,(\theta z + 2\theta\epsilon\pi)\,(z+\epsilon\pi) + u'(Y)\pi\; d\mu \\ H_\theta(\theta,\epsilon) &= \int_a^b u''(Y)\,(z+\epsilon\pi)^2\epsilon\; d\mu \end{aligned}$$

At $\epsilon = 0$, $H_\theta(\theta,0) = 0$ for all θ. The derivative $\theta'(0)$ can be well-defined in (10) only if $H_\epsilon(\theta,\,0) = 0$. Therefore, we look for θ_0 defined by $0 = H_\epsilon(\theta_0,\,0)$. At $\epsilon = 0$, this reduces to (using the fact that $\int_a^b z^2\; d\mu = \sigma_z^2 = 1$) $0 = u''(W)\,\theta_0 + u'(W)\pi$, which implies

$$\theta_0 = -\,\frac{u'(W)}{u''(W)}\,\pi \tag{11}$$

This is the simple portfolio rule indicating that θ is the product of risk tolerance and the risk premium per unit variance. If θ_0 is well-defined, then this must be its value.

Theorem 8 states the critical result.

Theorem 8. *Let (11) define θ_0. If $H(\theta, \epsilon)$ is analytic at $(\theta_0, 0)$, then there is an analytic function $\theta(\epsilon)$ that satisfies (9) such that $\theta(0) = \theta_0$ and $\theta(\epsilon) \neq 0$ for $\epsilon \neq 0$.*

Proof. Direct application of the Bifurcation Theorem. ■

The assumption in Theorem 8 that $H(\theta, \epsilon)$ is analytic at θ_0 is not trivially satisfied. $H(\theta, \epsilon)$ is an integral and is analytic if $u(c)$ is analytic over the set of c at which $u'(c)$ is evaluated in the integrand of $H(\theta, \epsilon)$, because the integral of a power series is a power series. If the support of μ is compact and u is analytic at W then $H(\theta, \epsilon)$ is analytic at $(\theta_0, 0)$ since for small ϵ, $u'(c)$ is evaluated only at values of c close to W. However, if μ has infinite support there may be problems because $u'(c)$ in the integrand of (9) is evaluated over an infinite range whenever $\epsilon, \theta \neq 0$. If the radius of convergence for the power series representation of $u'(c)$ based at W is finite, then it will not be valid at some points in the support of μ, rendering the power series approach invalid. This will be the case, for example, if $u(c) = \log c$ and μ is the measure for a log Normal random variable. The radius of convergence of power series approximations of $u(c)$ at $c = W$ is a critical element, as well as the analyticity of the density function of μ. The next corollary presents a sufficient condition for using the bifurcation approach on an open neighborhood $\mathcal{N}$.

Corollary 9. *Define θ_0 as in (11). If $u(c)$ is analytic at $c = W$ and the support of μ is compact, then there is a function $\theta(\epsilon)$ analytic and satisfies (9) on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ with $\theta(0) = \theta_0$ and $\theta(\epsilon) \neq 0$ for $\epsilon \neq 0$ in $(-\epsilon_0, \epsilon_0)$.*

In all formulas below, we will assume that the critical functions are locally analytic.

Computing $\theta'(0)$. Equation (11) is not an approximation to the portfolio choice at any particular variance. Instead, θ_0 is the limiting portfolio share as the variance vanishes. We generally need to compute several terms of the Taylor series expansion for $\theta(\epsilon)$

$$\theta(\epsilon) = \theta_0 + \theta'(0)\epsilon + \theta''(0)\frac{\epsilon^2}{2} + \theta'''(0)\frac{\epsilon^3}{6} + ... \tag{12}$$

In particular, the linear approximation is

$$\theta(\epsilon) \doteq \theta(0) + \epsilon\ \theta'(0). \tag{13}$$

To calculate $\theta^{'}(0)$, differentiate (10) with respect to ϵ to find $0 = H_{\theta\theta}\,\theta'\theta' + 2H_{\theta\epsilon}\,\theta' + H_\theta\theta'' + H_{\epsilon\epsilon}$. At $(\theta_0, 0)$, $H_{\epsilon\epsilon} = u^{'''}(W)\theta_0^2\,E\{z^3\}$, $H_{\theta\theta} = 0$, and $H_{\theta\epsilon} = u^{''}(W)$. Therefore,

$$\theta'(0) = -\frac{1}{2}H_{\theta\epsilon}^{-1}H_{\epsilon\epsilon} = -\frac{1}{2}\frac{u'''(W)}{u''(W)}E\{z^3\}\ \theta_0^2. \tag{14}$$

Again, we can use Corollary 9 to establish the existence of the derivatives of H for some random variables.

Equation (14) tells us how the share of wealth invested in equity changes as the riskiness increases. It highlights the importance of the third derivative of utility and the skewness of returns. If the distribution of Z is symmetric, then $E\{z^3\} = 0$, and the constant θ_0 is the linear approximation of $\theta(\epsilon)$ at $\epsilon = 0$. This is also true if $u'''(W) = 0$, such as in the quadratic utility case. The case of $\theta'(0) = 0$ corresponds to a pitchfork bifurcation point like B in Figure 1. However, if the utility function is not quadratic and the risky return is not symmetrically distributed, then $\theta'(0) \neq 0$, and the linear approximation is a nontrivial function of utility curvature and higher moments of the distribution. This indicates that the bifurcation point is transcritical like E in Figure 1.

Dividing both sides of (14) by θ_0 implies

$$\frac{\theta'(0)}{\theta_0} = \frac{1}{2}\frac{u^{'}(R)}{u''(R)}\frac{u'''(R)}{u''(R)}\ \pi\ E\{z^3\} \tag{15}$$

Equation (15) expresses the relative change in equity demand as ϵ increases in terms of skewness, $E\{z^3\}$, the risk premium, π, and utility derivatives. Our formulas would be unintuitive and cumbersome if we expressed them in terms of $u(c)$ and its derivatives. Fortunately, there are some useful utility parameters we can use. Define the functions

$$\begin{aligned}
\tau(c) &\equiv -\frac{u'(c)}{u''(c)} \\
\rho(c) &\equiv \frac{\tau^2}{2}\frac{u'''(c)}{u'(c)} = \frac{1}{2}\frac{u'(c)}{u''(c)}\frac{u'''(c)}{u''(c)}
\end{aligned}$$

The function $\tau(c)$ is the conventional risk tolerance. The bifurcation point θ_0 equals $\tau(W)\ \pi$, the product of risk tolerance at the deterministic consumption, $\tau(W)$, and the price of risk, π.

The definition of $\rho(c)$ implies that (15) can be expressed as

$$\frac{\theta'(0)}{\theta_0} = \rho(W)\ \pi\ E\{z^3\} \tag{16}$$

This motivates our definition of *skew tolerance*.[4]

Definition 10. *Skew tolerance at c is*

$$\rho(c) = \frac{1}{2}\frac{u'(c)}{u''(c)}\frac{u'''(c)}{u''(c)}$$

Skew tolerance has ambiguous sign since the sign of u''' is ambiguous. If there is more upside potential than downside risk, then skewness is positive. If $u''' > 0$, an increase in skewness will cause asset demand to increase as riskiness increases. We suspect that investors prefer positively skewed returns, holding mean and variance constant. For example, $u''' > 0$ for the CRRA and CARA families of utility functions. We never assume this, but this case provides us with some intuition for the results. There are many ways to manipulate the expression in (14). We chose our definition of skew tolerance because of the expression in (16) and the intuitive role it plays in critical expressions below.

The linear approximation (13) may not be sufficient. To compute $\theta''(0)$, differentiate (10) with respect to ϵ at $\epsilon = 0$ to find

$$3H_{\theta\epsilon}\theta''(0) = -(3H_{\theta\epsilon\epsilon}\theta'(0) + 3H_{\theta\theta\epsilon}(\theta'(0))^2 + H_{\epsilon\epsilon\epsilon}) \tag{17}$$

Equation (17) is linear in $\theta''(0)$. Since $H_{\theta\epsilon} \neq 0$ at $(\theta_0, 0)$, $\theta''(0)$ exists and is uniquely defined by (17). To express $\theta''(0)$, we define *kurtosis tolerance.*

Definition 11. *Kurtosis tolerance at c is*

$$\kappa(c) = -\frac{1}{3}\frac{u''''(c)}{u''(c)}\frac{u'(c)}{u''(c)}\frac{u'(c)}{u''(c)}$$

[4] Skew tolerance is obviously related to prudence, as defined in Kimball (1990), but we do not pursue those connections here.

Solving (17) at $\epsilon = 0$ shows that

$$\frac{\theta''(0)}{\theta_0} = \pi^2 \left((6\rho(W) - 2) + 4\rho(W)^2 E\{z^3\}^2 + \kappa(W) E\{z^4\}\right) \tag{18}$$

Equation (18) says that the impact of kurtosis on equity demand is proportional to the square of the price of risk and the kurtosis tolerance.

We could continue this indefinitely if u is locally analytic, an assumption satisfied by standard utility functions. Of course, the terms become increasingly complex. We end here since it illustrates the main ideas and these results are the only ones needed for the applications below. The general procedure is clear. Computing the higher-order terms is straightforward since any particular derivative is the solution to linear equations similar to (17) once we have computed lower-order derivatives.

Samuelson's Method. Samuelson [22] also examined the problem of asset demand with small risks. We now illustrate the relationships between our bifurcation approach and Samuelson's method. Samuelson's method replaced $u(Y)$ with a polynomial approximation based at the deterministic consumption, as in

$$\begin{aligned} u(W + \theta(\epsilon z + \epsilon^2 \pi)) \doteq\ & u(W) + \epsilon\theta z u'(W) \\ & + \frac{\epsilon^2}{2}\left(2\theta\pi^2 u'(W) + \theta^2 z^2 u''(W)\right) \\ & + \frac{\epsilon^3}{6}\left(6z\pi^2\theta^2 u''(W) + \theta^3 z^3 u'''(W)\right) + \ldots \end{aligned}$$

When we use the quadratic approximation in the first-order condition (8) we arrive at the equation $0 = (\pi u'(W) + \theta u''(W))\,\epsilon^2 + O(\epsilon^3)$, which, to $O(\epsilon)$, implies $\theta(\epsilon) \doteq -(u'(W)/u''(W))\pi$, our bifurcation point.

However, the Samuelson method differs from ours for higher-order approximations. Samuelson's second-order approximation is computed by using the third-order approximation of $u(Y)$ in the first-order condition (8), implying

$$0 \doteq (\pi u'(W) + \theta u''(W))\,\epsilon^2 + \epsilon^3 \frac{1}{2}\theta^2 E\{z^3\} u'''(W) \tag{19}$$

which is a quadratic equation with solution

$$\theta(\epsilon) \doteq \frac{-u''(W) + \sqrt{u''(W)^2 - 2\pi E\{z^3\}\epsilon u'''(W) u'(W)}}{E\{z^3\} u'''(W)\epsilon} \tag{20}$$

One could arrive at our first-order derivative in equation (14) by differentiating (20) with respect to ϵ at $\epsilon = 0$. The two methods are consistent and of similar complexity for the first-order approximation in a two-asset problem. However, the asymptotic approach we pursue here becomes relatively more efficient as we move to higher-order approximations and to more assets. Samuelson's approach generally requires solving nonlinear equations, as was the case in equation (19). The equations become more difficult to solve, and are impossible to solve exactly beyond the fourth order since there is no closed-form solution for polynomials of degree five and higher. Our bifurcation method uses linear operations to compute asymptotically valid approximations of the function $\theta(\epsilon)$. Therefore, we can easily derive each term and go to an arbitrary order as long as the necessary moments and derivatives exist.

The main reason for pursuing the asymptotic approach is its ability to derive economically interesting results. Equation (20) shows that linear-quadratic approximations would not be as good as higher-order approximations since equation (20) involves the skewness of Z and the third derivative of utility. However, Samuelson conjectured that LQ approximations are probably adequate in actual economic problems. This paper gives examples where the linear-quadratic approximation would be unreliable, and higher-order approximations are necessary to answer critical questions.

4.2. Demand with Three Assets. We applied the $\mathbb{R}^1$ version of the Bifurcation Theorem to the two-asset case. We next analyze the three-asset case to show the generality of the method and illustrate the key multivariate details. Consider again our investor model but with three assets. The bond yields one dollar per dollar invested and risky asset i yields Z_i dollars per dollar invested, for $i = 1, 2$. Let θ_i denote the proportion of wealth invested in risky asset i. Final wealth is $Y = (W - \theta_1 - \theta_2) + \theta_1 Z_1 + \theta_2 Z_2$. The investor chooses θ_i to maximize $E\{u(Y)\}$. To apply the Bifurcation Theorem, we assume that $Z_i = 1 + \epsilon z_i + \epsilon^2 \pi_i$. Without loss of generality, we assume that $E\{z_i\} = 0$. Let $\sigma_i^2 = E\{z_i^2\}$ be the variance of risky asset i's return and $\sigma_{12} = E\{z_1 z_2\}$ the covariance. We assume that the assets are not perfectly correlated; hence, $\sigma_i^2 \sigma_j^2 \neq (\sigma_{ij})^2$.

The first-order condition for risky asset i is $\epsilon E\left\{u'(Y)(\epsilon\pi_i + z_i)\right\} = 0$. The asset demand functions $\theta_i(\epsilon)$ are defined implicitly by $H(\theta_1, \theta_2, \epsilon) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ where $H^i(\theta_1, \theta_2, \epsilon) \equiv E\left\{u^{'}(Y)(\epsilon\pi_i + z_i)\right\}$, $i = 1, 2$. To invoke Theorem 7, we first note that $H_\theta(\theta_1, \theta_2, 0) = 0_{2\times 2}$ for all (θ_1, θ_2). We compute a candidate bifurcation point by solving $H_\epsilon(\theta_1, \theta_2, 0) = 0$. Direct computation shows

$$H_\epsilon(\theta_1, \theta_2, 0) = u^{'}(W)\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} + u^{''}(W)\Sigma\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

where Σ is the variance-covariance matrix of the risky returns (z_1, z_2). The solution of the bifurcation equation $H_\epsilon(\theta_1, \theta_2, 0) = 0$ is

$$\begin{bmatrix} \theta_1(0) \\ \theta_2(0) \end{bmatrix} = -\frac{u^{'}(W)}{u^{''}(W)}\Sigma^{-1}\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

We need to verify the nonsingularity of $H_{\theta\epsilon}$ at $(\theta_1(0), \theta_2(0), 0)$. Direct computation shows that $H_{\theta\epsilon}(\theta_1(0), \theta_2(0), 0) = u^{''}(W)\Sigma$ for all θ_1, θ_2. The determinant of $H_{\theta\epsilon}$ at $(\theta_1(0), \theta_2(0), 0)$ is $u^{''}(W)(\sigma_1^2\sigma_2^2 - (\sigma_{12})^2)$, which is nonzero as long as assets 1 and 2 are not perfectly correlated.

These calculations show that all the conditions in Theorem 7 hold for our model. Hence, the bifurcation theorem for $\mathbb{R}^2$ ensures the existence of analytic functions $\theta_1(\epsilon)$ and $\theta_2(\epsilon)$ which satisfy $H(\theta_1(\epsilon), \theta_2(\epsilon), \epsilon) = 0$ in some neighborhood of $\epsilon = 0$. This procedure can be applied for an arbitrary number of assets. We can also produce higher-order expansions as long as the necessary moments and derivatives exist. We next use these ideas to compute asset market equilibrium.

5. Asset Market Equilibrium with One Risky Asset

We now take our portfolio choice analysis and turn it into an equilibrium analysis[5]. We assume a two-period model, period 0 and period 1, with no consumption in period 0. Agents trade assets in period 0 and consume the asset payoffs in period 1. One bond yields 1 unit of consumption in period 1; the bond serves as our numeraire in period 0. Each share of equity has price p in period 0 and has a random period 1 value

[5] Chiappori et al. (1992) used similar methods to prove the existence of sunspot equilibria near deterministic steady states in overlapping generations models. We go through the details of our application since they are substantially different than the application in Chiappori et al.

of $1+\epsilon z$ units of consumption where z is a random variable with finite moments. We assume $E\{z\}=0$ and $E\{z^2\}=1$. For each value of ϵ we have an asset market with two assets; we call that economy the ϵ-economy.

We assume two types of traders. Type i traders have initial endowments of B_i^e units of the bond and θ_i^e shares of equity. The utility of a type i trader is $u_i(Y_i)$, a concave function, where Y_i is the final wealth and consumption of type i traders. The supply of equity is fixed at the endowment $\theta_1^e+\theta_2^e$. Without loss of generality, we assume $\theta_1^e+\theta_2^e=1$; this implies that z denotes aggregate risk in the aggregate endowment. Let θ_i be the shares of equity and B_i the value of bonds held by trader i after trading in period 0. The final wealth for trader i is $Y_i=\theta_i(1+\epsilon z)+B_i$. Each trader of type i chooses θ_i to maximize his expected utility $E\{u_i(Y_i)\}$, subject to the budget constraint $B_i+\theta_i p=B_i^e+\theta_i^e p$. His first-order condition for θ_i is $E\{u_i^{'}(Y_i)(1+\epsilon z-p)\}=0$. Market clearing implies $\theta_1+\theta_2=\theta_1^e+\theta_2^e=1$. Define $\theta=\theta_1$; then $\theta_2=1-\theta$. For each ϵ-economy, we want to find the equilibrium values of θ and p; let $\theta(\epsilon)$ and $p(\epsilon)$ be the equilibrium values of θ and p in the ϵ-economy. The equilibrium values of $\theta(\epsilon)$ and $p(\epsilon)$ must satisfy the equilibrium pair of equations

$$H^i(\theta(\epsilon),p(\epsilon),\epsilon)=E\{u_i^{'}(Y_i)(1+\epsilon z-p(\epsilon))\}=0,\ i=1,2 \tag{21}$$

which are implied by the agents' first-order conditions.

Equation (21) implicitly defines $(\theta(\epsilon),p(\epsilon))$. However, the IFT cannot be applied to analyze (21) around $\epsilon=0$. Since the assets are perfect substitutes at $\epsilon=0$, they must trade at the same price; hence, $p(0)=1$. However, $\theta(0)$ is indeterminate because $H(\theta,p,0)=0$, for all θ. The indeterminacy of θ implies that $H_\theta(\theta,1,0)=0$, ruling out application of the IFT.

We want to apply the Bifurcation Theorem, but we cannot apply it to $H(\theta,p,0)$ because $H_\theta(\theta,1,0)\neq 0$. Intuitively, the Bifurcation Theorem presented above requires that both θ and p are indeterminate at $\epsilon=0$. Moreover, we know $p'(0)$ if it exists. Implicit differentiation of $H(\theta(\epsilon),p(\epsilon),\epsilon)$ with respect to ϵ implies

$$H_\theta^i(\theta,p,\epsilon)\theta^{'}(\epsilon)+H_p^i(\theta,p,\epsilon)p^{'}(\epsilon)+H_\epsilon^i(\theta,p,\epsilon)=0.$$

For each i, $H_\theta^i(\theta,p(0),0)=0$ for all θ since $p(0)=1$. Therefore, if $p(\epsilon)$ is differentiable at $\epsilon = 0$, then

$$0 = H_p^i(\theta, p, 0)p^{'}(0) + H_\epsilon^i(\theta, p, 0) = \left(E\left\{z\right\} - p^{'}(0)\right) u_i^{'}(c_i)$$

for $i = 1, 2$, where $c_i = B_i^e + \theta_i^e$ is consumption in the no-risk case. Since $u_i^{'}(c_i)$ is never zero, $p^{'}(0) = E\left\{z\right\} = 0$ must hold if $\theta(\epsilon)$ and $p(\epsilon)$ are differentiable at $\epsilon = 0$. Therefore, we have indeterminacy of $\theta(0)$ but there is only a single possible value for both $p(0)$ and $p^{'}(0)$. This prevents us from using Theorem 7 directly since the Jacobian matrix $H_{(\theta,p)}^i$ is not a zero matrix.

This problem is solved by reformulating the problem in terms of the price of risk, not the price of the equity. More precisely, we assume the equity price parameterization

$$p(\epsilon) = 1 - \epsilon^2\pi(\epsilon) \tag{22}$$

where $\pi(\epsilon)$ is the risk premium in the ϵ-economy. Since $\sigma_z^2 = 1$, ϵ^2 is the variance of risk and $\pi(\epsilon)$ is the risk premium per unit variance. Since we expect the risk premium to depress the price of equity, we use the form in (22).

We have assumed the parameterization in (22) but we have not proved anything yet. We now need to show that this parameterization is consistent with Theorem 7. To check the sufficient conditions in Theorem 7, we reformulate equilibrium as the system of equations

$$0 = \mathcal{H}^i(\theta, \pi, \epsilon) \equiv E\left\{u_i^{'}(Y_i)\left(z - \epsilon\pi\right)\right\} = 0. \tag{23}$$

where $\mathcal{H}^i(\theta, \pi, \epsilon) = \epsilon^{-1}H^i(\theta, 1 - \epsilon^2\pi, \epsilon)$, $i = 1, 2$. It is clear that (θ, π, ϵ) satisfy (23) if and only if they also satisfy (21).

The parameterization in (22) and the equilibrium characterization in (23) now allow us to apply the Bifurcation Theorem. The functions $\mathcal{H}^i(\theta, \pi, \epsilon)$ have the degeneracy assumed in Theorem 7 since $\mathcal{H}_\theta^i(\theta, \pi, 0) = \mathcal{H}_\pi^i(\theta, \pi, \epsilon) = 0$ for all (θ, π). Intuitively, at $\epsilon = 0$, any portfolio satisfies the first-order conditions since all assets are perfect substitititutes and any price of risk, π, is consistent with equilibrium since the total amount of risk is zero. The Jacobian matrix

$$\mathcal{H}_{(\theta,\pi),\epsilon} = \left[\begin{array}{cc} \mathcal{H}_{\theta\epsilon}^1(\theta(0), \pi(0), 0) & \mathcal{H}_{\pi\epsilon}^1(\theta(0), \pi(0), 0) \\ \mathcal{H}_{\theta\epsilon}^2(\theta(0), \pi(0), 0) & \mathcal{H}_{\pi\epsilon}^2(\theta(0), \pi(0), 0) \end{array}\right] = \left[\begin{array}{cc} u_1^{''} & -u_1^{'} \\ u_2^{''} & u_2^{'} \end{array}\right]$$

has determinant $u_1^{''}u_2^{'}+u_1^{'}u_2^{''} < 0$. Therefore, all the sufficient conditions of Theorem 7 hold, and the Bifurcation Theorem provides a local proof of existence and uniqueness of solutions $(\theta(\epsilon), \pi(\epsilon))$ to (23). Theorem 12 summarizes the result.

Theorem 12. *If $u_i(c)$ is locally analytic for c near $B_i^e + \theta_i^e$, $i = 1, 2$, and $\mathcal{H}(\theta, \pi, \epsilon)$ is locally analytic near a solution (θ_0, π_0) to $\mathcal{H}_\epsilon(\theta, \pi, 0) = 0$, then there is some $\epsilon_0 > 0$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$ there is a unique analytic equilibrium selection $(\theta(\epsilon), \pi(\epsilon))$ such that $\mathcal{H}(\theta(\epsilon), \pi(\epsilon), \epsilon) = 0$.*

The basic approach to using the Bifurcation Theorem is to guess some parameterization for the unknown functions and then use the Bifurcation Theorem to check that it is correct and can produce a locally analytic approximation. Some of the choices we made, particularly the construction of (22) and (23), may appear arbitrary, but their use is validated by the Bifurcation Theorem. Our formulation is economically intuitive. For example, (22) just says that risk premia are proportional to variance. Therefore, application of these ideas to more complex problems is not difficult as long as we remember the intuition behind our construction. There are more complex versions of the Bifurcation theorem which would lead more directly to (22) and (23); see Zeidler [26]. We prefer the approach used here since it is straightforward once one uses economic intuition to arrive at (22) and (23).

Figure 2 displays the geometry of the bifurcation in (23). When $\epsilon = 0$, the entire $\theta - \pi$ plane constitutes an equilibrium. However, for nonzero ϵ we have a locally unique equilibrium. In Figure 2 the curve ABC represents the equilibrium manifold.

We can now proceed to compute asymptotic expressions for $(\theta(\epsilon), \pi(\epsilon))$. Direct computation shows that the bifurcation point $(\theta_0,\ \pi_0)$ for (23) is defined by $\mathcal{H}_\epsilon^i(\theta_0, \pi_0, \epsilon) = 0$, $i = 1, 2$, and satisfies the linear equations:

$$\begin{aligned} -u_1^{'}(c_1)\pi_0 + u_1^{''}(c_1)\theta_0 &= 0 \\ u_2^{'}(c_2)\pi_0 + u_2^{''}(c_2)\theta_0 &= u_2^{''}(c_2) \end{aligned} \tag{24}$$

where $c_i = B_i^e + \theta_i^e$. The linear equations in (24) imply the unique candidate bifurcation point

$$\theta_0 = \frac{\tau_1}{\tau_1 + \tau_2},\ \pi_0 = \frac{1}{\tau_1 + \tau_2} \tag{25}$$

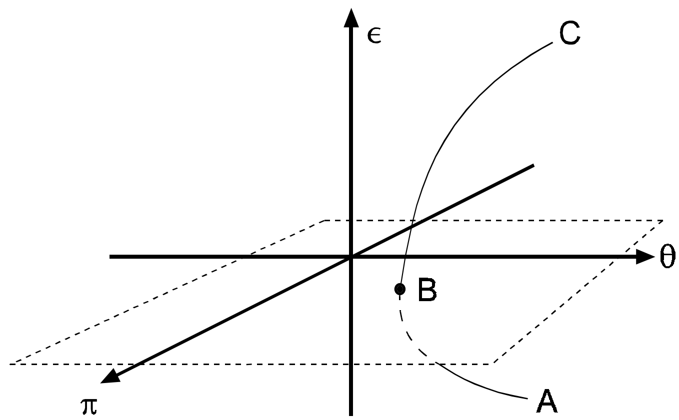


Figure 2: Bifurcation of Equilibrium Correspondance

where τ_i is evaluated at $c_i = B_i^e + \theta_i^e$, consumption in the deterministic limit. These formulas for θ_0 and π_0 are intuitive; the τ_i terms are the individual risk tolerances at $\epsilon = 0$, and the denominator is their sum, which is the social risk tolerance. The results are both very intuitive. The equilibrium risk premium is the inverse of total risk tolerance. Also, the fraction of equity held by investor 1 equals his contribution to social risk tolerance. These solutions resemble the intuitive results from mean-variance models.

The solution in (25) just tells us what the limit portfolio is as variance goes to zero. We want to know what the equilibrium portfolio is for nonzero variance. This requires computing the derivatives $\theta^{'}(0)$ and $\pi^{'}(0)$. Further implicit differentiations of $\mathcal{H}^i$ yield $(\theta^{'}(0), \pi^{'}(0))$ and any other higher-order derivative.

Theorem 13. *The first-order derivatives of the equilibrium correspondence* $(\theta(\epsilon),\ \pi(\epsilon))$ *at* $\epsilon = 0$ *are*

$$\theta'(0) = \frac{\tau_1}{\tau_1+\tau_2}\frac{\tau_2}{\tau_1+\tau_2}\frac{\rho_1-\rho_2}{\tau_1+\tau_2}E\left\{z^3\right\} \tag{26}$$

$$\pi'(0) = -\left(\frac{\tau_1}{\tau_1+\tau_2}\rho_1 + \frac{\tau_2}{\tau_1+\tau_2}\rho_2\right)\frac{E\left\{z^3\right\}}{\left(\tau_1+\tau_2\right)^2} \tag{27}$$

Therefore, type 1 investors increase their holdings of equity as ϵ *increases if* $(\rho_1 - \rho_2)E\left\{z^3\right\} > 0$, *and the risk premium per unit variance decreases as* ϵ *increases if* $E\left\{z^3\right\} > 0$.

Proof. Apply (6). ■

Theorem 13 gives us our first-order approximation to $\theta(\epsilon) = \theta_0 + \epsilon\theta'(0)$. We need to be clear what this tells us. For example, if $\theta'(0) > 0$ then we know that for all $\epsilon > 0$ sufficiently close to θ_0, $\theta(\epsilon)$ exceeds θ_0, and that $\theta(\epsilon)$ grows at rate $\theta'(0)$. We know this because $\theta(\epsilon)$ is locally analytic, implying that our Taylor series approximations are valid for ϵ sufficiently close to $\epsilon = 0$. This could be reversed for large ϵ with $\theta(\epsilon)$ less than θ_0. But, for sufficiently small ϵ, equations (26) and (27) tell us precisely how $\theta(\epsilon)$ and $\pi(\epsilon)$ behave.

Theorem 13 is economically intuitive. Equation (26) shows that the equity holdings of a type 1 investor are greater than θ_0 if ϵ is small and positive, if skewness, $E\{z^3\}$, is positive, and if his skew tolerance exceeds the skew tolerance of type 2 investors, where we evaluate skew tolerance at the $\epsilon = 0$ allocations. Equation (27) shows that the risk premium will decrease as ϵ increases (and the price of equity relative to bonds will increase) if skewness is positive. The magnitude of the change depends on a weighted sum of the skew tolerances, where the weights are the limit portfolio holdings. Notice that we get these results for any utility function, not just for CRRA utility functions or other families that have $u''' > 0$. The results in Theorem 13 resemble the style of analysis in Jones [12]. Jones examines the impact of changes in endowments on equilibrium, whereas we are examining the change in asset market equilibrium as we move away from the deterministic case. The problems are economically different but the mathematical idea is the same: use implicit function theorems or their generalizations to analyze the impact of small changes in parameters on equilibrium.

The derivatives $\theta^{'}(0)$ or $\pi^{'}(0)$ could be zero. This does not mean that $\theta(\epsilon)$ or $\pi(\epsilon)$ is constant for small ϵ. It just means that the local behavior is governed by higher-order terms in the expansion. For example, if $E\{z^3\} = 0$, then $\theta^{'}(0) = \pi^{'}(0) = 0$ and the local behavior of $\theta(0)$ and $\pi(0)$ is governed by $\theta''(0)$ and $\pi''(0)$, which depend on the kurtosis $E\{z^4\}$ and fourth-order properties of $u(c)$. We do not pursue these higher-order issues in this paper since Theorem 13 is adequate for the analysis below.

6. Asset Market Equilibrium with a Derivative Asset

The previous section examined a market with only a bond and a stock. In this section, we compare markets with different asset spans. In particular, we introduce a new derivative asset into the market and compute asymptotically valid expressions for equilibrium. The results allow us to single out important factors for these expressions.

We assume that the derivative pays ϵy and has price $q(\epsilon)$ in the ϵ-economy. We also assume that $y = f(z)$, which makes y a derivative security, such as an option. We force the payoff of the derivative to be zero when $\epsilon = 0$; hence, $q(0) = 0$. This implies no loss of generality since any portion of the asset's return which is deterministic given ϵ will be equivalent to the bond, adding nothing to the asset span. We assume that the net supply of the derivative is zero since we want to model the introduction of a derivative security. For instance, $y = \max[0, (z-S)]$ represents a call option, and ϵy is the call option $\max[0, \epsilon z - \epsilon S]$ with strike price ϵS. This may initially seem odd, but it is a standard option if $\epsilon = 1$. Also, if $F(z)$ is the cdf of z then the probability of exercise, $F(S)$, is unaffected by ϵ.

We decompose y into components that are spanned by the stock and bond, and a component orthogonal to the stock and bond. We assume

$$y = \overline{y} + \alpha z + \nu \tag{28}$$

where $\overline{y}$ is the mean of y, α is the covariance with z, the risky component of equity, and a nonzero random variable ν, the innovation in y. Therefore, $0 = E\{\nu\} = E\{z\nu\}$. This formulation implicitly assumes that markets are initially incomplete since we assume that ν is not spanned by 1 and z. For example, if z is a random variable with only two possible values, then the stock and bond span the market and there is no $y = f(z)$ such that ν in (28) is not identically equal to zero[6].

We compute the equilibrium holdings and prices of both assets. Let θ_i and B_i be the equity and bond holdings, and let ϕ_i be the units of y held by trader i after trading. The final wealth for trader i is $Y_i = \theta_i(1 + \epsilon z) + B_i + \phi_i \epsilon y$, and his budget constraint is $\theta_i p + B_i + \phi_i q = B_i^e + \theta_i^e p$. When we use the budget constraint to

[6] We could add securities which generate random shocks, such as pure gambling. Since investors are risk averse, there is no demand for such assets. Therefore, we ignore assets with pure noise payoffs.

eliminate B_i, the first-order conditions for θ_i and ϕ_i are

$$\begin{aligned} E\{u_i^{'}(Y_i)(1+\epsilon z - p(\epsilon))\} &= 0,\ i=1,2 \\ E\{u_i^{'}(Y_i)(\epsilon y - q(\epsilon))\} &= 0,\ i=1,2 \end{aligned} \tag{29}$$

Equilibrium is defined by combining the first-order conditions of type 1 and type 2 agents with the market clearing conditions; we shall compute the equilibrium values for θ_i, ϕ_i, p, and q as functions of ϵ in some neighborhood of $\epsilon = 0$. Let θ and ϕ denote θ_1 and ϕ_1; hence $\theta_2 = 1 - \theta$ and $\phi_2 = -\phi$. Similar to the analysis of previous section, $\theta(0)$ and $\phi(0)$ are indeterminate but $p(0) = 1$ and $q(0) = 0$.

We need to determine an appropriate parameterization for this problem, just as we did in the case of equilibrium with one asset. We implicitly differentiate the four first-order conditions in (29) with respect to ϵ, and find that differentiability of q and p at $\epsilon = 0$ requires $[E\{y\} - q'(0)]u_i^{'}((B_i^e + \theta_i^e)) = 0$ and $[E\{z\} - p'(0)]u_i^{'}((B_i^e + \theta_i^e)) = 0$. Therefore, if q and π are well behaved, $q'(0) = \overline{y}$ and $p'(0) = E\{z\} = 0$. We want to solve for θ, ϕ, p,and q as functions of ϵ, at least in some neighborhood of $\epsilon = 0$, and we need $p(0) = 1$, $p'(0) = E\{z\} = 0$ and $q(0) = 0$, $q'(0) = \overline{y}$. We choose the following parameterization:

$$p(\epsilon) = 1 - \epsilon^2\pi(\epsilon),\ q(\epsilon) = \epsilon\overline{y} - \epsilon^2\psi(\epsilon) \tag{30}$$

We next check if the parameterization in (30) is consistent with Theorem 7. The bifurcation point $(\phi_0, \theta_0, \pi_0, \psi_0)$ is computed by solving the system of linear equations

$$\begin{bmatrix} u_1^{''}\sigma_y^2 & u_1^{''}\sigma_{yz} & 0 & -u_1^{'} \\ u_1^{''}\sigma_{yz} & u_1^{''}\sigma_z^2 & -u_1^{'} & 0 \\ u_2^{''}\sigma_{yz} & u_1^{''}\sigma_z^2 & u_2^{'} & 0 \\ u_2^{''}\sigma_y^2 & u_2^{''}\sigma_{yz} & 0 & u_2^{'} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \theta_0 \\ \pi_0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2^{''}\sigma_z^2 \\ u_2^{''}\sigma_{yz} \end{bmatrix}$$

which has the unique solution

$$\theta_0 = \frac{\tau_1}{\tau_1 + \tau_2},\ \phi_0 = 0,\ \pi_0 = \frac{1}{\tau_1 + \tau_2},\ \psi_0 = \frac{\sigma_{yz}}{\tau_1 + \tau_2}. \tag{31}$$

The existence of solutions for $\phi(\epsilon)$, $\theta(\epsilon)$, $\pi(\epsilon)$, and $\psi(\epsilon)$ near the bifurcation point is established by applying Theorem 7 at the candidate bifurcation point (31). Furthermore, the first-order derivatives $(\theta'(0),\ \phi'(0),\ \pi'(0),\ \psi'(0))$, the second-order derivatives $(\theta^{''}(0),\ \phi''((0),\ \pi^{''}(0),\ \psi^{''}(0))$, and other derivatives can be obtained by solving linear systems of equations as long as the utility function u is analytic at the deterministic consumption. Since the solutions are cumbersome, we omit them except for the first-order derivatives.

The results follow standard intuition. The equilibrium price of the derivative security is asymptotically equal to

$$q(\epsilon) = \epsilon E\{y\} - \epsilon^2 \frac{\sigma_{yz}}{\tau_1 + \tau_2} + O(\epsilon^3), \tag{32}$$

which tells us that the derivative y carries a positive risk premium (modelled here as a discount in the price) only if $\sigma_{yz} > 0$, that is, y is positively correlated with aggregate risk z. The limit price and holdings of equity are unaffected by the presence of the derivative, and trading volume for the derivative is zero in the limit.

We see again that a key step is finding an appropriate parameterization of asset prices. There is no precise, generally applicable formula describing how we arrived at the parameterization in (30) which allowed us to apply the Bifurcation Theorem, but the steps we have followed in the one- and two-asset problems are clear. We first compute derivatives of the equilibrium equations and examine them to see if some terms in the Taylor series of the unknown functions are fixed. For example, we found that $q'(0) = \overline{y}$ and $p'(0) = 0$ must be true if there is to be a coherent Taylor expansion. If conventional IFT methods indicate the value of low-order terms in an expansion, we then focus on the next higher-order term. Since $q'(0) = \overline{y}$ and $p'(0) = 0$, we then examined the parameterization in (30) where $\pi(\epsilon)$ and $\psi(\epsilon)$ became the unknown terms which could not be determined by applying the logic of the conventional IFT. We continued this for each unknown function until we reach a point where the terms in its expansion could not be fixed by the IFT. At that point we can apply the Bifurcation Theorem.

6.1. Trading Patterns for the Derivative Asset. We next determine the trading patterns of y. Since $\phi(0) = 0$, the value of $\phi'(0)$ determines the trading patterns for nonzero ϵ. Direct computation produces Theorem 14.

Theorem 14. *Type 1 investors buy the derivative y if and only if* $(\rho_1 - \rho_2)\, Cov(\nu, z^2) >$

0. *In general,*

$$\phi'(0) = \frac{\tau_1 \tau_2 (\rho_1 - \rho_2)}{(\tau_1 + \tau_2)^3} \frac{Cov(\nu, z^2)}{E\{\nu^2\}} \tag{33}$$

Recall that $\phi'(0) > 0$ means that trader 1 buys and trader 2 sells the derivative asset y. If type 1 investors have more skew tolerance and y provides the market with a new risk that is positively correlated with the tails of equity returns, then type 1 investors buy y and type 2 investors sell it. If $Cov(\nu, z^2) > 0$, the new asset y adds a type of riskiness that appeals to individuals with relatively high skew tolerance, and type 1(2) agents will buy y if $\rho_1 > \rho_2$ ($\rho_1 < \rho_2$).

If $Cov(\nu, z^2) = 0$ then we would need to examine $\phi''(0)$ to determine who buys the derivative. We do not pursue that here since no financial institution has an interest in introducing a derivative with no first-order volume. We continue to focus on derivatives where $Cov(\nu, z^2) \neq 0$.

6.2. Change in Equity Holdings. The derivative asset y may change investors' holdings of equity. Let $\theta^b(\epsilon)$ and $\theta^a(\epsilon)$ denote the equilibrium holding of equity by type 1 investors without and with the derivative security.[7] At $\epsilon = 0$, $\theta^b(\epsilon)$ and $\theta^a(\epsilon)$ will be the same since all assets will be equivalent. To compare the equilibria across these market structures, we compute the series expansion of both $\theta^b(\epsilon)$ and $\theta^a(\epsilon)$, and then use the difference in their series expansions to express the difference between the two market equilibria. We can do this for any index of market equilibrium. Direct computation shows Theorem 15.

Theorem 15. *Let $\theta^b(\epsilon)$ ($\theta^a(\epsilon)$) denote the equilibrium equity demand of type 1 investors without (with) the derivative y. Then*

$$\theta^a(\epsilon) - \theta^b(\epsilon) = -\frac{\tau_1 \tau_2 (\rho_1 - \rho_2)}{(\tau_1 + \tau_2)^3} \, \alpha \, \frac{Cov(\nu, z^2)}{E\{\nu^2\}} \, \epsilon + O(\epsilon^2) \tag{34}$$

If ν and z are uncorrelated, (34) reduces to zero, implying that the introduction of y has only $O(\epsilon^3)$ effects on the demand for the equity. If $\alpha = Cov(\nu, z) > 0$ then the change in type 1 investors' holding of equity is negatively related to their demand for y since (33) and (34) imply that $\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = -\,\alpha\,\phi'(0) + O(\epsilon^2)$.

[7] Loosely speaking, θ^b is equilibrium equity holding "before" introduction of y and θ^a is holding "after" introduction.

6.3. Price Effects of the Derivative Asset. Our computations show that the equilibrium price for equity remains unchanged up to $O(\epsilon^3)$ in its Taylor expansion. The fourth-order term reveals the dominant effect of the derivative y on the price of equity.

Theorem 16. *Let $P^a(\epsilon)$ $(P^b(\epsilon))$ denote the equilibrium price of equity with (without) the derivative y. The price difference is*

$$P^a(\epsilon) - P^b(\epsilon) = 2\frac{\tau_1\tau_2\left(\rho_1-\rho_2\right)^2}{\left(\tau_1+\tau_2\right)^5}\frac{E\left\{\nu z^2\right\}^2}{E\left\{\nu^2\right\}}\epsilon^4 + O(\epsilon^5) > 0$$

In particular, the equity rises in value and rises more as the derivative is more correlated to the tails of equity returns, and as investors differ more in their skewness tolerance.

Theorem 16 shows the elements that affect the impact of the derivative on stock price. The price change is always positive, but depends on third-order properties of the utility function. The derivative asset y complements equity and allows investors to allocate tail risk independent of other risks. This makes equity more attractive.

Also, the magnitude is proportional to the covariance of the derivative's innovation ν with the extremes of equity returns. If ν is uncorrelated with those extremes then there is no price change to the order ϵ^4. There may be a price effect but it would be an order of magnitude smaller asymptotically.

6.4. Welfare Effects of the Derivative Asset. We next derive the effect of a derivative on the welfare of each trader. Theory tells us that in one-good models such as ours, individual investors may gain or lose utility from adding an asset, but someone must gain. Our solutions will add some precision to those statements.

With the derivatives computed by the bifurcation method, we can study the welfare effect of the derivative y. Precisely, we shall expand the utility functions in terms of ϵ and examine the dominated term. Let $U_i^b(\epsilon)$ and $U_i^a(\epsilon)$ denote trader i's optimal utility levels without and with y. The utility effect can be expressed by $[U_i^a(\epsilon) - U_i^b(\epsilon)]/u_i'(B_i^e + \theta_i^e)$, a measure of the welfare change in terms of a consumption equivalent. The following theorem summarizes the result of our perturbation analysis.

Theorem 17. *Let $U_1^a(\epsilon)$ and $U_1^b(\epsilon)$ denote the equilibrium expected utility of type 1 investors with and without the derivative y. Then*

$$\frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1'} = \frac{\tau_1^2 \tau_2^2 (\rho_1 - \rho_2)^2}{2(\tau_1 + \tau_2)^6} \left(4\left(\frac{\theta_1^e}{\tau_1} - \frac{\theta_2^e}{\tau_2}\right) + \frac{1}{\tau_1}\right) \frac{E\{\nu z^2\}^2}{E\{\nu^2\}} \epsilon^4 + O(\epsilon^5)$$

The second trader's welfare change is symmetrically expressed.

Again, the result corresponds to basic theory. The key term is $4(\theta_1^e/\tau_1 - \theta_2^e/\tau_2) + \tau_1^{-1}$, which may be positive or negative. The term $\theta_1^e/\tau_1 - \theta_2^e/\tau_2$ is proportional to the amount of equity type one investors sell to type two investors in the limit as ϵ goes to zero. If there is no equity trade asymptotically then the dominant impact on utility is the improved opportunity for risk-sharing provided by the introduction of y. The risk-sharing gain is proportional to τ^{-1}, which is absolute risk aversion, for type i investors. If $\theta_1^e/\tau_1 - \theta_2^e/\tau_2 \neq 0$, the investor type that sells shares also gains from the equity price increase caused by the introduction of the derivative asset. So, one type gains from the price increase and the other loses, but both gain from new risk-sharing opportunities. One of the investors may lose, but not both.

The results in Theorems 14, 15, 16, and 17 demonstrate the importance of higher-order expansions. Linear-quadratic expansions would completely miss all of the effects studied in these theorems since $\rho = 0$ for linear-quadratic utility functions. Approximation methods that only use the first two derivatives of utility functions would incorrectly predict that adding y would have no effect on equilibrium. The advantage of the approach used here is that one need not make a choice about how many derivatives to use since that decision is automatically made by the power series generated by the bifurcation (and the IFT) approach. The mechanical computation of the power series expansions of equilibrium prices and quantities tells us which power of ϵ contains the asymptotically dominant effects, and which derivatives of utility and which moments of returns should be used.

7. COMPUTATIONAL CONSIDERATIONS

The analysis above focused on applying the bifurcation method to a simple asset market model. The results were obtained only after much computational effort. Theorem 17 is a good example of why the computer is necessary. Since the effect of the derivative asset y on utility was zero at orders ϵ^2 and ϵ^3, we had to compute the fourth-order Taylor series expansion of utility. Also, equilibrium utility is a function of all four variables determined in equilibrium, the two premia and the two portfolio variables. These four equilibrium variables are locally analytic functions of ϵ. Therefore, Theorem 17 required a fourth-order expansion of a four-dimensional function where each argument is a fourth-order Taylor series in ϵ. This resulted in thousands of intermediate terms. The final result in Theorem 17 is compact since almost all of the intermediate terms disappear when they are evaluated at $\epsilon = 0$. However, the intermediate terms must be kept until that last step. The computations in this paper took only a few minutes using Mathematica on a 400 MHz machine, but would be impossible for us to do without a computer.

This paper used the computer to derive algebraic formulas and theoretical asymptotic results. The computational burden was particularly heavy since we were interested in general formulas expressing the results in terms of elasticities, shares, and prices. The computational costs will rise rapidly as we move to larger problems with more types of investors and/or more assets. However, as we gained experience with the simple model we discovered patterns which we can incorporate into the code to substantially improve performance and make possible examination of more complex models. For example, the definitions of risk tolerance and skew tolerance, and the decomposition in (28) substantially reduced the complexity and length of the formulas. With Mathematica and these simplifications, we can now handle larger problems, such as problems with four investor types and four assets.

The Taylor series expansions for equilibrium price correspondences $p(\epsilon)$ and portfolio allocations $\theta(\epsilon)$ could also be used to arrive at numerical approximations for specific utility functions and asset return distributions. The bifurcation method then reduces to computing the numerical values of all derivatives of the equations defining equilibrium up to the fourth order at $\epsilon = 0$, and then executing numerical linear operations instead of symbolic operations. Since numerical operations are faster and more compact than symbolic operations, computing expansions for specific examples would be far faster. The computer could handle much larger problems if we specify all utility functions and returns.

We would like to know how well these formulas do for nontrivial ϵ. In general, a power series constructed by the IFT for analytic functions will have a positive radius of convergence, but we know nothing about its magnitude in general. However, there is a simple diagnostic which can help. Suppose that $h(x)$ is implicitly defined by $H(x, h(x)) = 0$ and that we construct the degree k Taylor series approximation $h^*(x)$ based at $x = x_0$. If $h^*(x)$ is a good approximation to $h(x)$ then $H(x, h^*(x))$ should be nearly zero. Once we have computed $h^*(x)$, we can evaluate its quality by computing $H(x, h^*(x))$ for various values of x. The behavior of $H(x, h^*(x))$ as x moves away from x_0 will indicate where the approximation can be trusted. Judd and Guu [14] applied this approach to similar approximations of stochastic growth models. We have constructed examples of the asset models studied in this paper for which our Taylor series approximations for $p(\epsilon)$ and $\theta(\epsilon)$ imply very small Euler equation errors. Roughly, we found that the method does well if the disturbance z has compact support, but does poorly if z is log Normal, a finding consistent with the fact that making z a log Normal random variable makes it unlikely that $H(\theta, \epsilon)$ is analytic.

More generally, we could compare the results of our approach for large ϵ with the numerical approach in Schmedders [23]. If our formulas work, then they would produce results faster than Schmedders [23], but our formulas will not work for the large ϵ cases where Schmedders' algorith would work. There could be a partnership between the two approaches with our Taylor-style expansions used to produce an initial guess for Schmedders' algorithm. Further discussion and serious examination of these numerical issues must be left for another paper.

We used Mathematica to compute our results. Space limitations prevent us from presenting and explaining the code here. The reader can obtain the code by sending e-mail to judd@hoover.stanford.edu, or by going to the webpage http://bucky.stanford.edu/ or the *Economic Theory* webpage for this paper.

8. Generalizations

This paper has examined a few simple problems, but we believe that the same tools can be used to examine a large class of models. We briefly discuss those claims here.

This paper assumed a single good, two types of agents, and only one source of risk. Space limitations prevent us from presenting an analysis for more general cases, but we can outline the general approach. Adding more types of agents and more assets but staying with one good is a direct generalization of the methods above. The equilibrium in our examples were expressed as first-order conditions for each agent with respect to each asset. Adding agents and assets just implies a longer list of first-order conditions but the key elements are unchanged: the deterministic consumption levels are fixed at the endowment, the price of risk, π, and portfolio allocations, θ, are indeterminate in the deterministic model, and we can parameterize θ so that the Bifurcation theorem applies to a system of equations $H(\pi, \theta, \epsilon) = 0$ which include individual first-order conditions and market-clearing conditions.

The generalization to several goods is more complex. Let p be the price vector for goods, π the vector of prices of risk for the assets, and θ the allocation of assets across agents. In GEI models with several goods, equilibrium can be expressed as the solution to a system of equations $H(p, \pi, \theta, \epsilon) = 0$ where the components of H are the agents' first-order conditions over asset and consumption choices plus feasibility conditions. The excess demand for assets may not exist at some prices because of arbitrage; therefore, H may not be continuous. However, theory tells us that equilibrium will generically exist. If we let ϵ parameterize uncertainty then a system $H(p, \pi, \theta, \epsilon) = 0$ would represent equilibrium in the ϵ-economy and implicitly define equilibrium maps $p(\epsilon)$, $\pi(\epsilon)$, and $\theta(\epsilon)$. At $\epsilon = 0$, the economy reduces to a deterministic Arrow-Debreu general equilibrium. There will be trade in the goods in the deterministic limit economy, and goods' prices $p(0)$ will be determined by equilibrium conditions. Asset prices in the deterministic limit, $q(0)$, will also be determined by $p(0)$. The goods prices and asset prices would generically be locally determinate by the standard general equilibrium theory. However, the portfolio decisions $\theta(0)$ will be indeterminate in the $\epsilon = 0$ economy since all assets would be perfect substitutes. If asset prices in general can be represented as $q = q_0 - \epsilon^2 \pi(\epsilon)$, just as in equation (22) for the two-asset case, then the limit prices for risk, $\pi(0)$, measured in terms of excess return per unit variance, will be indeterminate since the level of risk is zero.

The geometrical structure of the GEI problem is illustrated in Figure 3. Let the axis labeled Δ denote the price simplex for goods, and the axis labeled (π, θ) represent the prices of risk and portfolio allocations of the risky assets. As in Figures 1 and 2, the ϵ axis in Figure 3 represents the level of risk. Suppose that the arc ABC

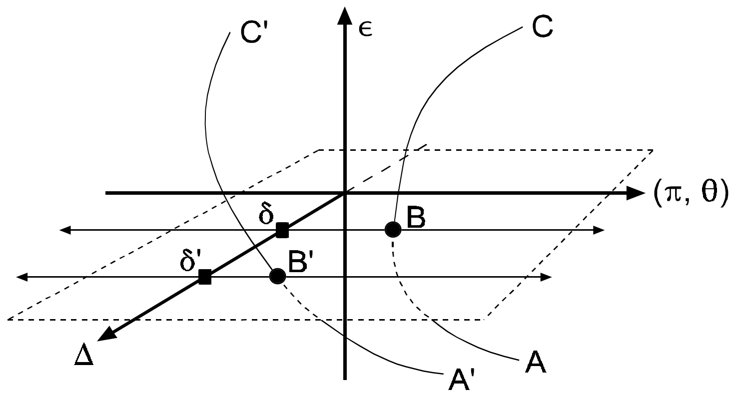


Figure 3: Bifurcation diagrams for general equilibrium problems

describes equilibrium values for p, π, and θ as ϵ changes. When $\epsilon = 0$, the problem reduces to an Arrow-Debreu model and equilibrium fixes goods' prices p at some point, say δ, in Δ, but the price of risk π and portfolio holdings would be indeterminate. Therefore, any point along the line $\overleftrightarrow{\delta B}$ would be an equilibrium. In order to analyze the arc ABC we need to find B. We analyze the Jacobian $H_{(p,\pi,\theta)}$ to find some suitable parameterization for $p(\epsilon)$, $\pi(\epsilon)$, and $\theta(\epsilon)$ such that the Bifurcation Theorem applies and produces B. The parameterization $q(\epsilon) = q_0 - \epsilon^2\pi(\epsilon)$ corresponds to the robust result that risk premia are related to the variance of risk, indicating that the Bifurcation Theorem should continue to apply. There may be cases where the bifurcation method used above does not apply, but we conjecture that this approach will often succeed since, generically, equilibrium does exist for endowment economies with incomplete asset markets.

The multicommodity case would produce more complex results. For example, there could be a second equilibrium arc, such as $A'B'C'$ which corresponds to a second set of equilibrium prices at δ' for goods in the deterministic economy. That does not present any essential difficulty as long as the local properties of the system of equilibrium equations $H(p, \pi, \theta, \epsilon) = 0$ satisfies the bifurcation theorem. Other complex possibilities may arise, such as multiple equilibrium arcs passing through a bifurcation point B. The bifurcation methods presented in this paper cannot handle such a case, but, fortunately, there are more powerful tools from bifurcation and singularity theory which could handle some of these problems. Presumably, the variety of welfare results in Hart, Elul, and Cass and Citanna, would also arise asymptotically in multigood economies. The key point is that the situation in Figure 3 is conceptually similar to the structure in Figures 1 and 2, and basic tools from bifurcation theory should be able to handle many multicommodity models.

9. Conclusion

We have used bifurcation approximation methods to examine simple asset market problems with small noise. The analysis produces a mean-variance-skewness-etc. theory of asset demand and asset market equilibrium, and found several interesting results. We found that the addition of derivative asset will increase the price of the underlying equity stock. Also, the demand for a derivative asset depends on skewness properties of asset returns and the relative skew tolerance of investors. These results indicate that skewness and skew tolerance will be important determinants of asset innovation in more general contexts and indicate that results from linear-quadratic or mean-variance models are of limited relevance. The approach also shows that, in small noise economies, equilibrium depends on the utility properties of traders and the moments of returns, not on the number of contingent states. The asymptotic approach provides more intuitive results than the usual state-contingent approach.

The mathematical tools are quite general and can be applied to far more complex problems. Zeidler shows that the critical bifurcation theorems hold in Banach spaces. For example, partial differential equations that characterize asset prices in continuous time can also be approximated by examining bifurcations of deterministic cases. The steps in such an application of the bifurcation theorem require the solution of linear partial differential equations.

This paper focussed on qualitative analyses, but the expansions derived here could have value as a numerical method for solving specific cases; we leave that possibility for another study. This paper focussed on applications of bifurcation methods but many of the same points could be made for applications of the IFT. Economists are familiar with comparative statics analysis, such as that in Jones [12], but that is generally limited to first-order expansions. Higher-order approximations could often be used to improve qualitative and quantitative analysis of economic models.

The necessary mathematics for deriving expansions have been known for a long time, but the cumbersome algebra made them impractical until now. Fortunately, the speed of modern computers and the availability of symbolic language software now makes bifurcation methods, and similar perturbation methods, a practical way to address important economic problems.

References

[1] Bensoussan, A.: Perturbation methods in optimal control. John Wiley and Sons, Chichester, 1988.

[2] Brown, D. J., DeMarzo, P. M., Eaves, B. C.: Computing equilibria when asset markets are incomplete. Econometrica **64**, 1-27 (1996)

[3] Cass, D., Citanna, A.: Pareto improving financial innovation in incomplete markets. Economic Theory **11,** (1998): 467-494

[4] Chiappori, P.-A., Geoffard, P. Y., and Guesnerie, R.: Sunspot fluctuations around a steady state: The case of multidimensional, one-step forward looking economic models. Econometrica **60,** 1097-1126 (1992).

[5] Chow, S.–N., and Hale J. K.: Methods of bifurcation theory, Springer–Verlag, New York, 1982.

[6] DeMarzo, P. M, and Eaves, B. C.: Computing equilibria of GEI by relocalization on a Grassmann manifold. Journal of Mathematical Economics, **26,** 479-497 (1996).

[7] Elul, R.: Welfare effects of financial innovation in incomplete markets economies with several consumption goods, Journal of Economic Theory **65**, 43-78 (1995).

[8] Fleming, W.: Stochastic control for small noise intensities. SIAM Journal of Control **9,** (1971).

[9] Fleming, W., and Souganides P. E.: Asymptotic series and the method of vanishing viscosity. Indiana University Mathematics Journal **35,** 425-447 (1986).

[10] Gaspar, J., and Judd, K. L.: Solving large-scale rational expectations models, Macroeconomic Dynamics **1**, 45-75 (1997).

[11] Hart, O.: On the optimality of equilibrium when the market structure is incomplete. Journal of Economic Theory **11**, 418-443 (1975).

[12] Jones, R. W.: The structure of simple general equilibrium models. Journal of Political Economy, **73,** 557-572 (1965).

[13] Judd, K. L.: Approximation, perturbation, and projection methods in economic analysis, in Handbook of Computational Economics, H. Amman, D. Kendrick, and J. Rust, eds. North Holland. 1996.

[14] Judd, K. L., and Guu, S.-M.: Perturbation solution methods for economic growth models, in Economic and Financial Modeling with Mathematica, Hal Varian, ed., Springer-Verlag: New York, 1993.

[15] Judd, K. L., and Guu, S.-M.: An asymptotic theory of financial innovation, mimeo, 2000.

[16] Kim, J., and Kim, S. H.: Spurious welfare reversals in international business cycle models, mimeo, 1999. (http:// www.stanford.edu/ group/SITE/ kim.pdf)

[17] Kimball, M. S.: Precautionary saving in the small and in the large. Econometrica **58,** 53-73 (1990).

[18] Kydland, F. E., and Prescott, E. C.: Time to build and aggregate fluctuations. Econometrica **50,** 1345-1370 (1982).

[19] Leisen, D., and Judd, K. L.: Pricing and hedging options in a stochastic volatility Model. mimeo, 1999.

[20] Magill, J. P. M.: A local analysis of N-sector capital accumulation under uncertainty. Journal of Economic Theory, **15,** 211–219 (1977).

[21] Magill, J. P. M., and Quinzii, M.: Theory of incomplete markets. Volume 1, Cambridge: MIT Press 1996.

[22] Samuelson, P. A.: The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments. Review of Economic Studies **37,** 537–542 (1970).

[23] Schmedders, K.: Computing equilibria in the general equilibrium model with incomplete asset markets. Journal of Economic Dynamics and Control, **22**, 1375-1401 (1998).

[24] Schmedders, K.: Monopolistic security design in finance economies. Economic Theory, this issue.

[25] Tesar, L.: Evaluating the gains from international risksharing. Carnegie-Rochester Conference Series on Public Policy, **42,** 95-143 (1995).

[26] Zeidler, E.: Nonlinear functional analysis and its applications: Volume I, Springer-Verlag: New York, 1986.

Mathematica Notebook for "Asymptotic Methods for Asset Market Equilibrium Analysis" by Judd and Guu

The following *Mathematica* program computes asset market equilibrium for two investors, one safe asset, and one or two risky assets.

The first command turns off annoying spelling queries.

```
Off[General::spell1]
```

■ The initial setup

■ Economic Model

Define the two risky assets' returns, Z and Y, in terms of zero-mean random variables z and y. ϵ is the scaling parameter, equal to the standard deviation in the ϵ-economy. Z is an asset with mean return R and variance ϵ^2, and is in positive net supply. It represents equity. To keep the formulas simple, we assume that the return on bonds, R, is 1. This is equivalent to assuming that the bond in the second period is the numeraire.

These facts imply that Z can be decomposed in the following manner:

```
Z = 1 + ϵ z;
```

Y is the derivative asset and has zero net supply. It has mean return $\epsilon\ \mu\text{Y}$, is partially correlated with Z through a $\epsilon\,\alpha\,z$ term, and has an orthogonal component $\epsilon\,y$. These facts imply that Y can be represented as

```
Y = ϵ μY  + ϵ α z + ϵ y;
```

The price of $Z\,(Y)$ is $p\,(q)$. We parametrize them in terms of the scaling parameter ϵ, the mean of Y, μY, and the premia of Z and Y, denoted Π and Ψ:

```
p = 1 - ϵ^2 Π ; q = ϵ μY - ϵ^2 Ψ ;
```

Note: The notation in the notebook corresponds to the notation in the paper except for a minor change. In *Mathematica*, the letter π is reserved for 3.14159... and so we could not use it as the risk premium for Z. Therefore, we use Π instead. To maintain the symmetry we also let Ψ denote the premium for Y.

W_i is final wealth of type i investor. θ is type 1 demand for equity. θe_i is type i endowment of equity and Be_i is type i endowment of bonds. ϕ is type 1's demand for Y; - ϕ is type 2 demand. In equilibrium, type 2 investors will hold

$\theta e_1 + \theta e_2 - \theta$ shares of stock. (in *Mathematica* prevent us from using θ_i^e (B_i^e) to denote type *i* endowment of the risky asset (bond).)

```
θe₁ = θe₁; θe₂ = θe₂;

W₁ = (Be₁ + p θe₁ - p θ - q ϕ) + θ * Z + ϕ Y
W₂ = (Be₂ + p θe₂ - p (θe₁ + θe₂ - θ) - q (-ϕ)) + (θe₁ + θe₂ - θ) * Z + (-ϕ) Y;
W₁ = Expand[W₁];
W₂ = Expand[W₂];
```

$$(1 + z\,\epsilon)\,\theta - \theta\,(1 - \epsilon^2\,\Pi) + (y\,\epsilon + z\,\alpha\,\epsilon + \epsilon\,\mu Y)\,\phi - \phi\,(\epsilon\,\mu Y - \epsilon^2\,\Psi) + Be_1 + (1 - \epsilon^2\,\Pi)\,\theta e_1$$

We compute the four first-order conditions for the two investors and the two risky assets. The bond demand is determined by substracting risky asset demand from initial wealth. We divide each first-order condition by ϵ to eliminate one degree of degeneracy. FOCij is the first-order condition of type *i* investor with respect to risky asset *j*, where *j* = 1 is equity and *j* = 2 refers to the derivative.

```
FOC11 = (1 / ϵ) D[u₁[W₁], θ] // Simplify
FOC12 = (1 / ϵ) D[u₁[W₁], ϕ] // Simplify
FOC21 = (1 / ϵ) D[u₂[W₂], θ] // Simplify
FOC22 = (1 / ϵ) D[u₂[W₂], ϕ] // Simplify
```

$$(z + \epsilon\,\Pi)\,u_1'[\epsilon\,(z\,\theta + \epsilon\,\theta\,\Pi + y\,\phi + z\,\alpha\,\phi + \epsilon\,\phi\,\Psi) + Be_1 + (1 - \epsilon^2\,\Pi)\,\theta e_1]$$

$$(y + z\,\alpha + \epsilon\,\Psi)\,u_1'[\epsilon\,(z\,\theta + \epsilon\,\theta\,\Pi + y\,\phi + z\,\alpha\,\phi + \epsilon\,\phi\,\Psi) + Be_1 + (1 - \epsilon^2\,\Pi)\,\theta e_1]$$

$$-(z + \epsilon\,\Pi)\,u_2'[-z\,\epsilon\,\theta - \epsilon^2\,\theta\,\Pi - y\,\epsilon\,\phi - z\,\alpha\,\epsilon\,\phi - \epsilon^2\,\phi\,\Psi + Be_2 + \epsilon\,(z + \epsilon\,\Pi)\,\theta e_1 + \theta e_2 + z\,\epsilon\,\theta e_2]$$

$$-(y + z\,\alpha + \epsilon\,\Psi)\,u_2'[-z\,\epsilon\,\theta - \epsilon^2\,\theta\,\Pi - y\,\epsilon\,\phi - z\,\alpha\,\epsilon\,\phi - \epsilon^2\,\phi\,\Psi + Be_2 + \epsilon\,(z + \epsilon\,\Pi)\,\theta e_1 + \theta e_2 + z\,\epsilon\,\theta e_2]$$

Even though we have divided by ϵ, these first-order conditions describe the situation at ϵ=0.
These four first-order conditions define equilibrium. Define *G* to be the vector of first-order conditions.

■ Moment substitutions and other substitutions

We next define lists of substitutions that will allow us to compute the moments of z and y. The doubly subscripted term $\mu_{m,n}$ dentoes the expectation of $z^m\,y^n$. We will often need to compute expectations of our Taylor series expansions. We could invoke *Mathematica*'s Integrate command, but we can do better by searching for specific terms in our polynomials and replacing them with their integrals. For example, whenever we see an isolated *z* in an expression we know that when the integral just it is replaces the *z* term by the mean of *z*, which is denoted as $\mu_{1,0}$. This works better than *Mathematica* since we know that our function is a polynomial whereas *Mathematica* needs to spend time to ascertain that fact. The following replacement rules implement this approach to integration.

```
ownmoments = {z → μ1,0, z^m_ → μm,0, y → μ0,1, y^m_ → μ0,m}
crossmoments = {z^m_ y^n_ → μm,n, z y^n_ → μ1,n, z^m_ y → μm,1}
```

$$\{z \to \mu_{1,0},\ z^{m\_} \to \mu_{m,0},\ y \to \mu_{0,1},\ y^{m\_} \to \mu_{0,m}\}$$

$$\{y^{n\_}\,z^{m\_} \to \mu_{m,n},\ y^{n\_}\,z \to \mu_{1,n},\ y\,z^{m\_} \to \mu_{m,1}\}$$

Our definition of Y decomposed it into its mean, covariance with Z, and residual. Without loss of generality, we set the mean of z and y to zero.

```
zeromean = {μ0,1 → 0, μ1,0 → 0};
```

We can also make z and y orthogonal and set the variance of z equal to 1.

```
orthogonal = {μ1,1 → 0, μ2,0 → 1};
```

We now gather all the substitions into one list.

```
moments = Join[ownmoments, crossmoments, zeromean, orthogonal]
```

```
{z → μ1,0, z^m_ → μm,0, y → μ0,1, y^m_ → μ0,m, y^n_ z^m_ → μm,n,
 y^n_ z → μ1,n, y z^m_ → μm,1, μ0,1 → 0, μ1,0 → 0, μ1,1 → 0, μ2,0 → 1}
```

The following substitutions apply in the case where there is no derivative security. They say that all cross-moments between z and y are zero. Note that y must be nonzero since we set its variance to 1. However, if all of its moments are uncorrelated with all moments of the endowment then it is a pure gamble and will not be traded in equilibrium.

```
NoDeriv = Table[μm,n → 0, {m, 1, 6}, {n, 1, 6}] // Flatten
```

```
{μ1,1 → 0, μ1,2 → 0, μ1,3 → 0, μ1,4 → 0, μ1,5 → 0, μ1,6 → 0, μ2,1 → 0, μ2,2 → 0, μ2,3 → 0,
 μ2,4 → 0, μ2,5 → 0, μ2,6 → 0, μ3,1 → 0, μ3,2 → 0, μ3,3 → 0, μ3,4 → 0, μ3,5 → 0, μ3,6 → 0,
 μ4,1 → 0, μ4,2 → 0, μ4,3 → 0, μ4,4 → 0, μ4,5 → 0, μ4,6 → 0, μ5,1 → 0, μ5,2 → 0, μ5,3 → 0,
 μ5,4 → 0, μ5,5 → 0, μ5,6 → 0, μ6,1 → 0, μ6,2 → 0, μ6,3 → 0, μ6,4 → 0, μ6,5 → 0, μ6,6 → 0}
```

We define a convenient list of simplifying substitutions which *Mathematica* did not automatically execute.

$$\text{sub1} = \left\{\frac{1}{2}\,(2\,\text{Be}_1 + 2\,\theta\text{e}_1) \to (\text{Be}_1 + \theta\text{e}_1),\ \frac{1}{2}\,(2\,\text{Be}_2 + 2\,\theta\text{e}_2) \to (\text{Be}_2 + \theta\text{e}_2)\right\}$$

$$\left\{\frac{1}{2}\,(2\,\text{Be}_1 + 2\,\theta\text{e}_1) \to \text{Be}_1 + \theta\text{e}_1,\ \frac{1}{2}\,(2\,\text{Be}_2 + 2\,\theta\text{e}_2) \to \text{Be}_2 + \theta\text{e}_2\right\}$$

■ Solvability Condition in Theorem 4

We check for the nonsingularity of the solvability matrix, $G_{\epsilon,\Lambda}$ where $\Lambda=(\theta,\phi,\Pi,\Psi)$. First define the solvability matrix:

```
Gϵ,Λ =
  {{D[FOC11, {ϵ, 1}, θ], D[FOC11, {ϵ, 1}, ϕ], D[FOC11, {ϵ, 1}, Ψ], D[FOC11, {ϵ, 1}, Π]},
   {D[FOC12, {ϵ, 1}, θ], D[FOC12, {ϵ, 1}, ϕ], D[FOC12, {ϵ, 1}, Ψ], D[FOC12, {ϵ, 1}, Π]},
   {D[FOC21, {ϵ, 1}, θ], D[FOC21, {ϵ, 1}, ϕ], D[FOC21, {ϵ, 1}, Ψ], D[FOC21, {ϵ, 1}, Π]},
   {D[FOC22, {ϵ, 1}, θ], D[FOC22, {ϵ, 1}, ϕ], D[FOC22, {ϵ, 1}, Ψ], D[FOC22, {ϵ, 1}, Π]}};
```

Evaluate $G_{\epsilon,\Lambda}$ at ϵ=0 and display it in matrix form:

```
Gϵ,Λ = Gϵ,Λ /. {ϵ -> 0};
Gϵ,Λ // MatrixForm
```

$$\begin{pmatrix} z^2\, u_1''[Be_1+\theta e_1] & z\,(y+z\,\alpha)\, u_1''[Be_1+\theta e_1] & 0 & u_1'[Be_1+\theta e_1] \\ z\,(y+z\,\alpha)\, u_1''[Be_1+\theta e_1] & (y+z\,\alpha)^2\, u_1''[Be_1+\theta e_1] & u_1'[Be_1+\theta e_1] & 0 \\ z^2\, u_2''[Be_2+\theta e_2] & -z\,(-y-z\,\alpha)\, u_2''[Be_2+\theta e_2] & 0 & -u_2'[Be_2+\theta e_2] \\ z\,(y+z\,\alpha)\, u_2''[Be_2+\theta e_2] & -(-y-z\,\alpha)\,(y+z\,\alpha)\, u_2''[Be_2+\theta e_2] & -u_2'[Be_2+\theta e_2] & 0 \end{pmatrix}$$

Integrate the elements of $G_{\epsilon,\Lambda}$ by replacing powers of z and y with their moments.

```
ExpG = Expand[Gϵ,Λ] //. moments;
ExpG // MatrixForm
```

$$\begin{pmatrix} u_1''[Be_1+\theta e_1] & \alpha\, u_1''[Be_1+\theta e_1] & 0 & u_1'[Be_1+\theta e_1] \\ \alpha\, u_1''[Be_1+\theta e_1] & \alpha^2\, u_1''[Be_1+\theta e_1]+\mu_{0,2}\, u_1''[Be_1+\theta e_1] & u_1'[Be_1+\theta e_1] & 0 \\ u_2''[Be_2+\theta e_2] & \alpha\, u_2''[Be_2+\theta e_2] & 0 & -u_2'[Be_2+\theta e_2] \\ \alpha\, u_2''[Be_2+\theta e_2] & \alpha^2\, u_2''[Be_2+\theta e_2]+\mu_{0,2}\, u_2''[Be_2+\theta e_2] & -u_2'[Be_2+\theta e_2] & 0 \end{pmatrix}$$

Compute the determinant of $E\,\{G_{\epsilon,\Lambda}\}$.

```
(Det[ExpG] // Simplify) //. moments
```

$$-\mu_{0,2}\,(u_2'[Be_2+\theta e_2]\, u_1''[Be_1+\theta e_1]+u_1'[Be_1+\theta e_1]\, u_2''[Be_2+\theta e_2])^2$$

As long as $\mu_{0,2}$ is not zero, this determinant is strictly positive and $G_{\epsilon,\Lambda}$ is nonsingular

■ Bifurcation equations: series construction

We construct substitutions that will define the equilibrium mappings from the scaling parameter ϵ to the equilibrium values of (θ,ϕ,Π,Ψ).

```
PortfolioSubs = {θ → θ[ϵ], ϕ → ϕ[ϵ]}
```

$\{\theta \to \theta[\epsilon],\ \phi \to \phi[\epsilon]\}$

```
PremiaSubs = {Π → Π[ϵ], Ψ → Ψ[ϵ]}
```

$\{\Pi \to \Pi[\epsilon],\ \Psi \to \Psi[\epsilon]\}$

To construct the equilibrium equations we take the first-order conditions and replace the the portfolio variables, θ and ϕ, and the risk premium variables, Π and Ψ, with their equilibrium maps.

```
EQM11 = FOC11 /. PortfolioSubs /. PremiaSubs
EQM21 = FOC21 /. PortfolioSubs /. PremiaSubs
EQM12 = FOC12 /. PortfolioSubs /. PremiaSubs
EQM22 = FOC22 /. PortfolioSubs /. PremiaSubs
```

$(z + \epsilon \Pi[\epsilon])\, u_1'[Be_1 + \theta e_1 (1 - \epsilon^2 \Pi[\epsilon]) + \epsilon (z\,\theta[\epsilon] + \epsilon\,\theta[\epsilon]\,\Pi[\epsilon] + y\,\phi[\epsilon] + z\,\alpha\,\phi[\epsilon] + \epsilon\,\phi[\epsilon]\,\Psi[\epsilon])]$

$-(z + \epsilon \Pi[\epsilon])\, u_2'[Be_2 + \theta e_2 + z\,\epsilon\,\theta e_2 - z\,\epsilon\,\theta[\epsilon] - \epsilon^2\,\theta[\epsilon]\,\Pi[\epsilon] + \epsilon\,\theta e_1 (z + \epsilon \Pi[\epsilon]) - y\,\epsilon\,\phi[\epsilon] - z\,\alpha\,\epsilon\,\phi[\epsilon] - \epsilon^2\,\phi[\epsilon]\,\Psi[\epsilon]]$

$(y + z\,\alpha + \epsilon \Psi[\epsilon])\, u_1'[Be_1 + \theta e_1 (1 - \epsilon^2 \Pi[\epsilon]) + \epsilon (z\,\theta[\epsilon] + \epsilon\,\theta[\epsilon]\,\Pi[\epsilon] + y\,\phi[\epsilon] + z\,\alpha\,\phi[\epsilon] + \epsilon\,\phi[\epsilon]\,\Psi[\epsilon])]$

$-(y + z\,\alpha + \epsilon \Psi[\epsilon])\, u_2'[Be_2 + \theta e_2 + z\,\epsilon\,\theta e_2 - z\,\epsilon\,\theta[\epsilon] - \epsilon^2\,\theta[\epsilon]\,\Pi[\epsilon] + \epsilon\,\theta e_1 (z + \epsilon \Pi[\epsilon]) - y\,\epsilon\,\phi[\epsilon] - z\,\alpha\,\epsilon\,\phi[\epsilon] - \epsilon^2\,\phi[\epsilon]\,\Psi[\epsilon]]$

The equilibrium equations are really the expectations 0=E{EQMij}. We will execute the integrations later, after the Taylor series expansions, since it is permissible to interchange the integration and differentiation operations.

We define abbrevations for the derivatives; this helps make the formulas more understandable than the ones automatically generated by *Mathematica.*

```
UtilDerivs = {Derivative[n_][ui_][Bei_ + θei_] -> Ai[n]}
```

$\{u_{i\_}^{(n\_)}[Be_{i\_} + \theta e_{i\_}] \to A_i[n]\}$

We want to replace derivatives of the utility function, the $A_i[m]$ terms, with risk tolerance, τ_i, skew tolerance, ρ_i, and kurtosis tolerance, κ_i. The next substitution rule does that.

$$\texttt{utilparams} = \left\{A_{i\_}[2] \to -\frac{A_i[1]}{\tau_i},\ A_{i\_}[3] \to \frac{2\,\rho_i\,A_i[1]}{\tau_i{}^2},\ A_{i\_}[4] \to \frac{3\,\kappa_i\,A_i[1]}{\tau_i{}^3},\ A_{i\_}[1] \to 1\right\}$$

$$\left\{A_{i\_}[2] \to -\frac{A_i[1]}{\tau_i},\ A_{i\_}[3] \to \frac{2\,\rho_i\,A_i[1]}{\tau_i^2},\ A_{i\_}[4] \to \frac{3\,\kappa_i\,A_i[1]}{\tau_i^3},\ A_{i\_}[1] \to 1\right\}$$

It is often useful to refer to the social risk tolerance, T, and express type i risk tolerance as a share, ν_i, of total risk tolerance. The following replacement rule allows us to do that.

```
taureps = {τ1 + τ2 → T, τi_ → υi T, υ1 + υ2 → 1};
```

We now compute the power series expansions of the four equilibrium conditions. We fully expand each series so that the various powers of z and y are gathered together. This is necessary for our integration approach to work.

```
EqmPow11 = (Normal[Series[EQM11 , {ϵ, 0, 6}]] /. sub1) //. UtilDerivs;
EqmPow11 = Expand[EqmPow11];
EqmPow12 = (Normal[Series[EQM12 , {ϵ, 0, 6}]] /. sub1) //. UtilDerivs;
EqmPow12 = Expand[EqmPow12];
EqmPow21 = (Normal[Series[EQM21 , {ϵ, 0, 6}]] /. sub1) //. UtilDerivs;
EqmPow21 = Expand[EqmPow21];
EqmPow22 = (Normal[Series[EQM22 , {ϵ, 0, 6}]] /. sub1) //. UtilDerivs;
EqmPow22 = Expand[EqmPow22];
```

We now take the expectation of each equilibrium equation's power series. Since z and y are the only random variables, we replace each power of z and y and each crossproduct by the appropriate moment.

```
EqmPow11 = EqmPow11 //. moments;
EqmPow12 = EqmPow12 //. moments;
EqmPow21 = EqmPow21 //. moments;
EqmPow22 = EqmPow22 //. moments;
```

The final step in constructing the equilibrium expressions is to collect terms of like powers of ϵ and list the coefficients of each power of ϵ. This puts the equations in the proper arrangement for solving the problem.

```
eqns11 = CoefficientList[EqmPow11, ϵ];
eqns12 = CoefficientList[EqmPow12, ϵ];
eqns21 = CoefficientList[EqmPow21, ϵ];
eqns22 = CoefficientList[EqmPow22, ϵ];
```

■ Solve the individual equations in sequence

■ Equation 1:

The coefficients of the ϵ^0 components of the equilibrium power series should be zero. We now check what they are.

```
{eqns11[[1]], eqns21[[1]], eqns12[[1]], eqns22[[1]]}
```

```
{0, 0, 0, 0}
```

This shows that we can continue. If this were not a vector of zeroes then we would know that our parameterization did not fulfill the necessary conditions of the Bifurcation Theorem.

■ Equation 2:

We compute the bifurcation point by choosing ($\theta[0],\phi[0],\Psi[0],\Pi[0]$) so that the ϵ components of the equilibrium equation expansion are zero. We first list the equations

```
eq1 = eqns11[[2]] // Simplify // Expand
eq2 = eqns21[[2]] // Simplify // Expand
eq3 = eqns12[[2]] // Simplify // Expand
eq4 = eqns22[[2]] // Simplify // Expand
```

$\Pi[0]\,A_1[1] + \theta[0]\,A_1[2] + \alpha\,\phi[0]\,A_1[2]$

$-\Pi[0]\,A_2[1] - \theta e_1\,A_2[2] - \theta e_2\,A_2[2] + \theta[0]\,A_2[2] + \alpha\,\phi[0]\,A_2[2]$

$\Psi[0]\,A_1[1] + \alpha\,\theta[0]\,A_1[2] + \alpha^2\,\phi[0]\,A_1[2] + \mu_{0,2}\,\phi[0]\,A_1[2]$

$-\Psi[0]\,A_2[1] - \alpha\,\theta e_1\,A_2[2] - \alpha\,\theta e_2\,A_2[2] + \alpha\,\theta[0]\,A_2[2] + \alpha^2\,\phi[0]\,A_2[2] + \mu_{0,2}\,\phi[0]\,A_2[2]$

We next solve for bifurcation point.

```
BifPt = Solve[{eq1 == 0, eq2 == 0, eq3 == 0, eq4 == 0}, {θ[0], ϕ[0], Ψ[0], Π[0]}][[1]]
```

$$\left\{\Psi[0] \to -\frac{\alpha A_1[2] (\theta e_1 A_2[2] + \theta e_2 A_2[2])}{A_1[2] A_2[1] + A_1[1] A_2[2]},\ \theta[0] \to \frac{A_1[1] (\theta e_1 A_2[2] + \theta e_2 A_2[2])}{A_1[2] A_2[1] + A_1[1] A_2[2]},\ \phi[0] \to 0,\ \Pi[0] \to -\frac{A_1[2] (\theta e_1 A_2[2] + \theta e_2 A_2[2])}{A_1[2] A_2[1] + A_1[1] A_2[2]}\right\}$$

We simplify the expression, bringing together common factors.

```
BifPt = BifPt // Simplify
```

$$\left\{\Psi[0] \to -\frac{\alpha (\theta e_1 + \theta e_2) A_1[2] A_2[2]}{A_1[2] A_2[1] + A_1[1] A_2[2]},\ \theta[0] \to \frac{(\theta e_1 + \theta e_2) A_1[1] A_2[2]}{A_1[2] A_2[1] + A_1[1] A_2[2]},\ \phi[0] \to 0,\ \Pi[0] \to -\frac{(\theta e_1 + \theta e_2) A_1[2] A_2[2]}{A_1[2] A_2[1] + A_1[1] A_2[2]}\right\}$$

We next simplify by using the equilibrium conditions. Essentially, we want to replace all occurrences of $\theta e_1 + \theta e_2$ terms, the aggregate endowment of the risky asset, with its value, 1.
Let Θ be the total endowment of the risky asset, which we set equal to 1. EqmSubs defines various substitutions that express the identity $\theta e_2 + \theta e_1 = \Theta$, and will allow us to simplify various expressions.

```
Θ = 1; EqmSubs = {θe₂ + θe₁ → Θ, θe₂ + θe₁ - Θ → 0, Θ - θe₂ - θe₁ → 0}
```

$$\{\theta e_1 + \theta e_2 \to 1,\ -1 + \theta e_1 + \theta e_2 \to 0,\ 1 - \theta e_1 - \theta e_2 \to 0\}$$

Simplify BifPt by uskng the substitutions in EqmSubs

```
BifPt = BifPt //. EqmSubs
```

$$\left\{\Psi[0] \to -\frac{\alpha A_1[2] A_2[2]}{A_1[2] A_2[1] + A_1[1] A_2[2]},\ \theta[0] \to \frac{A_1[1] A_2[2]}{A_1[2] A_2[1] + A_1[1] A_2[2]},\ \phi[0] \to 0,\ \Pi[0] \to -\frac{A_1[2] A_2[2]}{A_1[2] A_2[1] + A_1[1] A_2[2]}\right\}$$

We do not like the utility derivatives, $A_i[j]$. We apply substitutions contained in utilparams, defined above, that replace utility derivatives with indices such as risk tolerance.

```
BifPt = BifPt //. utilparams
```

$$\left\{\Psi[0] \to -\frac{\alpha}{\tau_1 \left(-\frac{1}{\tau_1} - \frac{1}{\tau_2}\right) \tau_2},\ \theta[0] \to -\frac{1}{\left(-\frac{1}{\tau_1} - \frac{1}{\tau_2}\right) \tau_2},\ \phi[0] \to 0,\ \Pi[0] \to -\frac{1}{\tau_1 \left(-\frac{1}{\tau_1} - \frac{1}{\tau_2}\right) \tau_2}\right\}$$

We need to simplify this expression for BifPt. The Simplify command can handle this for us. (In general, one must be careful using the Simplify command since it can often take a long time to find the desired simplification or it will find a simplification other than the one you want.)

```
BifPt = BifPt // Simplify
```

$$\left\{\Psi[0] \to \frac{\alpha}{\tau_1 + \tau_2},\ \theta[0] \to \frac{\tau_1}{\tau_1 + \tau_2},\ \phi[0] \to 0,\ \Pi[0] \to \frac{1}{\tau_1 + \tau_2}\right\}$$

We now have an intuitive expression for the bifurcation point. The procedure was direct: solve for the bifurcation point in terms of utility derivatives and endowments, and then apply various simplifications to transform the solution into an expression involving elasticities and shares. Below we will apply similar sequences of simplifications without explanation.

The bifurcation point conforms to basic intutions. The risk premium, `Π[0]`, equals the inverse of the total risk tolerance, $\frac{1}{\tau_1+\tau_2}$. Type 1 holdings of equity equals it share of social risk tolerance. The risk premium of the derivative asset Y, Ψ[0], equals $\frac{\alpha}{\tau_1+\tau_2}$ which is the product of the risk premium of equity and the covariance of Y with z. (This is equation (32) in the paper). The most surprising result is that the asymptotic holding of the derivative asset is zero. This does not mean that there is no demand for the derivative; it means that demand is smaller than order 0. We will see below that demand is of order ϵ.

■ Equation 3:

We next compute (θ'[0],ϕ'[0],Ψ'[0],Π'[0]) by setting the ϵ^2 components of the equilibrium equations Taylor series expansions equal to zero. We immediately make the substitutions stated in utilparams.

```
eq1 = eqns11[[3]] //. utilparams // Expand;
eq2 = eqns21[[3]] //. utilparams // Expand;
eq3 = eqns12[[3]] //. utilparams // Expand;
eq4 = eqns22[[3]] //. utilparams // Expand;
```

Here is the first equation; the rest are similar.

```
eq1
```

$$\frac{\rho_1\,\mu_{3,0}\,\theta[0]^2}{\tau_1^2}+\frac{2\,\rho_1\,\mu_{2,1}\,\theta[0]\,\phi[0]}{\tau_1^2}+\frac{2\,\alpha\,\rho_1\,\mu_{3,0}\,\theta[0]\,\phi[0]}{\tau_1^2}+\frac{\rho_1\,\mu_{1,2}\,\phi[0]^2}{\tau_1^2}+\frac{2\,\alpha\,\rho_1\,\mu_{2,1}\,\phi[0]^2}{\tau_1^2}+\frac{\alpha^2\,\rho_1\,\mu_{3,0}\,\phi[0]^2}{\tau_1^2}-\frac{\theta'[0]}{\tau_1}+\Pi'[0]-\frac{\alpha\,\phi'[0]}{\tau_1}$$

We now solve for the unknowns, θ'[0], ϕ'[0], Ψ'[0], and Π'[0]. We suppress printing the first set of results since they take up too much space. The substitutions we make below will produce comprehensible expressions.

```
Sol2 = Solve[{eq1 == 0, eq2 == 0, eq3 == 0, eq4 == 0}, {θ'[0], ϕ'[0], Ψ'[0], Π'[0]}][[1]];
```

The solution involves , θ[0], ϕ[0], Ψ[0], and Π[0]. We now replace these terms with the values found in our bifurcation point solution, BifPt.

```
Sol2 = Sol2 /. BifPt;
```

Note: we could have made these substitutions when we defined eq1, eq2, etc, but that would not be wise. If we had done that then the expressions in eq1, eq2, etc., would have been larger, making more work for *Mathematica*. This would not have been a major problem for this set of equations, but becomes important when we move to higher-order terms where the solutions become more complex.

We next simplify the result which produces a fairly compact form.

```
Sol2 = Sol2 // Simplify;
Sol2 // TableForm
```

$\Psi'[0] \to -\frac{(\rho_1\,\tau_1\,\tau_2+\rho_2\,((-1+\theta e_1+\theta e_2)\,\tau_1+(\theta e_1+\theta e_2)\,\tau_2)^2)\,(\mu_{2,1}+\alpha\,\mu_{3,0})}{\tau_2\,(\tau_1+\tau_2)^3}$

$\theta'[0] \to \frac{\tau_1\,(-\rho_1\,\tau_2^2+\rho_2\,((-1+\theta e_1+\theta e_2)\,\tau_1+(\theta e_1+\theta e_2)\,\tau_2)^2)\,(\alpha\,\mu_{2,1}-\mu_{0,2}\,\mu_{3,0})}{\tau_2\,(\tau_1+\tau_2)^3\,\mu_{0,2}}$

$\phi'[0] \to -\frac{\tau_1\,(-\rho_1\,\tau_2^2+\rho_2\,((-1+\theta e_1+\theta e_2)\,\tau_1+(\theta e_1+\theta e_2)\,\tau_2)^2)\,\mu_{2,1}}{\tau_2\,(\tau_1+\tau_2)^3\,\mu_{0,2}}$

$\Pi'[0] \to -\frac{(\rho_1\,\tau_1\,\tau_2+\rho_2\,((-1+\theta e_1+\theta e_2)\,\tau_1+(\theta e_1+\theta e_2)\,\tau_2)^2)\,\mu_{3,0}}{\tau_2\,(\tau_1+\tau_2)^3}$

We can further simplify it by applying the equilibrium substitutions.

```
Sol2 = Sol2 /. EqmSubs // Simplify;
Sol2 // TableForm
```

$\Psi'[0] \to -\frac{(\rho_1\,\tau_1+\rho_2\,\tau_2)\,(\mu_{2,1}+\alpha\,\mu_{3,0})}{(\tau_1+\tau_2)^3}$

$\theta'[0] \to \frac{(\rho_1-\rho_2)\,\tau_1\,\tau_2\,(-\alpha\,\mu_{2,1}+\mu_{0,2}\,\mu_{3,0})}{(\tau_1+\tau_2)^3\,\mu_{0,2}}$

$\phi'[0] \to \frac{(\rho_1-\rho_2)\,\tau_1\,\tau_2\,\mu_{2,1}}{(\tau_1+\tau_2)^3\,\mu_{0,2}}$

$\Pi'[0] \to -\frac{(\rho_1\,\tau_1+\rho_2\,\tau_2)\,\mu_{3,0}}{(\tau_1+\tau_2)^3}$

The solution for $\phi'[0]$ in Sol2 proves the assertions in Theorem 9.

If there is no derivative security, then the solution is

```
Sol2NoY = Sol2 //. NoDeriv
```

$$\left\{\Psi'[0] \to -\frac{\alpha\,(\rho_1\,\tau_1+\rho_2\,\tau_2)\,\mu_{3,0}}{(\tau_1+\tau_2)^3},\ \theta'[0] \to \frac{(\rho_1-\rho_2)\,\tau_1\,\tau_2\,\mu_{3,0}}{(\tau_1+\tau_2)^3},\ \phi'[0] \to 0,\ \Pi'[0] \to -\frac{(\rho_1\,\tau_1+\rho_2\,\tau_2)\,\mu_{3,0}}{(\tau_1+\tau_2)^3}\right\}$$

This proves Theorem 8. The solution for $\theta'[0]$ in Sol2NoY proves the assertion in equation (26) and the solution for $\Pi'[0]$ in Sol2NoY proves the assertion in equation (27).

■ Compute the change in equity demand from introduction of Y

Notice that the presence of Y has no impact on the equity premium derivative, $\Pi'[0]$, but it does affect equity demand, $\theta'[0]$. So, we now determine that effect.

If there are two assets, then $\theta'[0]$ is

```
twoasset = θ'[0] /. BifPt /. Sol2
```

$$\frac{(\rho_1-\rho_2)\,\tau_1\,\tau_2\,(-\alpha\,\mu_{2,1}+\mu_{0,2}\,\mu_{3,0})}{(\tau_1+\tau_2)^3\,\mu_{0,2}}$$

If α=0 then the derivative asset has no impact on equilibrium and it is as if it did not exist. The substitutions in NoDeriv (defined above) state that the derivative is uncorrelated with equity at all moments, representing the case where there is no derivative. Therefore, equity demand in its absence is

```
singleasset = twoasset /. NoDeriv
```

$$\frac{(\rho_1 - \rho_2)\ \tau_1\ \tau_2\ \mu_{3,0}}{(\tau_1 + \tau_2)^3}$$

The change in equity demand is the difference:

```
twoasset - singleasset // Simplify
```

$$\frac{\alpha\ (-\rho_1 + \rho_2)\ \tau_1\ \tau_2\ \mu_{2,1}}{(\tau_1 + \tau_2)^3\ \mu_{0,2}}$$

This is the result reported in Theorem 10.

■ Equation 4:

We next compute (θ"[0],ϕ"[0],Ψ"[0],Π"[0]) by setting the ϵ^3 components of the equilibrium equations equal to zero.

```
eq1 = eqns11[[4]] //. utilparams /. Sol2 /. BifPt /. {θe2 → Θ - θe1} // Expand;
eq2 = eqns21[[4]] //. utilparams /. Sol2 /. BifPt /. {θe2 → Θ - θe1} // Expand;
eq3 = eqns12[[4]] //. utilparams /. Sol2 /. BifPt /. {θe2 → Θ - θe1} // Expand;
eq4 = eqns22[[4]] //. utilparams /. Sol2 /. BifPt /. {θe2 → Θ - θe1} // Expand;
```

Here is the first equation; the rest are similar.

```
eq1
```

$$-\frac{1}{(\tau_1+\tau_2)^3} + \frac{3\,\rho_1}{(\tau_1+\tau_2)^3} + \frac{\theta e_1}{\tau_1\,(\tau_1+\tau_2)^2} - \frac{2\,\theta e_1\,\rho_1}{\tau_1\,(\tau_1+\tau_2)^2} + \frac{2\,\rho_1^2\,\tau_2\,\mu_{2,1}^2}{(\tau_1+\tau_2)^4\,\mu_{0,2}} - \frac{2\,\rho_1\,\rho_2\,\tau_2\,\mu_{2,1}^2}{(\tau_1+\tau_2)^4\,\mu_{0,2}} + \frac{2\,\rho_1^2\,\tau_2\,\mu_{3,0}^2}{(\tau_1+\tau_2)^4} - \frac{2\,\rho_1\,\rho_2\,\tau_2\,\mu_{3,0}^2}{(\tau_1+\tau_2)^4} + \frac{\kappa_1\,\mu_{4,0}}{2\,(\tau_1+\tau_2)^3} - \frac{\theta''[0]}{2\,\tau_1} + \frac{\Pi''[0]}{2} - \frac{\alpha\,\phi''[0]}{2\,\tau_1}$$

Solve for the second derivatives

```
Sol3 =
  Solve[{eq1 == 0, eq2 == 0, eq3 == 0, eq4 == 0}, {θ''[0], ϕ''[0], Ψ''[0], Π''[0]}][[1]];
```

```
Sol3 = Sol3 // Simplify;
Sol3
```

$$\Big\{\Psi''[0] \to -\frac{1}{(\tau_1+\tau_2)^5\,\mu_{0,2}}\,(2\,\alpha\,\rho_2\,(\tau_1+\tau_2)\,(2\,(-1+\theta e_1)\,\tau_1+(1+2\,\theta e_1)\,\tau_2)\,\mu_{0,2}+4\,\rho_1^2\,\tau_1\,\tau_2\,(\mu_{1,2}\,\mu_{2,1}+\alpha\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{2,1}\,\mu_{3,0}+\alpha\,\mu_{0,2}\,\mu_{3,0}^2)+4\,\rho_2^2\,\tau_1\,\tau_2\,(\mu_{1,2}\,\mu_{2,1}+\alpha\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{2,1}\,\mu_{3,0}+\alpha\,\mu_{0,2}\,\mu_{3,0}^2)-2\,\rho_1\,(\alpha\,(-3+2\,\theta e_1)\,\tau_1^2\,\mu_{0,2}+2\,\alpha\,\theta e_1\,\tau_2^2\,\mu_{0,2}+\tau_1\,\tau_2\,(4\,\rho_2\,\mu_{2,1}\,(\mu_{1,2}+\alpha\,\mu_{2,1})+\mu_{0,2}\,(-3\,\alpha+4\,\alpha\,\theta e_1+4\,\rho_2\,\mu_{3,0}\,(\mu_{2,1}+\alpha\,\mu_{3,0}))))+(\tau_1+\tau_2)\,(\kappa_1\,\tau_1+\kappa_2\,\tau_2)\,\mu_{0,2}\,(\mu_{3,1}+\alpha\,\mu_{4,0})),$$

$$\theta''[0] \to \frac{1}{(\tau_1+\tau_2)^5\,\mu_{0,2}^2}\,(-2\,(-1+\theta e_1)\,(-1+2\,\rho_2)\,\tau_1^3\,\mu_{0,2}^2+2\,\theta e_1\,(1-2\,\rho_1)\,\tau_2^3\,\mu_{0,2}^2+\tau_1^2\,\tau_2\,(4\,\alpha\,\rho_2\,(-\rho_1+\rho_2)\,\mu_{1,2}\,\mu_{2,1}+\mu_{0,2}\,(4\,\rho_1\,\rho_2\,\mu_{2,1}\,(\mu_{2,1}-\alpha\,\mu_{3,0})-4\,\rho_2^2\,\mu_{2,1}\,(\mu_{2,1}-\alpha\,\mu_{3,0})+\alpha\,(-\kappa_1+\kappa_2)\,\mu_{3,1})+\mu_{0,2}^2\,(-4+2\,\rho_2-2\,\theta e_1\,(-3+2\,\rho_1+4\,\rho_2)-4\,\rho_2^2\,\mu_{3,0}^2+\rho_1\,(6+4\,\rho_2\,\mu_{3,0}^2)+\kappa_1\,\mu_{4,0}-\kappa_2\,\mu_{4,0}))-\tau_1\,\tau_2^2\,(4\,\alpha\,\rho_1\,(\rho_1-\rho_2)\,\mu_{1,2}\,\mu_{2,1}+\mu_{0,2}\,(-4\,\rho_1^2\,\mu_{2,1}\,(\mu_{2,1}-\alpha\,\mu_{3,0})+4\,\rho_1\,\rho_2\,\mu_{2,1}\,(\mu_{2,1}-\alpha\,\mu_{3,0})+\alpha\,(\kappa_1-\kappa_2)\,\mu_{3,1})+\mu_{0,2}^2\,(2+2\,\rho_2+2\,\theta e_1\,(-3+4\,\rho_1+2\,\rho_2)-4\,\rho_1^2\,\mu_{3,0}^2+\rho_1\,(-6+4\,\rho_2\,\mu_{3,0}^2)-\kappa_1\,\mu_{4,0}+\kappa_2\,\mu_{4,0}))),$$

$$\phi''[0] \to \frac{1}{(\tau_1+\tau_2)^5\,\mu_{0,2}^2}\,(\tau_1\,\tau_2\,(-4\,\rho_2^2\,\tau_1\,\mu_{2,1}\,(\mu_{1,2}+\mu_{0,2}\,\mu_{3,0})+4\,\rho_1\,\rho_2\,(\tau_1-\tau_2)\,\mu_{2,1}\,(\mu_{1,2}+\mu_{0,2}\,\mu_{3,0})+4\,\rho_1^2\,\tau_2\,\mu_{2,1}\,(\mu_{1,2}+\mu_{0,2}\,\mu_{3,0})+(\kappa_1-\kappa_2)\,(\tau_1+\tau_2)\,\mu_{0,2}\,\mu_{3,1})),$$

$$\Pi''[0] \to -\frac{1}{(\tau_1+\tau_2)^5\,\mu_{0,2}}\,(2\,\rho_2\,(\tau_1+\tau_2)\,(2\,(-1+\theta e_1)\,\tau_1+(1+2\,\theta e_1)\,\tau_2)\,\mu_{0,2}+4\,\rho_1^2\,\tau_1\,\tau_2\,(\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}^2)+4\,\rho_2^2\,\tau_1\,\tau_2\,(\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}^2)-2\,\rho_1\,((-3+2\,\theta e_1)\,\tau_1^2\,\mu_{0,2}+2\,\theta e_1\,\tau_2^2\,\mu_{0,2}+\tau_1\,\tau_2\,(4\,\rho_2\,\mu_{2,1}^2+\mu_{0,2}\,(-3+4\,\theta e_1+4\,\rho_2\,\mu_{3,0}^2)))+(\tau_1+\tau_2)\,(\kappa_1\,\tau_1+\kappa_2\,\tau_2)\,\mu_{0,2}\,\mu_{4,0})\Big\}$$

▪ Change in equity risk premium

The presence of Y does affect Π"[0]. pi2 expresses this term.

```
pi2 = Π''[0] /. Sol3 // Together // Expand
```

$$-\frac{6\,\rho_1\,\tau_1^2}{(\tau_1+\tau_2)^5}+\frac{4\,\theta e_1\,\rho_1\,\tau_1^2}{(\tau_1+\tau_2)^5}+\frac{4\,\rho_2\,\tau_1^2}{(\tau_1+\tau_2)^5}-\frac{4\,\theta e_1\,\rho_2\,\tau_1^2}{(\tau_1+\tau_2)^5}-\frac{6\,\rho_1\,\tau_1\,\tau_2}{(\tau_1+\tau_2)^5}+\frac{8\,\theta e_1\,\rho_1\,\tau_1\,\tau_2}{(\tau_1+\tau_2)^5}+\frac{2\,\rho_2\,\tau_1\,\tau_2}{(\tau_1+\tau_2)^5}-\frac{8\,\theta e_1\,\rho_2\,\tau_1\,\tau_2}{(\tau_1+\tau_2)^5}+\frac{4\,\theta e_1\,\rho_1\,\tau_2^2}{(\tau_1+\tau_2)^5}-\frac{2\,\rho_2\,\tau_2^2}{(\tau_1+\tau_2)^5}-\frac{4\,\theta e_1\,\rho_2\,\tau_2^2}{(\tau_1+\tau_2)^5}-\frac{4\,\rho_1^2\,\tau_1\,\tau_2\,\mu_{2,1}^2}{(\tau_1+\tau_2)^5\,\mu_{0,2}}+\frac{8\,\rho_1\,\rho_2\,\tau_1\,\tau_2\,\mu_{2,1}^2}{(\tau_1+\tau_2)^5\,\mu_{0,2}}-\frac{4\,\rho_2^2\,\tau_1\,\tau_2\,\mu_{2,1}^2}{(\tau_1+\tau_2)^5\,\mu_{0,2}}-\frac{4\,\rho_1^2\,\tau_1\,\tau_2\,\mu_{3,0}^2}{(\tau_1+\tau_2)^5}+\frac{8\,\rho_1\,\rho_2\,\tau_1\,\tau_2\,\mu_{3,0}^2}{(\tau_1+\tau_2)^5}-\frac{4\,\rho_2^2\,\tau_1\,\tau_2\,\mu_{3,0}^2}{(\tau_1+\tau_2)^5}-\frac{\kappa_1\,\tau_1^2\,\mu_{4,0}}{(\tau_1+\tau_2)^5}-\frac{\kappa_1\,\tau_1\,\tau_2\,\mu_{4,0}}{(\tau_1+\tau_2)^5}-\frac{\kappa_2\,\tau_1\,\tau_2\,\mu_{4,0}}{(\tau_1+\tau_2)^5}-\frac{\kappa_2\,\tau_2^2\,\mu_{4,0}}{(\tau_1+\tau_2)^5}$$

Define the price function

```
PriceZ[ϵ_] = 1 - ϵ^2 Π[ϵ]
```

$$1-\epsilon^2\,\Pi[\epsilon]$$

Compute the Taylor series for the price function

```
TaylorPriceZ = Normal[Series[PriceZ[ϵ] , {ϵ, 0, 4}]]
```

$$1 - \epsilon^2 \Pi[0] - \epsilon^3 \Pi'[0] - \frac{1}{2} \epsilon^4 \Pi''[0]$$

TaylorPriceTwoAssetZ is the second-order expansion for the price of *Z* when there are two assets.

```
TaylorPriceTwoAssetZ =
 TaylorPriceZ //. utilparams /. Sol3 /. Sol2 /. BifPt /. {θe₂ → Θ - θe₁} // Simplify
```

$$1 - \frac{\epsilon^2}{\tau_1 + \tau_2} + \frac{\epsilon^3 (\rho_1 \tau_1 + \rho_2 \tau_2) \mu_{3,0}}{(\tau_1 + \tau_2)^3} + \frac{1}{2 (\tau_1 + \tau_2)^5 \mu_{0,2}} (\epsilon^4 (2 \rho_2 (\tau_1 + \tau_2) (2 (-1 + \theta e_1) \tau_1 + (1 + 2 \theta e_1) \tau_2) \mu_{0,2} + 4 \rho_1^2 \tau_1 \tau_2 (\mu_{2,1}^2 + \mu_{0,2} \mu_{3,0}^2) + 4 \rho_2^2 \tau_1 \tau_2 (\mu_{2,1}^2 + \mu_{0,2} \mu_{3,0}^2) - 2 \rho_1 ((-3 + 2 \theta e_1) \tau_1^2 \mu_{0,2} + 2 \theta e_1 \tau_2^2 \mu_{0,2} + \tau_1 \tau_2 (4 \rho_2 \mu_{2,1}^2 + \mu_{0,2} (-3 + 4 \theta e_1 + 4 \rho_2 \mu_{3,0}^2))) + (\tau_1 + \tau_2) (\kappa_1 \tau_1 + \kappa_2 \tau_2) \mu_{0,2} \mu_{4,0}))$$

TaylorPriceOneAssetZ is the second-order expansion for the price of Z when there is just Z. It equals TaylorPriceTwoAssetZ with the moments for *Y* and cross-moments between *Z* and *Y* zeroed out by the substitutions in NoDeriv.

```
TaylorPriceOneAssetZ = TaylorPriceTwoAssetZ //. NoDeriv // Simplify
```

$$1 - \frac{\epsilon^2}{\tau_1 + \tau_2} + \frac{\epsilon^3 (\rho_1 \tau_1 + \rho_2 \tau_2) \mu_{3,0}}{(\tau_1 + \tau_2)^3} + \frac{1}{2 (\tau_1 + \tau_2)^5} (\epsilon^4 (2 \rho_2 (\tau_1 + \tau_2) (2 (-1 + \theta e_1) \tau_1 + (1 + 2 \theta e_1) \tau_2) + 4 \rho_1^2 \tau_1 \tau_2 \mu_{3,0}^2 + 4 \rho_2^2 \tau_1 \tau_2 \mu_{3,0}^2 - 2 \rho_1 ((-3 + 2 \theta e_1) \tau_1^2 + 2 \theta e_1 \tau_2^2 + \tau_1 \tau_2 (-3 + 4 \theta e_1 + 4 \rho_2 \mu_{3,0}^2)) + (\tau_1 + \tau_2) (\kappa_1 \tau_1 + \kappa_2 \tau_2) \mu_{4,0}))$$

We now compute the difference to determine the impact of the derivative asset on the price of *Z*.

```
TaylorPriceTwoAssetZ - TaylorPriceOneAssetZ // Together // Simplify
```

$$\frac{2 \epsilon^4 (\rho_1 - \rho_2)^2 \tau_1 \tau_2 \mu_{2,1}^2}{(\tau_1 + \tau_2)^5 \mu_{0,2}}$$

This is the result in Theorem 11.

■ Equation 5:

We next compute ($\theta'''[0], \phi'''[0], \Psi'''[0], \Pi'''[0]$) by setting the ϵ^4 components equal to zero. We do this since these third derivatives may play a role in the utility analysis below.

```
eq1 = eqns11[[5]] //. utilparams /. Sol3 /. Sol2 /. BifPt /. {θe₂ → Θ - θe₁};
eq2 = eqns21[[5]] //. utilparams /. Sol3 /. Sol2 /. BifPt /. {θe₂ → Θ - θe₁};
eq3 = eqns12[[5]] //. utilparams /. Sol3 /. Sol2 /. BifPt /. {θe₂ → Θ - θe₁};
eq4 = eqns22[[5]] //. utilparams /. Sol3 /. Sol2 /. BifPt /. {θe₂ → Θ - θe₁};
```

We display the first equation as an example

```
eq1 // Simplify
```

$$\begin{aligned}&\frac{1}{24\,\tau_1\,(\tau_1+\tau_2)^6\,\mu_{0,2}^2}\,(120\,\rho_1^3\,\tau_1\,\tau_2^2\,(\mu_{1,2}\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}\,(2\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}^2))+\\
&\quad 24\,\rho_1^2\,((\tau_1+\tau_2)\,((-3+2\,\theta e_1)\,\tau_1^2-4\,(-3+\theta e_1)\,\tau_1\,\tau_2-6\,\theta e_1\,\tau_2^2)\,\mu_{0,2}^2\,\mu_{3,0}+\\
&\qquad 2\,\rho_2\,\tau_1\,(2\,\tau_1-3\,\tau_2)\,\tau_2\,(\mu_{1,2}\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}\,(2\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}^2)))-\\
&\quad 12\,\rho_1\,(2\,\rho_2\,(\tau_1+\tau_2)\,(4\,(-1+\theta e_1)\,\tau_1^2+11\,\tau_1\,\tau_2-4\,\theta e_1\,\tau_2^2)\,\mu_{0,2}^2\,\mu_{3,0}+\\
&\qquad 2\,\rho_2^2\,\tau_1\,(4\,\tau_1-\tau_2)\,\tau_2\,(\mu_{1,2}\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}\,(2\,\mu_{2,1}^2+\mu_{0,2}\,\mu_{3,0}^2))-\tau_2\,(\tau_1+\tau_2)\,\mu_{0,2}\\
&\qquad\quad (4\,\theta e_1\,\tau_2\,\mu_{0,2}\,\mu_{3,0}+\tau_1\,((5\,\kappa_1-2\,\kappa_2)\,\mu_{2,1}\,\mu_{3,1}+\mu_{0,2}\,\mu_{3,0}\,(-6+4\,\theta e_1+5\,\kappa_1\,\mu_{4,0}-2\,\kappa_2\,\mu_{4,0}))))-\\
&\quad (\tau_1+\tau_2)\,\mu_{0,2}\,(12\,\kappa_1\,((-4+3\,\theta e_1)\,\tau_1^2\,\mu_{0,2}\,\mu_{3,0}+3\,\theta e_1\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}+\\
&\qquad\quad \tau_1\,\tau_2\,(3\,\rho_2\,\mu_{2,1}\,\mu_{3,1}+\mu_{0,2}\,\mu_{3,0}\,(-4+6\,\theta e_1+3\,\rho_2\,\mu_{4,0})))+\\
&\qquad \mu_{0,2}\,(24\,\rho_2\,\tau_2\,((-3+2\,\theta e_1)\,\tau_1+2\,\theta e_1\,\tau_2)\,\mu_{3,0}-(\tau_1+\tau_2)\,(\tau_1^5\,(\mu_{5,0}\,\mathbb{A}_1[5]+4\,\Pi^{(3)}[0])-\\
&\qquad\qquad 4\,\tau_2^4\,(\theta^{(3)}[0]+\alpha\,\phi^{(3)}[0])-4\,\tau_1^4\,(\theta^{(3)}[0]-4\,\tau_2\,\Pi^{(3)}[0]+\alpha\,\phi^{(3)}[0])+\\
&\qquad\qquad 4\,\tau_1\,\tau_2^3\,(\tau_2\,\Pi^{(3)}[0]-4\,(\theta^{(3)}[0]+\alpha\,\phi^{(3)}[0]))+8\,\tau_1^2\,\tau_2^2\,(2\,\tau_2\,\Pi^{(3)}[0]-3\\
&\qquad\qquad\quad (\theta^{(3)}[0]+\alpha\,\phi^{(3)}[0]))+8\,\tau_1^3\,\tau_2\,(3\,\tau_2\,\Pi^{(3)}[0]-2\,(\theta^{(3)}[0]+\alpha\,\phi^{(3)}[0]))))))\end{aligned}$$

We now solve for the third derivatives. We do not display any of the results since they are very long and difficult to interpret.

```
Sol4 = Solve[{eq1 == 0, eq2 == 0, eq3 == 0, eq4 == 0},
    {θ'''[0], ϕ'''[0], Ψ'''[0], Π'''[0]}][[1]];
```

Utility Expansion

We want to evaluate the impact of the derivative security *Y* on utility. This is necessary to derive the results in Section 6.4.

■ Utility

W_1 is the final wealth and consumption of a type 1 agent. Express it in terms of the random variables *z* and *y*, and portfolio holdings.

```
W1 = Expand[W1]
```

$$z\,\epsilon\,\theta+\epsilon^2\,\theta\,\Pi+y\,\epsilon\,\phi+z\,\alpha\,\epsilon\,\phi+\epsilon^2\,\phi\,\Psi+Be_1+\theta e_1-\epsilon^2\,\Pi\,\theta e_1$$

Define equilibrium expected utility for type 1 agents by substituting the equilibrium functions for θ, ϕ, Π, and Ψ into our expression for final consumption. The result is equilibrium utility as a function of ϵ.

```
U1 = u1[W1] /. PortfolioSubs /. PremiaSubs
```

$$u_1[Be_1+\theta e_1+z\,\epsilon\,\theta[\epsilon]-\epsilon^2\,\theta e_1\,\Pi[\epsilon]+\epsilon^2\,\theta[\epsilon]\,\Pi[\epsilon]+y\,\epsilon\,\phi[\epsilon]+z\,\alpha\,\epsilon\,\phi[\epsilon]+\epsilon^2\,\phi[\epsilon]\,\Psi[\epsilon]]$$

Compute the degree 5 Taylor series of the utility of type 1 agents in terms of ϵ and call it U1pow.

```
U1pow = Normal[Series[U1 , {ϵ, 0, 5}]] //. UtilDerivs;
```

Expand it so that products of *z* and *y* are collected

```
U1pow = Expand[U1pow];
```

Compute expected utility by replacing instances of powers of *z* and *y* with their moment expressions

```
U1pow = U1pow //. moments;
```

Replace θ, ϕ, Π, and Ψ and their derivatives with the solutions derived above.

```
U1pow = U1pow /. Sol4 /. Sol3 /. Sol2 /. BifPt //. UtilDerivs;
```

Express in terms of τ, ρ, and κ.

```
U1pow = U1pow //. utilparams;
```

Combine like powers of ϵ and put the coefficients in the list U1ϵpows.

```
U1ϵpows = CoefficientList[U1pow, ϵ];
```

Display the coefficients of 1, ϵ, ϵ^2, and ϵ^3

```
Table[U1ϵpows[[i]], {i, 1, 4}] // Simplify
```

$$\left\{u_1[Be_1 + \theta e_1],\ 0,\ \frac{\tau_1 - 2\,\theta e_1\,(\tau_1 + \tau_2)}{2\,(\tau_1 + \tau_2)^2},\ \frac{(3\,\rho_2\,\tau_2\,((-1 + \theta e_1)\,\tau_1 + \theta e_1\,\tau_2) + \rho_1\,\tau_1\,((-2 + 3\,\theta e_1)\,\tau_1 + (1 + 3\,\theta e_1)\,\tau_2))\,\mu_{3,0}}{3\,(\tau_1 + \tau_2)^4}\right\}$$

This list shows that the moments of *Y* and its cross-moments with *Z* have no impact on utility up to the order ϵ^3. Therefore, we move to the fourth-order terms to determine the utility impact of the derivative security.

■ Utility difference - order 4

Compute the utility contribution of the new asset. The utility difference is the utility with a nontrivial *Y* minus the utility with trivial *Y*, that is, a *Y* with zero co-moments with z.

util4 is the degree four term in the utility expansion with two assets and equals

```
util4 = U1ϵpows[[5]] // Together
```

$$
\begin{aligned}
&\frac{1}{8\,\tau_1\,(\tau_1+\tau_2)^6\,\mu_{0,2}}
\Big(-4\,\tau_1^4\,\mu_{0,2}+8\,\theta e_1\,\tau_1^4\,\mu_{0,2}-4\,\theta e_1^2\,\tau_1^4\,\mu_{0,2}-16\,\rho_1\,\tau_1^4\,\mu_{0,2}+32\,\theta e_1\,\rho_1\,\tau_1^4\,\mu_{0,2}-16\,\theta e_1^2\,\rho_1\,\tau_1^4\,\mu_{0,2}+\\
&16\,\rho_2\,\tau_1^4\,\mu_{0,2}-32\,\theta e_1\,\rho_2\,\tau_1^4\,\mu_{0,2}+16\,\theta e_1^2\,\rho_2\,\tau_1^4\,\mu_{0,2}-8\,\tau_1^3\,\tau_2\,\mu_{0,2}+24\,\theta e_1\,\tau_1^3\,\tau_2\,\mu_{0,2}-\\
&16\,\theta e_1^2\,\tau_1^3\,\tau_2\,\mu_{0,2}-8\,\rho_1\,\tau_1^3\,\tau_2\,\mu_{0,2}+56\,\theta e_1\,\rho_1\,\tau_1^3\,\tau_2\,\mu_{0,2}-48\,\theta e_1^2\,\rho_1\,\tau_1^3\,\tau_2\,\mu_{0,2}+8\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}-\\
&56\,\theta e_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}+48\,\theta e_1^2\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}-4\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+24\,\theta e_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-24\,\theta e_1^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+\\
&8\,\rho_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+16\,\theta e_1\,\rho_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-48\,\theta e_1^2\,\rho_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-8\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-16\,\theta e_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+\\
&48\,\theta e_1^2\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+8\,\theta e_1\,\tau_1\,\tau_2^3\,\mu_{0,2}-16\,\theta e_1^2\,\tau_1\,\tau_2^3\,\mu_{0,2}-8\,\theta e_1\,\rho_1\,\tau_1\,\tau_2^3\,\mu_{0,2}-16\,\theta e_1^2\,\rho_1\,\tau_1\,\tau_2^3\,\mu_{0,2}+\\
&8\,\theta e_1\,\rho_2\,\tau_1\,\tau_2^3\,\mu_{0,2}+16\,\theta e_1^2\,\rho_2\,\tau_1\,\tau_2^3\,\mu_{0,2}-4\,\theta e_1^2\,\tau_2^4\,\mu_{0,2}-16\,\rho_1^2\,\tau_1^3\,\tau_2\,\mu_{2,1}^2+16\,\theta e_1\,\rho_1^2\,\tau_1^3\,\tau_2\,\mu_{2,1}^2+\\
&32\,\rho_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{2,1}^2-32\,\theta e_1\,\rho_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{2,1}^2-16\,\rho_2^2\,\tau_1^3\,\tau_2\,\mu_{2,1}^2+16\,\theta e_1\,\rho_2^2\,\tau_1^3\,\tau_2\,\mu_{2,1}^2+\\
&4\,\rho_1^2\,\tau_1^2\,\tau_2^2\,\mu_{2,1}^2+16\,\theta e_1\,\rho_1^2\,\tau_1^2\,\tau_2^2\,\mu_{2,1}^2-8\,\rho_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{2,1}^2-32\,\theta e_1\,\rho_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{2,1}^2+4\,\rho_2^2\,\tau_1^2\,\tau_2^2\,\mu_{2,1}^2+\\
&16\,\theta e_1\,\rho_2^2\,\tau_1^2\,\tau_2^2\,\mu_{2,1}^2-16\,\rho_1^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+16\,\theta e_1\,\rho_1^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+32\,\rho_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2-\\
&32\,\theta e_1\,\rho_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2-16\,\rho_2^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+16\,\theta e_1\,\rho_2^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+4\,\rho_1^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2+\\
&16\,\theta e_1\,\rho_1^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2-8\,\rho_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2-32\,\theta e_1\,\rho_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2+4\,\rho_2^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2+\\
&16\,\theta e_1\,\rho_2^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2-3\,\kappa_1\,\tau_1^4\,\mu_{0,2}\,\mu_{4,0}+4\,\theta e_1\,\kappa_1\,\tau_1^4\,\mu_{0,2}\,\mu_{4,0}-2\,\kappa_1\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}+\\
&8\,\theta e_1\,\kappa_1\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}-4\,\kappa_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}+4\,\theta e_1\,\kappa_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}+\kappa_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}+\\
&4\,\theta e_1\,\kappa_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}-4\,\kappa_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}+8\,\theta e_1\,\kappa_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}+4\,\theta e_1\,\kappa_2\,\tau_1\,\tau_2^3\,\mu_{0,2}\,\mu_{4,0}\Big)
\end{aligned}
$$

This term is very complex and difficult to interpret. However, since we are only interested in the impact of the new security on utility, we do not need to understand all of util4. We only need to understand how the new security affects util4. Examination of util4 shows that $\mu_{2,1}$ is the only moment involving the new asset in the expression util4. The degree four term in the utility expansion for the case with only one asset is computed by evaluating util4 with $\mu_{2,1}$ set equal to zero.

```
(util4 /. {μ2,1 -> 0})
```

$$
\begin{aligned}
&\frac{1}{8\,\tau_1\,(\tau_1+\tau_2)^6\,\mu_{0,2}}
\Big(-4\,\tau_1^4\,\mu_{0,2}+8\,\theta e_1\,\tau_1^4\,\mu_{0,2}-4\,\theta e_1^2\,\tau_1^4\,\mu_{0,2}-16\,\rho_1\,\tau_1^4\,\mu_{0,2}+32\,\theta e_1\,\rho_1\,\tau_1^4\,\mu_{0,2}-16\,\theta e_1^2\,\rho_1\,\tau_1^4\,\mu_{0,2}+\\
&16\,\rho_2\,\tau_1^4\,\mu_{0,2}-32\,\theta e_1\,\rho_2\,\tau_1^4\,\mu_{0,2}+16\,\theta e_1^2\,\rho_2\,\tau_1^4\,\mu_{0,2}-8\,\tau_1^3\,\tau_2\,\mu_{0,2}+24\,\theta e_1\,\tau_1^3\,\tau_2\,\mu_{0,2}-\\
&16\,\theta e_1^2\,\tau_1^3\,\tau_2\,\mu_{0,2}-8\,\rho_1\,\tau_1^3\,\tau_2\,\mu_{0,2}+56\,\theta e_1\,\rho_1\,\tau_1^3\,\tau_2\,\mu_{0,2}-48\,\theta e_1^2\,\rho_1\,\tau_1^3\,\tau_2\,\mu_{0,2}+\\
&8\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}-56\,\theta e_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}+48\,\theta e_1^2\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}-4\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+24\,\theta e_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-\\
&24\,\theta e_1^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+8\,\rho_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+16\,\theta e_1\,\rho_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-48\,\theta e_1^2\,\rho_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-8\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}-\\
&16\,\theta e_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+48\,\theta e_1^2\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}+8\,\theta e_1\,\tau_1\,\tau_2^3\,\mu_{0,2}-16\,\theta e_1^2\,\tau_1\,\tau_2^3\,\mu_{0,2}-\\
&8\,\theta e_1\,\rho_1\,\tau_1\,\tau_2^3\,\mu_{0,2}-16\,\theta e_1^2\,\rho_1\,\tau_1\,\tau_2^3\,\mu_{0,2}+8\,\theta e_1\,\rho_2\,\tau_1\,\tau_2^3\,\mu_{0,2}+16\,\theta e_1^2\,\rho_2\,\tau_1\,\tau_2^3\,\mu_{0,2}-\\
&4\,\theta e_1^2\,\tau_2^4\,\mu_{0,2}-16\,\rho_1^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+16\,\theta e_1\,\rho_1^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+32\,\rho_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2-\\
&32\,\theta e_1\,\rho_1\,\rho_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2-16\,\rho_2^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+16\,\theta e_1\,\rho_2^2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{3,0}^2+4\,\rho_1^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2+\\
&16\,\theta e_1\,\rho_1^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2-8\,\rho_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2-32\,\theta e_1\,\rho_1\,\rho_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2+4\,\rho_2^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2+\\
&16\,\theta e_1\,\rho_2^2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{3,0}^2-3\,\kappa_1\,\tau_1^4\,\mu_{0,2}\,\mu_{4,0}+4\,\theta e_1\,\kappa_1\,\tau_1^4\,\mu_{0,2}\,\mu_{4,0}-2\,\kappa_1\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}+\\
&8\,\theta e_1\,\kappa_1\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}-4\,\kappa_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}+4\,\theta e_1\,\kappa_2\,\tau_1^3\,\tau_2\,\mu_{0,2}\,\mu_{4,0}+\kappa_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}+\\
&4\,\theta e_1\,\kappa_1\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}-4\,\kappa_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}+8\,\theta e_1\,\kappa_2\,\tau_1^2\,\tau_2^2\,\mu_{0,2}\,\mu_{4,0}+4\,\theta e_1\,\kappa_2\,\tau_1\,\tau_2^3\,\mu_{0,2}\,\mu_{4,0}\Big)
\end{aligned}
$$

We take the difference in the past two expressions to compute the impact of the new security.

```
UtilDiff = (util4 - (util4 /. { μ2,1 -> 0})) // Expand
```

$$-\frac{2\rho_1^2\tau_1^2\tau_2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{2\theta e_1\rho_1^2\tau_1^2\tau_2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{4\rho_1\rho_2\tau_1^2\tau_2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}-\frac{4\theta e_1\rho_1\rho_2\tau_1^2\tau_2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}-\frac{2\rho_2^2\tau_1^2\tau_2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{2\theta e_1\rho_2^2\tau_1^2\tau_2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{\rho_1^2\tau_1\tau_2^2\mu_{2,1}^2}{2(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{2\theta e_1\rho_1^2\tau_1\tau_2^2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}-\frac{\rho_1\rho_2\tau_1\tau_2^2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}-\frac{4\theta e_1\rho_1\rho_2\tau_1\tau_2^2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{\rho_2^2\tau_1\tau_2^2\mu_{2,1}^2}{2(\tau_1+\tau_2)^6\mu_{0,2}}+\frac{2\theta e_1\rho_2^2\tau_1\tau_2^2\mu_{2,1}^2}{(\tau_1+\tau_2)^6\mu_{0,2}}$$

This is a much simpler expression. We simplify the difference to arrive at the result in the paper.

```
UtilDiff = Simplify[UtilDiff]
```

$$\frac{(\rho_1-\rho_2)^2\tau_1\tau_2(4(-1+\theta e_1)\tau_1+(1+4\theta e_1)\tau_2)\mu_{2,1}^2}{2(\tau_1+\tau_2)^6\mu_{0,2}}$$

Use the identity $-1+\theta e_1-\theta e_2$:

```
UtilDiff = UtilDiff /. {-1 + θe1 → -θe2}
```

$$\frac{(\rho_1-\rho_2)^2\tau_1\tau_2(-4\theta e_2\tau_1+(1+4\theta e_1)\tau_2)\mu_{2,1}^2}{2(\tau_1+\tau_2)^6\mu_{0,2}}$$

The expression in Theorem 12 is

$$\mathbf{Thm12} = \frac{(\rho_1-\rho_2)^2\tau_1{}^2\tau_2{}^2}{2(\tau_1+\tau_2)^6}\left(4\left(\frac{\theta e_1}{\tau_1}-\frac{\theta e_2}{\tau_2}\right)+\frac{1}{\tau_1}\right)\frac{\mu_{2,1}^2}{\mu_{0,2}}$$

$$\frac{(\rho_1-\rho_2)^2\tau_1^2\left(\frac{1}{\tau_1}+4\left(\frac{\theta e_1}{\tau_1}-\frac{\theta e_2}{\tau_2}\right)\right)\tau_2^2\mu_{2,1}^2}{2(\tau_1+\tau_2)^6\mu_{0,2}}$$

```
UtilDiff - Thm12 // Simplify
```

0

Therefore, our UtilDiff expression is equivalent to the one for utility change in Theorem 12.

Automating the Implicit Function Theorem

```
x = 0; Remove["Global`*"]
```

Introduction

Economics revolves around solving equilibrium models

Common features of equilibrium models
(1) They are often solutions to systems of equations of analytic functions
(2) Different models often have common structure in terms of functional forms, and differ only in terms of parameter values
(3) There are usually a few special (often degenerate) cases where we can solve for the solution explicitly
(4) The implicit function theorem tells us that there is an analytic map between exogenous parameters and the equilibrium outcome near these cases.

Our objective today:
(1) Display the structure of basic economic models, using simple examples
(2) Show how one can use AD and IFT to compute equilibria for pieces of the parameter space
(3) Argue that for many purposes this approach may dominate numerical methods for solving specific instances

A Specific Example - Easy one

Let's examine a very simple example. Suppose that p is the price of a good and the demand function for that good is

```
Dmd[p_] = p^-3;
```

Suppose that producers pay a tax of τ p for each unit it sells where τ is the tax rate (like a VAT) and that supply is a function of the after-tax price received by the producer

```
Supply[p_, τ_] = (p (1 - τ))^1/2;
```

The excess demand for price p and tax rate τ is

```
ExDmd[p_, τ_] = Dmd[p] - Supply[p, τ]
```

$$\frac{1}{p^3} - \sqrt{p\,(1 - \tau)}$$

The true solution is

```
Ptrue[τ_] = (1 - τ)^-1/7;
```

Examination of the excess demand function shows that price is 1 when the tax is τ=0. Therefore

```
P[0] = 1;
```

Write the excess demand as a function of the tax τ and the equilibrium price P[τ]

```
ExDmdTax[τ_] = ExDmd[P[τ], τ]
```

$$\frac{1}{P[\tau]^3} - \sqrt{(1-\tau)\, P[\tau]}$$

ExDmdTax[τ]=0 for all tax rates τ.

The task: Use the parameterized equilibrium equation, the P[0]=1 condition, and the IFT to trace out the equilibrium manifold, P[τ], for τ close to zero.

Differentiate ExDmdTax[τ] at τ=0 to compute P'[0]

```
ExDmdTax'[0] // Expand
```

$$\frac{1}{2} - \frac{7\, P'[0]}{2}$$

```
solp1 = Solve[% == 0, P'[0]][[1]]
```

$$\left\{P'[0] \to \frac{1}{7}\right\}$$

We do the same to get P''[0] and P'''[0].

```
Solve[ExDmdTax''[0] == 0, P''[0]][[1]]
```

$$\left\{P''[0] \to \frac{1}{14}\left(1 + 2\,P'[0] + 49\,P'[0]^2\right)\right\}$$

```
solp2 = % //. solp1
```

$$\left\{P''[0] \to \frac{8}{49}\right\}$$

```
Solve[ExDmdTax'''[0] == 0, P'''[0]][[1]];
solp3 = % //. solp1 //. solp2
```

$$\left\{P^{(3)}[0] \to \frac{120}{343}\right\}$$

The degree 3 Taylor series of P[τ] at τ=0 is

```
Series[P[τ], {τ, 0, 3}]
% /. solp3
% /. solp2
% /. solp1
% // Normal
```

$$1 + P'[0]\,\tau + \frac{1}{2} P''[0]\,\tau^2 + \frac{1}{6} P^{(3)}[0]\,\tau^3 + O[\tau]^4$$

$$1 + P'[0]\,\tau + \frac{1}{2} P''[0]\,\tau^2 + \frac{20\,\tau^3}{343} + O[\tau]^4$$

$$1 + P'[0]\,\tau + \frac{4\,\tau^2}{49} + \frac{20\,\tau^3}{343} + O[\tau]^4$$

$$1 + \frac{\tau}{7} + \frac{4\,\tau^2}{49} + \frac{20\,\tau^3}{343} + O[\tau]^4$$

$$1 + \frac{\tau}{7} + \frac{4\,\tau^2}{49} + \frac{20\,\tau^3}{343}$$

SolveAlways Command

The degree 3 Taylor series of ExDmdTax[τ] at τ=0 is

```
ser = Series[ExDmdTax[τ], {τ, 0, 3}]
```

$$\left(\frac{1}{2} - \frac{7\,P'[0]}{2}\right)\tau + \frac{1}{8}\left(1 + 2\,P'[0] + 49\,P'[0]^2 - 14\,P''[0]\right)\tau^2 +$$
$$\frac{1}{48}\left(3 + 3\,P'[0] - 3\,P'[0]^2 - 483\,P'[0]^3 + 6\,P''[0] + 294\,P'[0]\,P''[0] - 28\,P^{(3)}[0]\right)\tau^3 + O[\tau]^4$$

We will use the SolveAlways command to solve out for the derivatives of P at τ=0.

```
sol = SolveAlways[ser == 0, τ][[1]]
```

$$\left\{P^{(3)}[0] \to \frac{120}{343},\ P''[0] \to \frac{8}{49},\ P'[0] \to \frac{1}{7}\right\}$$

We now substitute this into the the Taylor series expansion for P[τ] at τ=0.

```
Series[P[τ], {τ, 0, 3}] /. sol
```

$$1 + \frac{\tau}{7} + \frac{4\,\tau^2}{49} + \frac{20\,\tau^3}{343} + O[\tau]^4$$

The degree 3 Taylor series of P[τ] at τ=0 is

```
solPp[τ_] = Series[P[τ], {τ, 0, 3}] /. sol // Normal
```

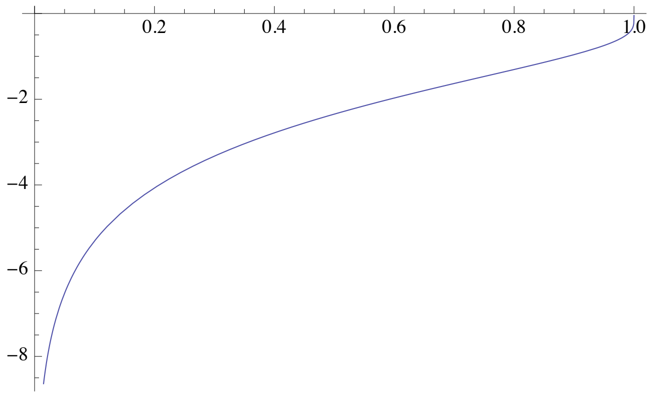
$$1 + \frac{\tau}{7} + \frac{4\,\tau^2}{49} + \frac{20\,\tau^3}{343}$$

■ Quality check

Question: How good is our Taylor series approximation.

Answer 1: Since we know the solution, display the relative error of the approximation. The error is less than 0.1 per cent for tax rates below 0.30; not bad. Higher order series will do better.

```
Plot[Log[10, 1 - solPp[τ] / Ptrue[τ]], {τ, 0, 1}]
```



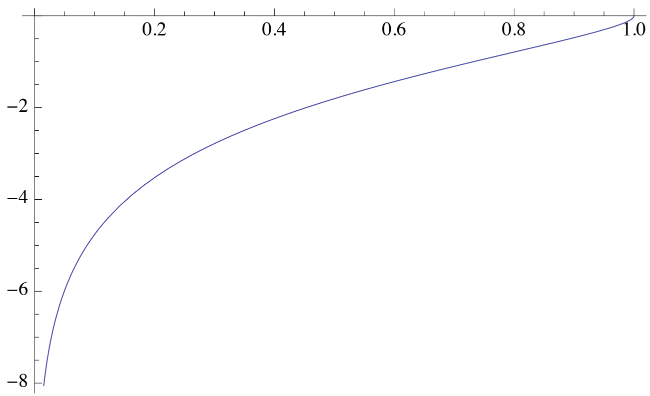
But we generally do not know the answer!

Answer 2: Check the residual. We can always do that.

```
Residual[τ_] = ExDmdTax[τ] /. P → solPp
```

$$\frac{1}{\left(1+\frac{\tau}{7}+\frac{4\,\tau^2}{49}+\frac{20\,\tau^3}{343}\right)^3}-\sqrt{(1-\tau)\left(1+\frac{\tau}{7}+\frac{4\,\tau^2}{49}+\frac{20\,\tau^3}{343}\right)}$$

```
Plot[Log[10, Residual[τ] / Dmd[solPp[τ]]], {τ, 0, 1}]
```



The residual, normalized relative to demand, is less than 0.1 percent for tax rates below 0.30.

In practice, we check residual to determine the order of the Taylor series necessary for a good solution.

An Abstract Example in General Style

In general, we will want to compute the approximation for many different parameter sets. Therefore, we take the following approach.

Consider a constant elasticity specification, a generalization of our simple example

```
Dmd[p_] = A p^-ν; Supply[p_, τ_] = B (p (1 - τ))^η;
ExDmd[p_, τ_] = Dmd[p] - Supply[p, τ];
ExDmdTax[τ_] = ExDmd[P[τ], τ]
```

$A\,P[\tau]^{-\nu} - B\,((1-\tau)\,P[\tau])^{\eta}$

We know the solution at τ=0:

```
P[0] = (A / B)^(1/(η+ν));
```

Compute Derivatives

We first compute derivatives of ExDmdTax[t] at t=0 in symbolic form

```
ExDmdTax'[0]
% // PowerExpand
% // Simplify
```

$$-A\left(\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}}\right)^{-1-\nu}\nu\,P'[0]-\left(\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}}\right)^{-1+\eta}B\,\eta\left(-\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}}+P'[0]\right)$$

$$-A^{1+\frac{-1-\nu}{\eta+\nu}}\,B^{-\frac{-1-\nu}{\eta+\nu}}\,\nu\,P'[0]-A^{\frac{-1+\eta}{\eta+\nu}}\,B^{1-\frac{-1+\eta}{\eta+\nu}}\,\eta\left(-A^{\frac{1}{\eta+\nu}}\,B^{-\frac{1}{\eta+\nu}}+P'[0]\right)$$

$$A^{-\frac{1+\nu}{\eta+\nu}}\,B^{\frac{\nu}{\eta+\nu}}\left(A^{\frac{1+\eta+\nu}{\eta+\nu}}\,\eta-A\,B^{\frac{1}{\eta+\nu}}\,(\eta+\nu)\,P'[0]\right)$$

```
ExDmdTax''[0] // PowerExpand // Simplify
```

$$A^{-\frac{2+\nu}{\eta+\nu}}\,B^{-\frac{\eta}{\eta+\nu}}\left(-A^{\frac{2+\eta+\nu}{\eta+\nu}}\,B\,(-1+\eta)\,\eta-\right.$$
$$\left.A\,B^{1+\frac{2}{\eta+\nu}}\left(-\eta+\eta^2-\nu\,(1+\nu)\right)P'[0]^2+A^{\frac{1+\eta+\nu}{\eta+\nu}}\,B^{1+\frac{1}{\eta+\nu}}\,\eta\,(2\,\eta\,P'[0]-P''[0])-A^{1+\frac{1}{\eta+\nu}}\,B^{\frac{1+\eta+\nu}{\eta+\nu}}\,\nu\,P''[0]\right)$$

```
ExDmdTax'''[0] // PowerExpand // Simplify
```

$$A^{\frac{-3+\eta}{\eta+\nu}} B^{\frac{\nu}{\eta+\nu}} \left(-B^{\frac{3}{\eta+\nu}} \nu (1+\nu) (2+\nu) P'[0]^3 + (-2+\eta)(-1+\eta)\eta \left(A^{\frac{1}{\eta+\nu}} - B^{\frac{1}{\eta+\nu}} P'[0]\right)^3 - 3 A^{\frac{1}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} (-1+\eta)\eta \left(A^{\frac{1}{\eta+\nu}} - B^{\frac{1}{\eta+\nu}} P'[0]\right)(2 P'[0] - P''[0]) + 3 A^{\frac{1}{\eta+\nu}} B^{\frac{2}{\eta+\nu}} \nu (1+\nu) P'[0] P''[0] + A^{\frac{2}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} \eta \left(3 P''[0] - P^{(3)}[0]\right) - A^{\frac{2}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} \nu P^{(3)}[0]\right)$$

Collect and store these expressions.

```
derivs = {ExDmdTax'[0], ExDmdTax''[0], ExDmdTax'''[0]} // PowerExpand // Simplify;
```

■ Using Series to get derivatives

Sometimes, the following is a faster way to get these derivatives

```
ser = Series[ExDmdTax[τ], {τ, 0, 3}] // Normal;

ser = ser // PowerExpand // Simplify
```

$$A^{-\frac{1+\nu}{\eta+\nu}} B^{\frac{\nu}{\eta+\nu}} \tau \left(A^{\frac{1+\eta+\nu}{\eta+\nu}} \eta - A B^{\frac{1}{\eta+\nu}} (\eta+\nu) P'[0]\right) -$$
$$\frac{1}{2} A^{-\frac{2+\nu}{\eta+\nu}} B^{-\frac{\eta}{\eta+\nu}} \tau^2 \left(A^{\frac{2+\eta+\nu}{\eta+\nu}} B (-1+\eta) \eta + A B^{1+\frac{2}{\eta+\nu}} \left(-\eta+\eta^2-\nu(1+\nu)\right) P'[0]^2 -\right.$$
$$\left. A^{\frac{1+\eta+\nu}{\eta+\nu}} B^{1+\frac{1}{\eta+\nu}} \eta (2\eta P'[0] - P''[0]) + A^{1+\frac{1}{\eta+\nu}} B^{\frac{1+\eta+\nu}{\eta+\nu}} \nu P''[0]\right) +$$
$$\frac{1}{6} A^{\frac{-3+\eta}{\eta+\nu}} B^{\frac{\nu}{\eta+\nu}} \tau^3 \left(-B^{\frac{1}{\eta+\nu}} \nu \left(B^{\frac{2}{\eta+\nu}} \left(2+3\nu+\nu^2\right) P'[0]^3 - 3 A^{\frac{1}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} (1+\nu) P'[0] P''[0] + A^{\frac{2}{\eta+\nu}} P^{(3)}[0]\right) +\right.$$
$$\eta \left(A^{\frac{3}{\eta+\nu}} \left(2-3\eta+\eta^2\right) - B^{\frac{3}{\eta+\nu}} \left(2-3\eta+\eta^2\right) P'[0]^3 + 3 A^{\frac{1}{\eta+\nu}} B^{\frac{2}{\eta+\nu}} (-1+\eta) P'[0] (\eta P'[0] - P''[0]) -\right.$$
$$\left.\left. A^{\frac{2}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} \left(3\eta^2 P'[0] - 3\eta (P'[0] + P''[0]) + P^{(3)}[0]\right)\right)\right)$$

```
coeffs = CoefficientList[ser, τ]
```

$$\left\{0,\ A^{-\frac{1+\nu}{\eta+\nu}}\ B^{\frac{\nu}{\eta+\nu}}\ \left(A^{\frac{1+\eta+\nu}{\eta+\nu}}\ \eta - A\ B^{\frac{1}{\eta+\nu}}\ (\eta+\nu)\ P'[0]\right),\right.$$

$$-\frac{1}{2}\ A^{-\frac{2+\nu}{\eta+\nu}}\ B^{-\frac{\eta}{\eta+\nu}}\ \left(A^{\frac{2+\eta+\nu}{\eta+\nu}}\ B\ (-1+\eta)\ \eta + A\ B^{1+\frac{2}{\eta+\nu}}\ \left(-\eta+\eta^2-\nu\ (1+\nu)\right)\ P'[0]^2 - A^{\frac{1+\eta+\nu}{\eta+\nu}}\ B^{1+\frac{1}{\eta+\nu}}\ \eta\ (2\ \eta\ P'[0] - P''[0]) + A^{1+\frac{1}{\eta+\nu}}\ B^{\frac{1+\eta+\nu}{\eta+\nu}}\ \nu\ P''[0]\right),$$

$$\frac{1}{6}\ A^{\frac{-3+\eta}{\eta+\nu}}\ B^{\frac{\nu}{\eta+\nu}}\ \left(-B^{\frac{1}{\eta+\nu}}\ \nu\ \left(B^{\frac{2}{\eta+\nu}}\ \left(2+3\ \nu+\nu^2\right)\ P'[0]^3 - 3\ A^{\frac{1}{\eta+\nu}}\ B^{\frac{1}{\eta+\nu}}\ (1+\nu)\ P'[0]\ P''[0] + A^{\frac{2}{\eta+\nu}}\ P^{(3)}[0]\right) + \eta\ \left(A^{\frac{3}{\eta+\nu}}\ \left(2-3\ \eta+\eta^2\right) - B^{\frac{3}{\eta+\nu}}\ \left(2-3\ \eta+\eta^2\right)\ P'[0]^3 + 3\ A^{\frac{1}{\eta+\nu}}\ B^{\frac{2}{\eta+\nu}}\ (-1+\eta)\ P'[0]\ (\eta\ P'[0] - P''[0]) - A^{\frac{2}{\eta+\nu}}\ B^{\frac{1}{\eta+\nu}}\ \left(3\ \eta^2\ P'[0] - 3\ \eta\ (P'[0] + P''[0]) + P^{(3)}[0]\right)\right)\right)\Big\}$$

```
derivs = Rest[coeffs]
```

$$\left\{A^{-\frac{1+\nu}{\eta+\nu}} B^{\frac{\nu}{\eta+\nu}} \left(A^{\frac{1+\eta+\nu}{\eta+\nu}} \eta - A B^{\frac{1}{\eta+\nu}} (\eta+\nu) P'[0]\right),\right.$$

$$-\frac{1}{2} A^{-\frac{2+\nu}{\eta+\nu}} B^{-\frac{\eta}{\eta+\nu}} \left(A^{\frac{2+\eta+\nu}{\eta+\nu}} B (-1+\eta) \eta + A B^{1+\frac{2}{\eta+\nu}} \left(-\eta+\eta^2-\nu (1+\nu)\right) P'[0]^2 - A^{\frac{1+\eta+\nu}{\eta+\nu}} B^{1+\frac{1}{\eta+\nu}} \eta (2 \eta P'[0] - P''[0]) + A^{1+\frac{1}{\eta+\nu}} B^{\frac{1+\eta+\nu}{\eta+\nu}} \nu P''[0]\right),$$

$$\frac{1}{6} A^{\frac{-3+\eta}{\eta+\nu}} B^{\frac{\nu}{\eta+\nu}} \left(-B^{\frac{1}{\eta+\nu}} \nu \left(B^{\frac{2}{\eta+\nu}} \left(2+3\nu+\nu^2\right) P'[0]^3 - 3 A^{\frac{1}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} (1+\nu) P'[0] P''[0] + A^{\frac{2}{\eta+\nu}} P^{(3)}[0]\right) + \eta \left(A^{\frac{3}{\eta+\nu}} \left(2-3\eta+\eta^2\right) - B^{\frac{3}{\eta+\nu}} \left(2-3\eta+\eta^2\right) P'[0]^3 + 3 A^{\frac{1}{\eta+\nu}} B^{\frac{2}{\eta+\nu}} (-1+\eta) P'[0] (\eta P'[0] - P''[0]) - A^{\frac{2}{\eta+\nu}} B^{\frac{1}{\eta+\nu}} \left(3 \eta^2 P'[0] - 3 \eta (P'[0] + P''[0]) + P^{(3)}[0]\right)\right)\right)\left.\right\}$$

■ Power Series Solution

We want to solve for the first three derivatives of P[τ] at τ=0. This is accomplished by using the derivatives for ExDmdTax at τ=0 and solving out for P'[0], P''[0], and P'''[0].

```
Pderivs = Solve[derivs == 0, {P'[0], P''[0], P'''[0]}];

% // Simplify
```

$$\left\{\left\{P^{(3)}[0] \to \frac{A^{\frac{1}{\eta+\nu}} B^{-\frac{1}{\eta+\nu}} \eta \left(6 \eta^2 + 7 \eta \nu + 2 \nu^2\right)}{(\eta+\nu)^3},\ P''[0] \to \frac{A^{\frac{1}{\eta+\nu}} B^{-\frac{1}{\eta+\nu}} \eta (2 \eta + \nu)}{(\eta+\nu)^2},\ P'[0] \to \frac{A^{\frac{1}{\eta+\nu}} B^{-\frac{1}{\eta+\nu}} \eta}{\eta+\nu}\right\}\right\}$$

Create the degree three power series for P[τ] at τ=0 using the solutions in Pderivs.

```
Series[P[τ], {τ, 0, 3}] /. Pderivs[[1]];
% // Simplify
```

$$\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}} + \frac{A^{\frac{1}{\eta+\nu}} B^{-\frac{1}{\eta+\nu}} \eta \tau}{\eta+\nu} + \frac{A^{\frac{1}{\eta+\nu}} B^{-\frac{1}{\eta+\nu}} \eta (2 \eta + \nu) \tau^2}{2 (\eta+\nu)^2} + \frac{A^{\frac{1}{\eta+\nu}} B^{-\frac{1}{\eta+\nu}} \eta \left(6 \eta^2 + 7 \eta \nu + 2 \nu^2\right) \tau^3}{6 (\eta+\nu)^3} + O[\tau]^4$$

This is an asymptotically valid third-order solution to computing the equilibrium as we change the tax τ.

■ Numerical Applications

It is normally impractical to compute the abstract form of the derivatives of P. Of course, these closed-form solutions are not the objective. We generally will want to compute the Taylor series for P[τ] for some specific parameter values. The issue is when do we make those substitutions. We now follow the following strategy: compute the abstract derivatives of the implicit expression that defines P, then replace the parameters with numerical values, and solve for the derivatives of P.

Repeat the construction of the derivatives:

```
ExDmdTax'[0]
```

$$-A\left(\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}}\right)^{-1-\nu}\nu\,P'[0]-\left(\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}}\right)^{-1+\eta}B\,\eta\left(-\left(\frac{A}{B}\right)^{\frac{1}{\eta+\nu}}+P'[0]\right)$$

```
derivs = {ExDmdTax'[0], ExDmdTax''[0], ExDmdTax'''[0]} // PowerExpand // Simplify
```

$$\Big\{A^{-\frac{1+\nu}{\eta+\nu}}B^{\frac{\nu}{\eta+\nu}}\left(A^{\frac{1+\eta+\nu}{\eta+\nu}}\eta-A\,B^{\frac{1}{\eta+\nu}}(\eta+\nu)\,P'[0]\right),\ A^{-\frac{2+\nu}{\eta+\nu}}B^{-\frac{\eta}{\eta+\nu}}\Big(-A^{\frac{2+\eta+\nu}{\eta+\nu}}B\,(-1+\eta)\,\eta-$$

$$A\,B^{1+\frac{2}{\eta+\nu}}\left(-\eta+\eta^2-\nu\,(1+\nu)\right)P'[0]^2+A^{\frac{1+\eta+\nu}{\eta+\nu}}B^{1+\frac{1}{\eta+\nu}}\eta\,(2\,\eta\,P'[0]-P''[0])-A^{1+\frac{1}{\eta+\nu}}B^{\frac{1+\eta+\nu}{\eta+\nu}}\nu\,P''[0]\Big),$$

$$A^{\frac{-3+\eta}{\eta+\nu}}B^{\frac{\nu}{\eta+\nu}}\Big(-B^{\frac{3}{\eta+\nu}}\nu\,(1+\nu)\,(2+\nu)\,P'[0]^3+(-2+\eta)\,(-1+\eta)\,\eta\left(A^{\frac{1}{\eta+\nu}}-B^{\frac{1}{\eta+\nu}}P'[0]\right)^3-$$

$$3\,A^{\frac{1}{\eta+\nu}}B^{\frac{1}{\eta+\nu}}(-1+\eta)\,\eta\left(A^{\frac{1}{\eta+\nu}}-B^{\frac{1}{\eta+\nu}}P'[0]\right)(2\,P'[0]-P''[0])+$$

$$3\,A^{\frac{1}{\eta+\nu}}B^{\frac{2}{\eta+\nu}}\nu\,(1+\nu)\,P'[0]\,P''[0]+A^{\frac{2}{\eta+\nu}}B^{\frac{1}{\eta+\nu}}\eta\left(3\,P''[0]-P^{(3)}[0]\right)-A^{\frac{2}{\eta+\nu}}B^{\frac{1}{\eta+\nu}}\nu\,P^{(3)}[0]\Big)\Big\}$$

Then, when one wants to compute specific cases, evaluate these expressions with parameter values. This is not as easy as it sounds due to round-off error.

```
vals = Thread[{ν, η, A, B} → {2., 3., 1., 5.}];
```

In general, we need to solve the equations in a sequential linear manner.

The first derivative expression gives us a linear equation for P'[0], which we then use to solve for P'[0].

```
eq1 = derivs[[1]] /. vals // Expand
```

```
5.71096 - 13.1326 P′[0]
```

```
sol1 = Solve[eq1 == 0, P'[0]][[1]]
```

```
{P′[0] → 0.434868}
```

The second derivative implies

```
eq2 = derivs[[2]] /. vals // Expand
```

$$-11.4219 + 47.2775\,P'[0] + 0.\,P'[0]^2 - 13.1326\,P''[0]$$

Solving for P''[0] gives us

```
sol2 = Solve[eq2 == 0, P''[0]][[1]]
```

$$\{P''[0] \to -0.0761462\,(11.4219 - 47.2775\,P'[0] + 0.\,P'[0]^2)\}$$

which contains P'[0]. We now substitute the solution for P'[0] to complete the solution for P''[0]

```
sol2 = sol2 /. sol1
```

```
{P″[0] → 0.695788}
```

The third derivative is solved by the following sequence:

```
eq3 = derivs[[3]] /. vals // Expand
```

$$11.4219 - 141.833\, P'[0] + 195.691\, P'[0]^2 - 150.\, P'[0]^3 + 70.9163\, P''[0] - 1.35263 \times 10^{-14}\, P'[0]\, P''[0] - 13.1326\, P^{(3)}[0]$$

```
sol3 = Solve[eq3 == 0, P'''[0]][[1]]
```

$$\{P^{(3)}[0] \to -0.0761462\, (-11.4219 + 141.833\, P'[0] - 195.691\, P'[0]^2 + 150.\, P'[0]^3 - 70.9163\, P''[0] + 1.35263 \times 10^{-14}\, P'[0]\, P''[0])\}$$

```
sol3 = sol3 /. sol2 /. sol1
```

$$\{P^{(3)}[0] \to 1.80905\}$$

Multiple Consumers and Firms

We represent demand and supply implicitly through optimality conditions implied by the utility and production functions:
Ui[di, p]=0 iff di is agent i's demand at price p.
Fi[qi, p(1-t)]=0 iff qi is firm i's output when after-tax price is p(1-t)

```
eqns = {U1[d1, p], U2[d2, p], U3[d3, p],
    F1[q1, p (1 - τ)], F2[q2, p (1 - τ)], (q1 + q2) - (d1 + d2 + d3)} /. p → P[τ];
eqns = % /. d1 → D1[τ] /. d2 → D2[τ] /. d3 → D3[τ] /. q1 → Q1[τ] /. q2 → Q2[τ];
```

```
eqns // TableForm
```

```
U1[D1[τ], P[τ]]
U2[D2[τ], P[τ]]
U3[D3[τ], P[τ]]
F1[Q1[τ], (1 - τ) P[τ]]
F2[Q2[τ], (1 - τ) P[τ]]
-D1[τ] - D2[τ] - D3[τ] + Q1[τ] + Q2[τ]
```

Initialize the solutions at t=0.

```
P[0] = p0;
Q1[0] = q10; Q2[0] = q20; D1[0] = d10; D2[0] = d20; D3[0] = d30;
U1[d10, p0] = 0; U2[d20, p0] = 0; U3[d30, p0] = 0;
F1[q10, p0] = 0; F2[q20, p0] = 0;
```

```
eqns0 = (D[eqns, τ]) /. τ → 0;
```

```
% // MatrixForm
```

$$\begin{pmatrix} P'[0]\, U1^{(0,1)}[d10, p0] + D1'[0]\, U1^{(1,0)}[d10, p0] \\ P'[0]\, U2^{(0,1)}[d20, p0] + D2'[0]\, U2^{(1,0)}[d20, p0] \\ P'[0]\, U3^{(0,1)}[d30, p0] + D3'[0]\, U3^{(1,0)}[d30, p0] \\ (-p0 + P'[0])\, F1^{(0,1)}[q10, p0] + Q1'[0]\, F1^{(1,0)}[q10, p0] \\ (-p0 + P'[0])\, F2^{(0,1)}[q20, p0] + Q2'[0]\, F2^{(1,0)}[q20, p0] \\ -D1'[0] - D2'[0] - D3'[0] + Q1'[0] + Q2'[0] \end{pmatrix}$$

```
vars = {P'[0], D1'[0], D2'[0], D3'[0], Q1'[0], Q2'[0]};
```

Let's use a substitution that reduces size of expressions.

```
sbs = Derivative[jj__][gg__][xx__] → Derivative[jj][gg];
```

This substitution makes our variable list simpler

```
vars /. sbs
```

$\{P', D1', D2', D3', Q1', Q2'\}$

LinSystem is the linear system implied by IFT.

```
LinSystem = CoefficientArrays[(eqns0 /. sbs), (vars /. sbs)];
```

LinSystem[[2]] is the Jacobian, and LinSystem[[1]] is the vector term.

```
jac = LinSystem[[2]];
jac // MatrixForm
```

$$\begin{pmatrix} U1^{(0,1)} & U1^{(1,0)} & 0 & 0 & 0 & 0 \\ U2^{(0,1)} & 0 & U2^{(1,0)} & 0 & 0 & 0 \\ U3^{(0,1)} & 0 & 0 & U3^{(1,0)} & 0 & 0 \\ F1^{(0,1)} & 0 & 0 & 0 & F1^{(1,0)} & 0 \\ F2^{(0,1)} & 0 & 0 & 0 & 0 & F2^{(1,0)} \\ 0 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

The vector term is:

```
vec = LinSystem[[1]];
vec // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -p0\, F1^{(0,1)} \\ -p0\, F2^{(0,1)} \\ 0 \end{pmatrix}$$

We can use LinearSolve to solve this system, but the result is not pretty. Delete the apostrophe in the next command to see the result.

```
LinearSolve[jac, vec];
```

■ General problem

In general
(1) There will be many goods, making the di's and p's vectors.
(2) The diagonal terms involving Ui will be blocks corresponding to Jacobians of equations, one block for each demander.
(3) Similarly for the blocks with Fi

Functional forms

We will want to solve such systems with thousands of demanders and hundreds of goods. This is quite feasible.

Typically, the functional forms of the Ui's are the same across people; the differences are really only in the parameter values. For example, Ui could be

$$\mathrm{Ui}[q, p] = A_i (a_i + b_i q^{\rho_i})^{\gamma_i}$$

Similarly for the production equation, Fi.

For multiple goods, it is often impossible to solve out for the demand vector explicitly; must use implicit form. Same for the output vectors for firms.

So, even if you have thousands of demanders and hundreds of goods, applying AD to construct the derivatives needed to construct LinSystem (which is done only when one has substituted in the parameters for Ui's and Fi's) is feasible.

Higher-order terms

We can go to higher-order derivatives

```
eqnstt = D[eqns, τ, τ]; eqns0 = eqnstt /. τ → 0; eqns0 = eqns0 /. sbs;
vars = {P''[0], D1''[0], D2''[0], D3''[0], Q1''[0], Q2''[0]} /. sbs;
LinSystem = CoefficientArrays[eqns0, vars];
```

The solvability matrix is unchanged.

```
LinSystem[[2]] // MatrixForm
```

$$\begin{pmatrix} U1^{(0,1)} & U1^{(1,0)} & 0 & 0 & 0 & 0 \\ U2^{(0,1)} & 0 & U2^{(1,0)} & 0 & 0 & 0 \\ U3^{(0,1)} & 0 & 0 & U3^{(1,0)} & 0 & 0 \\ F1^{(0,1)} & 0 & 0 & 0 & F1^{(1,0)} & 0 \\ F2^{(0,1)} & 0 & 0 & 0 & 0 & F2^{(1,0)} \\ 0 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

The vector term is more complex, but still fits our framework.

Perturbation Methods for a 2D Growth Models

Deterministic Model

```
x = 0; Remove["Global`*"]
```

■ Setup

We define the Euler equation for a simple growth model.
fi[ki] is the production function; ci[k1,k2] is the (unknown) consumption policy function.
kiplus and ciplus are the next period's capital stock and consumption

```
k1plus = f1[k1] - c1[k1, k2];
c1plus = c1[k1plus, k2plus];
k2plus = f2[k2] - c2[k1, k2];
c2plus = c2[k1plus, k2plus];
EulerEq =
 {u1[c1[k1, k2], c2[k1, k2]] - β u1[c1plus, c2plus] f1'[k1plus],
  u2[c1[k1, k2], c2[k1, k2]] - β u2[c1plus, c2plus] f2'[k2plus]}
```

```
{u1[c1[k1, k2], c2[k1, k2]] - β u1[c1[-c1[k1, k2] + f1[k1], -c2[k1, k2] + f2[k2]],
    c2[-c1[k1, k2] + f1[k1], -c2[k1, k2] + f2[k2]]] f1′[-c1[k1, k2] + f1[k1]],
 u2[c1[k1, k2], c2[k1, k2]] - β u2[c1[-c1[k1, k2] + f1[k1], -c2[k1, k2] + f2[k2]],
    c2[-c1[k1, k2] + f1[k1], -c2[k1, k2] + f2[k2]]] f2′[-c2[k1, k2] + f2[k2]]}
```

Choose utility and production functions. Put a free parameter, A, in f[k] so that we can later fix the steady state capital stock.

```
u[x_, y_] = Log[x] + Sqrt[y];
u1[x_, y_] = D[u[x, y], x];
u2[x_, y_] = D[u[x, y], y];
f1[x_] = x + A x^α;
f2[x_] = x + A x^α;
α = 1. / 4;
β = 95 / 100;
```

We want the steady state capital stock to be k=1 since it makes it easier to understand the results.
Choose A so that βf'[1]=1.

```
A = A //. Solve[f1'[1] == 1 / β, A][[1]]
```

```
0.210526
```

Let's look at our Euler equation

```
EulerEq
```

$$\left\{\frac{1}{c1[k1,\ k2]} - \frac{19\left(1 + \frac{0.0526316}{(0.210526\ k1^{0.25}+k1-c1[k1,k2])^{0.75}}\right)}{20\ c1\left[0.210526\ k1^{0.25} + k1 - c1[k1,\ k2],\ 0.210526\ k2^{0.25} + k2 - c2[k1,\ k2]\right]},\right.$$
$$\left.\frac{1}{2\sqrt{c2[k1,\ k2]}} - \frac{19\left(1 + \frac{0.0526316}{(0.210526\ k2^{0.25}+k2-c2[k1,k2])^{0.75}}\right)}{40\sqrt{c2\left[0.210526\ k1^{0.25} + k1 - c1[k1,\ k2],\ 0.210526\ k2^{0.25} + k2 - c2[k1,\ k2]\right]}}\right\}$$

ss is a list of substitutions that impose the steady state, k=1, and some substitutions that convert floating point versions of 1 and 0 to integer versions.

```
ss = {k1 → 1, k2 → 1, 1. → 1, 0. → 0}
```

```
{k1 → 1, k2 → 1, 1. → 1, 0. → 0}
```

The steady state consumption is defined next

```
c1[1, 1] = c2[1, 1] = css = f1[1] - 1
```

```
0.210526
```

Now check the Euler equation at the steady state.

```
EulerEq //. ss
```

$$\{8.88178\times 10^{-16},\ 2.22045\times 10^{-16}\}$$

sol will be the list of solutions for derivatives of c[k]. We begin the construction by setting sol to be the empty set.

```
sol = {};
```

prep for pert

```
EulerEqϵ = EulerEq /. k1 → 1 + ϵ κ1 /. k2 → 1 + ϵ κ2
```

$$\left\{\frac{1}{c1[1+\epsilon\kappa1,\,1+\epsilon\kappa2]} - \left(19\left(1+\frac{0.0526316}{\left(1+\epsilon\kappa1+0.210526\,(1+\epsilon\kappa1)^{0.25}-c1[1+\epsilon\kappa1,\,1+\epsilon\kappa2]\right)^{0.75}}\right)\right)\Big/ \left(20\,c1\left[1+\epsilon\kappa1+0.210526\,(1+\epsilon\kappa1)^{0.25}-c1[1+\epsilon\kappa1,\,1+\epsilon\kappa2],\ 1+\epsilon\kappa2+0.210526\,(1+\epsilon\kappa2)^{0.25}-c2[1+\epsilon\kappa1,\,1+\epsilon\kappa2]\right]\right),\right.$$

$$\left.\frac{1}{2\sqrt{c2[1+\epsilon\kappa1,\,1+\epsilon\kappa2]}} - \left(19\left(1+\frac{0.0526316}{\left(1+\epsilon\kappa2+0.210526\,(1+\epsilon\kappa2)^{0.25}-c2[1+\epsilon\kappa1,\,1+\epsilon\kappa2]\right)^{0.75}}\right)\right)\Big/ \left(40\sqrt{c2\left[1+\epsilon\kappa1+0.210526\,(1+\epsilon\kappa1)^{0.25}-c1[1+\epsilon\kappa1,\,1+\epsilon\kappa2],\ 1+\epsilon\kappa2+0.210526\,(1+\epsilon\kappa2)^{0.25}-c2[1+\epsilon\kappa1,\,1+\epsilon\kappa2]\right]}\right)\right\}$$

```
subs = {cc_^(i_,j_)[1, 1] → cc_{i,j}}
```

$$\{cc\_^{(i\_,j\_)}[1,\,1] \to cc_{i,j}\}$$

k pert

The Euler equation must hold at all k. Therefore, its derivative w.r.t. k must also be zero at all k. Compute the derivative of the Eulereq

```
D[EulerEqϵ, ϵ];
% /. ϵ → 0 /. 1. → 1
```

$$\{0.178125\,(1.05263\,\kappa1-\kappa2\,c1^{(0,1)}[1,1]-\kappa1\,c1^{(1,0)}[1,1]) - 22.5625\,(\kappa2\,c1^{(0,1)}[1,1]+\kappa1\,c1^{(1,0)}[1,1]) + 22.5625\,(c1^{(1,0)}[1,1]\,(1.05263\,\kappa1-\kappa2\,c1^{(0,1)}[1,1]-\kappa1\,c1^{(1,0)}[1,1]) + c1^{(0,1)}[1,1]\,(1.05263\,\kappa2-\kappa2\,c2^{(0,1)}[1,1]-\kappa1\,c2^{(1,0)}[1,1])),$$

$$0.0408647\,(1.05263\,\kappa2-\kappa2\,c2^{(0,1)}[1,1]-\kappa1\,c2^{(1,0)}[1,1]) - 2.5881\,(\kappa2\,c2^{(0,1)}[1,1]+\kappa1\,c2^{(1,0)}[1,1]) + 2.5881\,((1.05263\,\kappa1-\kappa2\,c1^{(0,1)}[1,1]-\kappa1\,c1^{(1,0)}[1,1])\,c2^{(1,0)}[1,1] + c2^{(0,1)}[1,1]\,(1.05263\,\kappa2-\kappa2\,c2^{(0,1)}[1,1]-\kappa1\,c2^{(1,0)}[1,1]))\}$$

```
% /. subs // Simplify
```

$$\{-22.5625\,\kappa1\,(-0.116233+c1_{1,0})\,(0.0714964+c1_{1,0}) + c1_{0,1}\,(1.00937\,\kappa2-22.5625\,\kappa2\,c1_{1,0}-22.5625\,\kappa2\,c2_{0,1}-22.5625\,\kappa1\,c2_{1,0}),$$

$$0.0430155\,\kappa2-2.5881\,\kappa2\,c2_{0,1}^2+(0.0953509\,\kappa1-2.5881\,\kappa2\,c1_{0,1}-2.5881\,\kappa1\,c1_{1,0})\,c2_{1,0} + c2_{0,1}\,(0.0953509\,\kappa2-2.5881\,\kappa1\,c2_{1,0})\}$$

```
SolveAlways[% == 0, {κ1, κ2}]
```

$$\{\{c1_{0,1}\to 0.,\ c1_{1,0}\to -0.0714964,\ c2_{0,1}\to -0.111809,\ c2_{1,0}\to 0.\},$$
$$\{c1_{0,1}\to 0.,\ c1_{1,0}\to -0.0714964,\ c2_{0,1}\to 0.148651,\ c2_{1,0}\to 0.\},$$
$$\{c1_{0,1}\to 0.,\ c1_{1,0}\to 0.116233,\ c2_{0,1}\to -0.111809,\ c2_{1,0}\to 0.\},$$
$$\{c1_{0,1}\to 0.,\ c1_{1,0}\to 0.116233,\ c2_{0,1}\to 0.148651,\ c2_{1,0}\to 0.\}\}$$

There are two solutions, one corresponding to the stable manifold and the other one corresponding to the unstable manifold. We choose the second one because we know that c'[1]>0, and put it in the solution set.

```
sol = Union[sol, % // Last]
```

$\{c1_{0,1} \to 0., c1_{1,0} \to 0.116233, c2_{0,1} \to 0.148651, c2_{1,0} \to 0.\}$

```
sol = sol /. 0. → 0
```

$\{c1_{0,1} \to 0, c1_{1,0} \to 0.116233, c2_{0,1} \to 0.148651, c2_{1,0} \to 0\}$

▪ k pert - degree 2

We now move on to the second derivative

```
D[EulerEqϵ, {ϵ, 2}];
% /. ϵ → 0 /. 1. → 1 /. subs;
```

```
% /. sol
```

$$\{2.72335\,\kappa 1^2 - 22.5625\,(\kappa 2\,(\kappa 2\,c1_{0,2} + \kappa 1\,c1_{1,1}) + \kappa 1\,(\kappa 2\,c1_{1,1} + \kappa 1\,c1_{2,0})) - 0.2375\,\left(1.15086\,\kappa 1^2 - 0.75\,\left(-0.0394737\,\kappa 1^2 - \kappa 2\,(\kappa 2\,c1_{0,2} + \kappa 1\,c1_{1,1}) - \kappa 1\,(\kappa 2\,c1_{1,1} + \kappa 1\,c1_{2,0})\right)\right) - 1.\,\left(2.53917\,\kappa 1^2 - 22.5625\,\left(0.903981\,\kappa 2\,(0.903981\,\kappa 2\,c1_{0,2} + 0.936398\,\kappa 1\,c1_{1,1}) + 0.936398\,\kappa 1\,(0.903981\,\kappa 2\,c1_{1,1} + 0.936398\,\kappa 1\,c1_{2,0}) + 0.116233\,\left(-0.0394737\,\kappa 1^2 - \kappa 2\,(\kappa 2\,c1_{0,2} + \kappa 1\,c1_{1,1}) - \kappa 1\,(\kappa 2\,c1_{1,1} + \kappa 1\,c1_{2,0})\right)\right)\right),\ -0.0235791\,\kappa 2^2 - 0.0544862\,\left(1.07255\,\kappa 2^2 - 0.75\,\left(-0.0394737\,\kappa 2^2 - \kappa 2\,(\kappa 2\,c2_{0,2} + \kappa 1\,c2_{1,1}) - \kappa 1\,(\kappa 2\,c2_{1,1} + \kappa 1\,c2_{2,0})\right)\right) + \frac{1}{2}\left(0.81495\,\kappa 2^2 - 5.17619\,(\kappa 2\,(\kappa 2\,c2_{0,2} + \kappa 1\,c2_{1,1}) + \kappa 1\,(\kappa 2\,c2_{1,1} + \kappa 1\,c2_{2,0}))\right) - 0.5\,\left(0.665961\,\kappa 2^2 - 5.17619\,\left(0.903981\,\kappa 2\,(0.903981\,\kappa 2\,c2_{0,2} + 0.936398\,\kappa 1\,c2_{1,1}) + 0.936398\,\kappa 1\,(0.903981\,\kappa 2\,c2_{1,1} + 0.936398\,\kappa 1\,c2_{2,0}) + 0.148651\,\left(-0.0394737\,\kappa 2^2 - \kappa 2\,(\kappa 2\,c2_{0,2} + \kappa 1\,c2_{1,1}) - \kappa 1\,(\kappa 2\,c2_{1,1} + \kappa 1\,c2_{2,0})\right)\right)\right)\}$$

```
CoefficientList[%, {κ1, κ2}]
```

$$\{\{\{0, 0, -6.92549\,c1_{0,2}\}, \{0, -12.5286\,c1_{1,1}, 0\}, \{-0.199701 - 5.57939\,c1_{2,0}, 0, 0\}\}, \{\{0, 0, -0.0243237 - 0.898741\,c2_{0,2}\}, \{0, -1.64579\,c2_{1,1}, 0\}, \{-0.744333\,c2_{2,0}, 0, 0\}\}\}$$

```
% // Flatten // Union // Rest
```

$$\{-6.92549\,c1_{0,2}, -12.5286\,c1_{1,1}, -0.199701 - 5.57939\,c1_{2,0}, -0.0243237 - 0.898741\,c2_{0,2}, -1.64579\,c2_{1,1}, -0.744333\,c2_{2,0}\}$$

```
eqs = %
```

$$\{-6.92549\,c1_{0,2}, -12.5286\,c1_{1,1}, -0.199701 - 5.57939\,c1_{2,0}, -0.0243237 - 0.898741\,c2_{0,2}, -1.64579\,c2_{1,1}, -0.744333\,c2_{2,0}\}$$

```
vars = Variables[eqs]
```

$$\{c1_{0,2}, c1_{1,1}, c1_{2,0}, c2_{0,2}, c2_{1,1}, c2_{2,0}\}$$

This is a linear expression in terms of the unknown c''[1]. Solve it

```
Solve[eqs == 0, vars]
```

$$\{\{c1_{0,2} \to 0., c1_{1,1} \to 0., c1_{2,0} \to -0.0357926, c2_{0,2} \to -0.0270642, c2_{1,1} \to 0., c2_{2,0} \to 0.\}\}$$

and add this solution to our solution set

```
sol = Union[sol, %[[1]]] // Simplify
```

$$\{c1_{0,1} \to 0, c1_{0,2} \to 0., c1_{1,0} \to 0.116233, c1_{1,1} \to 0., c1_{2,0} \to -0.0357926, c2_{0,1} \to 0.148651, c2_{0,2} \to -0.0270642, c2_{1,0} \to 0, c2_{1,1} \to 0., c2_{2,0} \to 0.\}$$

```
sol = sol /. 0. → 0
```

$$\{c1_{0,1} \to 0, c1_{0,2} \to 0, c1_{1,0} \to 0.116233, c1_{1,1} \to 0, c1_{2,0} \to -0.0357926, c2_{0,1} \to 0.148651, c2_{0,2} \to -0.0270642, c2_{1,0} \to 0, c2_{1,1} \to 0, c2_{2,0} \to 0\}$$

■ k pert - degree 3

We will now express the steps in a more compact manner

```
D[EulerEqϵ, {ϵ, 3}];
% /. ϵ → 0 /. 1. → 1 /. subs;
% /. sol;
CoefficientList[%, {κ1, κ2}];
% // Flatten // Union // Rest;
eqs = %
vars = Variables[eqs]
Solve[eqs == 0, vars]
 sol = Union[sol, %[[1]]] // Simplify
```

$\{-8.69586\, c1_{0,3}, -24.2945\, c1_{1,2}, -22.437\, c1_{2,1}, 0.370006 - 6.83767\, c1_{3,0}, 0.0436706 - 1.10182\, c2_{0,3}, -3.09976\, c2_{1,2}, -2.8867\, c2_{2,1}, -0.888667\, c2_{3,0}\}$

$\{c1_{0,3}, c1_{1,2}, c1_{2,1}, c1_{3,0}, c2_{0,3}, c2_{1,2}, c2_{2,1}, c2_{3,0}\}$

$\{\{c1_{0,3} \to 0., c1_{1,2} \to 0., c1_{2,1} \to 0., c1_{3,0} \to 0.0541129, c2_{0,3} \to 0.0396351, c2_{1,2} \to 0., c2_{2,1} \to 0., c2_{3,0} \to 0.\}\}$

$\{c1_{0,1} \to 0, c1_{0,2} \to 0, c1_{0,3} \to 0., c1_{1,0} \to 0.116233, c1_{1,1} \to 0, c1_{1,2} \to 0., c1_{2,0} \to -0.0357926, c1_{2,1} \to 0., c1_{3,0} \to 0.0541129, c2_{0,1} \to 0.148651, c2_{0,2} \to -0.0270642, c2_{0,3} \to 0.0396351, c2_{1,0} \to 0, c2_{1,1} \to 0, c2_{1,2} \to 0., c2_{2,0} \to 0, c2_{2,1} \to 0., c2_{3,0} \to 0.\}$

```
sol = sol /. 0. → 0
```

$\{c1_{0,1} \to 0, c1_{0,2} \to 0, c1_{0,3} \to 0, c1_{1,0} \to 0.116233, c1_{1,1} \to 0, c1_{1,2} \to 0, c1_{2,0} \to -0.0357926, c1_{2,1} \to 0, c1_{3,0} \to 0.0541129, c2_{0,1} \to 0.148651, c2_{0,2} \to -0.0270642, c2_{0,3} \to 0.0396351, c2_{1,0} \to 0, c2_{1,1} \to 0, c2_{1,2} \to 0, c2_{2,0} \to 0, c2_{2,1} \to 0, c2_{3,0} \to 0\}$

■ k pert - degree 4

We will now express the steps in a more compact manner

```
D[EulerEqϵ, {ϵ, 4}];
% /. ϵ → 0 /. 1. → 1 /. subs;
% /. sol;
CoefficientList[%, {κ1, κ2}];
% // Flatten // Union // Rest;
eqs = %
vars = Variables[eqs];
Solve[eqs == 0, vars];
 sol = Union[sol, %[[1]]] /. 0. → 0 // Simplify
```

$\{-10.2962\, c1_{0,4}, -39.0237\, c1_{1,3}, -55.1774\, c1_{2,2}, -34.4659\, c1_{3,1}, -1.08824 - 8.01592\, c1_{4,0}, -0.12473 - 1.28539\, c2_{0,4}, -4.89366\, c2_{1,3}, -6.95528\, c2_{2,2}, -4.37084\, c2_{3,1}, -1.02382\, c2_{4,0}\}$

$\{c1_{0,1} \to 0, c1_{0,2} \to 0, c1_{0,3} \to 0, c1_{0,4} \to 0, c1_{1,0} \to 0.116233, c1_{1,1} \to 0, c1_{1,2} \to 0, c1_{1,3} \to 0, c1_{2,0} \to -0.0357926, c1_{2,1} \to 0, c1_{2,2} \to 0, c1_{3,0} \to 0.0541129, c1_{3,1} \to 0, c1_{4,0} \to -0.135759, c2_{0,1} \to 0.148651, c2_{0,2} \to -0.0270642, c2_{0,3} \to 0.0396351, c2_{0,4} \to -0.0970362, c2_{1,0} \to 0, c2_{1,1} \to 0, c2_{1,2} \to 0, c2_{1,3} \to 0, c2_{2,0} \to 0, c2_{2,1} \to 0, c2_{2,2} \to 0, c2_{3,0} \to 0, c2_{3,1} \to 0, c2_{4,0} \to 0\}$

■ k pert - degree 5

We will now express the steps in a more compact manner

```
D[EulerEqϵ, {ϵ, 5}];
% /. ϵ → 0 /. 1. → 1 /. subs;
% /. sol;
CoefficientList[%, {κ1, κ2}];
% // Flatten // Union // Rest;
eqs = %
vars = Variables[eqs];
Solve[eqs == 0, vars];
 sol = Union[sol, %[[1]]] /. 0. → 0 // Simplify
```

$\{-11.743\, c1_{0,5},\ -56.2726\, c1_{1,4},\ -107.486\, c1_{2,3},\ -102.245\, c1_{3,2},\ -48.4079\, c1_{4,1},\ 4.34358 - 9.11923\, c1_{5,0},\ 0.485097 - 1.45134\, c2_{0,5},\ -6.97658\, c2_{1,4},\ -13.3728\, c2_{2,3},\ -12.7716\, c2_{3,2},\ -6.07443\, c2_{4,1},\ -1.15038\, c2_{5,0}\}$

$\{c1_{0,1} \to 0,\ c1_{0,2} \to 0,\ c1_{0,3} \to 0,\ c1_{0,4} \to 0,\ c1_{0,5} \to 0,\ c1_{1,0} \to 0.116233,\ c1_{1,1} \to 0,\ c1_{1,2} \to 0,\ c1_{1,3} \to 0,\ c1_{1,4} \to 0,\ c1_{2,0} \to -0.0357926,\ c1_{2,1} \to 0,\ c1_{2,2} \to 0,\ c1_{2,3} \to 0,\ c1_{3,0} \to 0.0541129,\ c1_{3,1} \to 0,\ c1_{3,2} \to 0,\ c1_{4,0} \to -0.135759,\ c1_{4,1} \to 0,\ c1_{5,0} \to 0.47631,\ c2_{0,1} \to 0.148651,\ c2_{0,2} \to -0.0270642,\ c2_{0,3} \to 0.0396351,\ c2_{0,4} \to -0.0970362,\ c2_{0,5} \to 0.33424,\ c2_{1,0} \to 0,\ c2_{1,1} \to 0,\ c2_{1,2} \to 0,\ c2_{1,3} \to 0,\ c2_{1,4} \to 0,\ c2_{2,0} \to 0,\ c2_{2,1} \to 0,\ c2_{2,2} \to 0,\ c2_{2,3} \to 0,\ c2_{3,0} \to 0,\ c2_{3,1} \to 0,\ c2_{3,2} \to 0,\ c2_{4,0} \to 0,\ c2_{4,1} \to 0,\ c2_{5,0} \to 0\}$

▪ k pert - degree more

We will now express the steps in a more compact manner

```
sol5 = sol;
```

```
sol = sol5
```

$$\{c1_{0,1}\to 0,\ c1_{0,2}\to 0,\ c1_{0,3}\to 0,\ c1_{0,4}\to 0,\ c1_{0,5}\to 0,\ c1_{1,0}\to 0.116233,\ c1_{1,1}\to 0,\ c1_{1,2}\to 0,\ c1_{1,3}\to 0,\ c1_{1,4}\to 0,\ c1_{2,0}\to -0.0357926,\ c1_{2,1}\to 0,\ c1_{2,2}\to 0,\ c1_{2,3}\to 0,\ c1_{3,0}\to 0.0541129,\ c1_{3,1}\to 0,\ c1_{3,2}\to 0,\ c1_{4,0}\to -0.135759,\ c1_{4,1}\to 0,\ c1_{5,0}\to 0.47631,\ c2_{0,1}\to 0.148651,\ c2_{0,2}\to -0.0270642,\ c2_{0,3}\to 0.0396351,\ c2_{0,4}\to -0.0970362,\ c2_{0,5}\to 0.33424,\ c2_{1,0}\to 0,\ c2_{1,1}\to 0,\ c2_{1,2}\to 0,\ c2_{1,3}\to 0,\ c2_{1,4}\to 0,\ c2_{2,0}\to 0,\ c2_{2,1}\to 0,\ c2_{2,2}\to 0,\ c2_{2,3}\to 0,\ c2_{3,0}\to 0,\ c2_{3,1}\to 0,\ c2_{3,2}\to 0,\ c2_{4,0}\to 0,\ c2_{4,1}\to 0,\ c2_{5,0}\to 0\}$$

```
Do[
 d0 = D[EulerEqϵ, {ϵ, jjj}];
 d1 = d0 /. ϵ → 0 /.  1. → 1 /. subs;
d2 = d1 /. sol;
d3 = CoefficientList[d2, {ϰ1, ϰ2}];
eqs = d3 // Flatten // Union // Rest;
vars = Variables[eqs];
solnew = Solve[eqs == 0, vars];
sol = Union[sol, solnew[[1]]] /.  0. → 0 // Simplify,
 {jjj, 6, 8}]
```

```
Length[sol]
```

88

```
sol
```

$$\{c1_{0,1}\to 0,\ c1_{0,2}\to 0,\ c1_{0,3}\to 0,\ c1_{0,4}\to 0,\ c1_{0,5}\to 0,\ c1_{0,6}\to 0,\ c1_{0,7}\to 0,\ c1_{0,8}\to 0,\ c1_{1,0}\to 0.116233,\ c1_{1,1}\to 0,\ c1_{1,2}\to 0,\ c1_{1,3}\to 0,\ c1_{1,4}\to 0,\ c1_{1,5}\to 0,\ c1_{1,6}\to 0,\ c1_{1,7}\to 0,\ c1_{2,0}\to -0.0357926,\ c1_{2,1}\to 0,\ c1_{2,2}\to 0,\ c1_{2,3}\to 0,\ c1_{2,4}\to 0,\ c1_{2,5}\to 0,\ c1_{2,6}\to 0,\ c1_{3,0}\to 0.0541129,\ c1_{3,1}\to 0,\ c1_{3,2}\to 0,\ c1_{3,3}\to 0,\ c1_{3,4}\to 0,\ c1_{3,5}\to 0,\ c1_{4,0}\to -0.135759,\ c1_{4,1}\to 0,\ c1_{4,2}\to 0,\ c1_{4,3}\to 0,\ c1_{4,4}\to 0,\ c1_{5,0}\to 0.47631,\ c1_{5,1}\to 0,\ c1_{5,2}\to 0,\ c1_{5,3}\to 0,\ c1_{6,0}\to -2.14781,\ c1_{6,1}\to 0,\ c1_{6,2}\to 0,\ c1_{7,0}\to 11.8364,\ c1_{7,1}\to 0,\ c1_{8,0}\to -77.0954,\ c2_{0,1}\to 0.148651,\ c2_{0,2}\to -0.0270642,\ c2_{0,3}\to 0.0396351,\ c2_{0,4}\to -0.0970362,\ c2_{0,5}\to 0.33424,\ c2_{0,6}\to -1.48609,\ c2_{0,7}\to 8.10074,\ c2_{0,8}\to -52.3138,\ c2_{1,0}\to 0,\ c2_{1,1}\to 0,\ c2_{1,2}\to 0,\ c2_{1,3}\to 0,\ c2_{1,4}\to 0,\ c2_{1,5}\to 0,\ c2_{1,6}\to 0,\ c2_{1,7}\to 0,\ c2_{2,0}\to 0,\ c2_{2,1}\to 0,\ c2_{2,2}\to 0,\ c2_{2,3}\to 0,\ c2_{2,4}\to 0,\ c2_{2,5}\to 0,\ c2_{2,6}\to 0,\ c2_{3,0}\to 0,\ c2_{3,1}\to 0,\ c2_{3,2}\to 0,\ c2_{3,3}\to 0,\ c2_{3,4}\to 0,\ c2_{3,5}\to 0,\ c2_{4,0}\to 0,\ c2_{4,1}\to 0,\ c2_{4,2}\to 0,\ c2_{4,3}\to 0,\ c2_{4,4}\to 0,\ c2_{5,0}\to 0,\ c2_{5,1}\to 0,\ c2_{5,2}\to 0,\ c2_{5,3}\to 0,\ c2_{6,0}\to 0,\ c2_{6,1}\to 0,\ c2_{6,2}\to 0,\ c2_{7,0}\to 0,\ c2_{7,1}\to 0,\ c2_{8,0}\to 0\}$$

Perturbation Methods for General Dynamic Stochastic Models

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Abstract. We describe a general Taylor series method for computing asymptotically valid approximations to deterministic and stochastic rational expectations models near the deterministic steady state. Contrary to conventional wisdom, the higher-order terms are conceptually no more difficult to compute than the conventional deterministic linear approximations. We display the solvability conditions for second- and third-order approximations and show how to compute the solvability conditions in general. We use an implicit function theorem to prove a local existence theorem for the general stochastic model given existence of the degenerate deterministic model. We describe an algorithm which takes as input the equilibrium equations and an integer k, and computes the order k Taylor series expansion along with diagnostic indices indicating the quality of the approximation. We apply this algorithm to some multidimensional problems and show that the resulting nonlinear approximations are far superior to linear approximations.

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Economists are using increasingly complex dynamic stochastic models and need more powerful and reliable computational methods for their analysis. We describe a general perturbation method for computing asymptotically valid approximations to general stochastic rational expectations models based on their deterministic steady states. These approximations go beyond the normal "linearize around the steady state" approximations by adding both higher-order terms and deviations from certainty equivalence. The higher-order terms and corrections for risk will likely improve the accuracy of the approximations and their useful range. Also, some questions, such as welfare effects of security markets, can be answered only with higher-order approximations; see Judd and Guu (2001) for ,models where higher-order terms are essential. Contrary to conventional wisdom, these higher-order terms are no more difficult to compute than the conventional deterministic linear approximations; in fact, they are conceptually easier. However, we show that one cannot just assume that the higher-order terms create a better approximation. We examine the relevant implicit function theorems that justify perturbation methods in some cases and point out cases where perturbation methods may fail. Since perturbation methods are not perfectly reliable, we also present diagnostic procedures which will indicate the reliability of any specific approximation. Since the diagnostic procedures consume little computational effort compared with the construction of the approximation, they produce critical information at little cost.

Linearizations methods for dynamic models have been a workhorse of macroeconomic analysis. Magill (1977) showed how to compute a linear approximation around deterministic steady states and apply them to approximate spectral properties of stochastic models. Kydland and Prescott (1982) applied a special case of the Magill method to a real business cycle model. However, the approximations in Magill, and Kydland and Prescott were just linear approximations of the deterministic model applied to stochastic models; they ignored higher-order terms and were certainty equivalent approximations, that is, variance had no impact on decision rules. The motivating intuition was also specific to the case of linear, certainty equivalent, approximations. Kydland and Prescott (1982) motivated their procedure by replacing the nonlinear law of motion with a linear law of motion and replacing the nonlinear payoff function with a quadratic approximation, and then applying linear-quadratic dynamic programming methods to the approximate model. This motivation gives the impression that it is not easy to compute higher-order approximations, particularly since computing the first-order terms requires solving a quadratic matrix equation. In fact, Marcet(1994) dismissed the possibility that higher-order approximations be computed, stating that "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..."

Furthermore, little effort has been made to determine the conditions under which certainty equivalent linearizations are valid. Linearization methods are typically used in an application without examining whether they are valid in that case. This raises questions about many of the applications, particularly since the conventional linearization approach sometimes produces clearly erroneous results. For example, Tesar (1995) uses the standard Kydland-Prescott method and found an example where completing asset markets will make all agents worse off. This result violates general equilibrium theory and can only be attributed to the numerical method used. Kim and Kim (forthcoming) show that this will often occur in simple stochastic models. Below we will present a portfolio-like example which shows that casual applications of higher-order procedures (such as those advocated by Sims, 2002, and Campbell and Viciera, 2002) can easily produce nonsensical answers. These examples emphasize two important points. First, more flexible, robust, and accurate methods based on sound mathematical principles are needed. Second, we cannot blindly accept the results of a Taylor series approximation but need ways to test an approximation's reliability. This paper addresses both issues.

We will show that it is practical to compute higher-order terms to the multivariate Taylor series approximation based at the deterministic steady state. The basic fact shown below is that all the higher–order terms of the Taylor series expansion, even in the stochastic multidimensional case, are solutions to linear problems once one computes the first–order terms. This implies that the higher–order terms are *easier* to compute in the sense that linear problems are conceptually less complex. In previous papers, Judd and Guu (1993, 1997) examined perturbation methods for deterministic, continuous- and discrete-time growth models in one capital stock, and stochastic growth models in continuous time with one state. They find that the high-order approximations can be used to compute highly accurate approximations which avoid the certainty equivalence property of the standard linearization method. Judd and Gaspar (1997) described perturbation methods for multidimensional stochastic models in continuous time, and produced Fortran computer code for fourth-order expansions. Judd (1998) presented the general method for deterministic discrete-time models and presented a discrete-time stochastic example indicating the critical adjustments necessary to move from continuous time to discrete time. In particular, the natural perturbation parameter is the instantaneous variance in the continuous-time case, but the standard deviation is the natural perturbation parameter for discrete-time stochastic models. The reader is referred to these papers and their mathematical sources for key definitions and introductions to these methods. In this paper, we will outline how these methods can be adapted to handle more general rational expectations problems.

There has recently been an increase in the interest in higher-order approximation methods. Collard and Juillard (2001a) computed a higher-order perturbation approximation of an asset-pricing model and Collard and Juillard (2001b). Chen and Zadrozny (forthcoming) computed higher-order approximations for a family of optimal control problems. Kim and Kim (forthcoming) applied second-order approximation methods to welfare questions in international trade. Sims (2000) and Grohe-Schmidt and Uribe (2002) have generalized Judd (1998), Judd and Gaspar (1997), and Judd and Guu (1993) by examining second-order approximations of multidimensional discrete-time models.

The first key step is to express the problem formally as two different kinds of perturbation problems and apply the appropriate implicit function theorems. Even though we are applying ideas from implicit function theory, there are unique difficulties which arise in stochastic dynamic models. Perturbation methods revolve around *solvability conditions,* that is, conditions which guarantee a unique solution to terms in an asymptotic expansion. We display the solvability conditions for Taylor series expansions of arbitrary orders for both deterministic and stochastic problems, showing that they reduce to the invertibility of a series of matrices. The implicit function theorem for the deterministic problem is straightforward, but the stochastic components produce novel problems. We give an example where a casual approach will produce a nonsensical result. We use an implicit function theorem to prove a local existence theorem for the general stochastic model given existence of the degenerate deterministic model. This is a nontrivial step and an important one since it is easy for economists to specify models which lack a local existence theorem justifying perturbation methods.

We then describe an algorithm which takes as input the equilibrium equations and an integer k, and computes the order k Taylor series expansion along with diagnostic indices indicating the quality of the approximation. We apply this algorithm to some multidimensional problems and show that the resulting nonlinear approximations are far superior to linear approximations over a large range of states. We also emphasize the importance of error estimation along with computation of the approximation.

1. A Perturbation Approach to the General Rational Expectations Problem

We examine general stochastic problems of the form

$$\begin{aligned} 0 &= E\left\{g^{\mu}(x_t, y_t, x_{t+1}, y_{t+1}, \varepsilon z)|x_t\right\},\ \mu = 1, ..., m \\ x^i_{t+1} &= F^i(x_t, y_t, \varepsilon z_t),\ i = 1, ..., n \end{aligned} \quad (1)$$

where $x_t \in \mathbb{R}^n$ are the predetermined variables at the beginning of time t, such as capital stock, lagged variables and productivity, $y_t \in \mathbb{R}^m$ are the endogenous variables at time t, such as consumption, labor supply and prices, and $F^i(x_t, y_t, \varepsilon z_t)$: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \rightarrow \mathbb{R}$, $i = 1, ..., n$ is the law of motion for x^i, and

$$g^{\mu}(x_t, y_t, x_{t+1}, y_{t+1}, \varepsilon z) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R},\ \mu = 1, ..., q$$

are the equations defining equilibrium, including Euler equations and market clearing conditions. The scalar ε is a scaling parameter for the disturbance terms z. We assume that the components of z are i.i.d. with mean zero and unit variance, making ε the common standard deviation. Since correlation and heteroscedasticity can be built into the function g, we can do this without loss of generality. Different values for ε represent economies with different levels of uncertainty. The objective is to find some equilibrium rule, $Y(x, \varepsilon)$, such that in the ε-economy the endogenous and predetermined variables satisfy

$$y_t = Y(x_t, \varepsilon)$$

This implies that $Y(x, \varepsilon)$ must satisfy the functional equation

$$E\left\{g^{\mu}(x, Y(x, \varepsilon), F(x, Y(x, \varepsilon), \varepsilon z), Y(F(x, Y(x, \varepsilon), \varepsilon z), \varepsilon), \varepsilon z)|x\right\} = 0 \quad (2)$$

Our perturbation method will approximate $Y(x, \varepsilon)$ with a polynomial in (x, ε) in a neighborhood of the deterministic steady state. The deterministic steady state is the solution to

$$\begin{aligned} 0 &= g(x_*, y_*, x_*, y_*, 0) \qquad (3) \\ x_* &= F(x_*, y_*, 0) \end{aligned}$$

The steady state y_* is the first step in approximating $Y(x, \varepsilon)$ since $Y(x_*, 0) = y_*$. The task is to find the derivatives of $Y(x, \varepsilon)$ with respect to x and ε at the deterministic steady state, and use that information to construct a degree k Taylor series approximation of $Y(x, \varepsilon)$, such as in

$$\begin{aligned} Y(x_* + v, \varepsilon) &\doteq y_* + Y_x(x_*, 0)\, v + \varepsilon Y_\varepsilon(x_*, 0) \qquad (4) \\ &+ v^\top Y_{xx}(x_*, 0)\, v + \varepsilon Y_{x\varepsilon}(x_*, 0)\, v + \varepsilon^2 Y_{\varepsilon\varepsilon}(x_*, 0) \\ &+ \ldots \\ &+ o\left(\varepsilon^k + \|v\|^k\right) \qquad (5) \end{aligned}$$

If Y is analytic in the neighborhood of $(x_*, 0)$ then this series has an infinite number of terms and it is locally convergent. The objective is also to be able to use the Taylor series approximation in simulations of the nonlinear model and be able to produce uniformly valid approximations of the long-run and short-run behavior of the true nonlinear model. This is a long list of requirements but we will develop diagnostics to check out the performance of our Taylor series approximations.

Equation (1) includes a broad range of dynamic stochastic models, but does leave out some models. For example, models with intertemporally nonseparable preferences, like those in Epstein-Zin (1989), are functional equations and do not obviously reduce to a dynamic system in $\mathbb{R}^m$. However, with modest modifications, our methods can be applied to any problem of the form in (2), a larger set of problems than those expressible as (1). We also assume that any solution to (3) is locally unique. This rules out many interesting models, particularly models with portfolio choices and models where income distribution may matter. Portfolio problems can probably be handled with dynamic extensions of Judd and Guu (2001), and income distribution problems can probably be handled by application of the center manifold theorem, but we leave these developments for later work.

Computing and evaluating the approximation in (4) is accomplished in five steps. The first is to solve (3) for steady state values (x_*, y_*). This is presumably accomplished by applying some nonlinear equation solver to (3) and will not be further discussed here. The second is to compute the linear approximation terms, $Y_x(x_*, 0)$. This is done by analyzing the the deterministic system formed by setting $\varepsilon = 0$ in (1) to create the perfect foresight system

$$\begin{aligned} 0 &= g^{\mu}(x_t, y_t, x_{t+1}, y_{t+1}, 0) \\ x^i_{t+1} &= F^i(x_t, y_t, 0) \end{aligned} \tag{6}$$

Computing the linear terms is a standard computation, solvable by a variety of techniques. See the literature on linear rational expectations models for solution methods (Anderson, et al. , 1996, is a survey of this literature); we will not discuss this step further.

This paper is concerned with the next three steps. Third, we compute the higher-order deterministic terms; that is, we compute perturbations of $Y(x, 0)$ in the x directions. Formally, we want to compute $\frac{\partial}{\partial x^k} Y(x_*, 0)$, $k = 1, 2, \ldots$. This produces the Taylor series approximation for the deterministic problem

$$Y(x, 0) \doteq y_* + Y_x(x_*, 0)(x - x_*) + (x - x_*)^{\top} Y_{xx}(x_*, 0)(x - x_*) + \ldots \tag{7}$$

for the solution to (6).

Fourth, with the Taylor series for $Y(x, 0)$ in hand, we examine the general stochastic problem $Y(x, \varepsilon)$. We use the expansion (7) of the deterministic problem to compute the ε derivatives, $\left(\frac{\partial}{\partial \varepsilon}\right)^{\ell} \left(\frac{\partial}{\partial x}\right)^{k} Y(x_*, 0)$. More generally, we show that how to take a solution of $Y(x,0)$ and use it to construct a solution to $Y(x,\varepsilon)$ for small ε. This last step raises the possibility that we have an approximation which is not just locally valid around the deterministic steady state point $(x_*,0)$ but instead around a large portion of the stable manifold defined by $Y(x,0)$.

This four-stage approach is the proper procedure since each step requires solutions from the previous steps. Also, by separating the stochastic step from the deterministic steps we see the main point that we can perturb around the deterministic stable manifold, not just the deterministic steady state.

Before we accept the resulting candidate Taylor series, we must test its reliability. Let $\widehat{Y}(x,\varepsilon)$ be the computed finite order Taylor series we have computed. We evaluate it by computing

$$E\left\{\widetilde{g}^{\mu}(x,\widehat{Y}(x,\varepsilon),F(x,\widehat{Y}(x,\varepsilon),\varepsilon z),\widehat{Y}(F(x,\widehat{Y}(x,\varepsilon),\varepsilon z),\varepsilon),\varepsilon z)|x\right\}=0$$

for a range of values of (x,ε) that we want to use, where $\widetilde{g}^{\mu}(x_t,y_t,x_{t+1},y_{t+1},\varepsilon z)$ is a unit-free version of $g^{\mu}(x_t,y_t,x_{t+1},y_{t+1},\varepsilon z)$. That is, each component of $E\{\widetilde{g}^{\mu}\}$ is transformed so that any deviation from zero represent a relative error. For example, if one component of g^{μ} is supply equals demand then the corresponding component of $\widetilde{g}^{\mu}$ will express excess demand as a fraction of total demand and any deviation of $E\{\widetilde{g}^{\mu}(x_t,y_t,x_{t+1},y_{t+1},\varepsilon z)|x_t\}$ from zero represents the relative error in the supply equals demand condition. If these relative errors are sufficiently small then we will accept $\widehat{Y}(x,\varepsilon)$. This last step is critical since Taylor series expansions have only a finite range of validity and we have no a priori way of knowing the range of validity.

Before continuing, we warn the reader of the nontrivial notational challenge which awaits him in the sections below where we develop the theoretical properties of our perturbation method and present the formal justification of our algorithm. After being introduced to tensor notation and its application to multivariate stochastic control, the reader may decide that this approach is far too burdensome to be of value. If one had to go through these manipulations for each and every application, we might agree. Fortunately, all of the algebra discussed below has been automated, executing *all* the necessary computations, including analytic derivatives and error indices, and produce the Taylor series approximation discussed below. This will relieve the user of executing all the algebra we discuss below.

2. MULTIDIMENSIONAL COMPARATIVE STATICS AND TENSOR NOTATION

We first review the tensor notation necessary to efficiently express the critical multivariate formulas. We will follow the tensor notation conventions used in mathematics (see, for example, Bishop and Goldberg, 1981, and Misner et al., 1973) and statistics (see McCullagh, 1987), and use standard adaptations to deal with idiosyncratic features of rational expectations models. We then review the implicit function theorem, and higher-order applications of the implicit function theorem.

2.1. Tensor Notation. Multidimensional perturbation problems use the multidimensional chain rule. Unfortunately, the chain rule in R^n produces a complex sequence of summations, and conventional notation becomes unwieldy. The *Einstein summation notation* for tensors and its adaptations will give us a natural way to address the notational problems.[1] Tensor notation is a powerful way of dealing with multidimensional collections of numbers and operations involving them. We will present the elements of tensor notation necessary for our task; see Judd (1998) for more discussion.

Suppose that a_i is a collection of numbers indexed by $i = 1, \ldots, n$, and that x^i is a singly indexed collection of real numbers. Then

$$a_i\, x^i \equiv \sum_i a_i\, x^i.$$

[1]The dubious reader should try to read the formulas in Bensoussan(1988) where conventional notation is used.

is the tensor notation for the inner product of the vectors represented by a_i and x^i. This notation is desirable since it eliminates the unnecessary Σ symbol. Similarly suppose that a_{ij} is a collection of numbers indexed by $i, j = 1, \ldots, n$. Then

$$a_{ij}\, x^i\, y^j \equiv \sum_i \sum_j \ a_{ij}\, x^i\, y^j.$$

is the tensor notation for a quadratic form. Similarly, $a^i_j x_i y^j$ is the quadratic form of the matrix a^i_j with the vectors x and y, and the expression $z_j = a^i_j x_i$ can also be thought of as a matrix multiplying a vector. We will often make use of the Kronecker tensor, which is defined as

$$\delta^i_j \equiv \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$$

and is a representation of the identity matrix. δ^α_β, δ^I_J, etc., are similarly defined.

More formally, we let x^i denote any vector in R^n and let a_i denote any element in the dual space of R^n, that is, a linear map on vectors x^iin $\mathbb{R}^n$. Of course, the dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$. However, it is useful in tensor algebra to keep the distinction between a vector and an element in the dual space.

In general, $a^{i_1,i_2,\ldots,i_\ell}_{j_1,j_2,\ldots,j_m}$ is a $\ell - m$ *tensor*, a set of numbers indexed by ℓ superscripts and m subscripts. It can be thought of as a scalar-valued multilinear map on $(\mathbb{R}^n)^\ell \times (\mathbb{R}^n)^m$. This generalizes the idea that matrices are bilinear maps on $(\mathbb{R}^n)^2$. The summation convention becomes particularly useful for higher-order tensors. For example, in $\mathbb{R}^n$,

$$c^{i_3,j_4}_{j_3,i_4} = a^{i_1,i_2,i_3}_{j_1,j_2,j_3} b^{j_1,j_2,j_4}_{i_1,i_2,i_4} \equiv \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{j_1=1}^n \sum_{j_2=1}^n a^{i_1,i_2,i_3}_{j_1,j_2,j_3} b^{j_1,j_2,j_4}_{i_1,i_2,i_4}$$

In our applications if $f : R \rightarrow \mathbb{R}^m$, then f_j will be the derivative of f with respect to x^j. The following equation expresses the multivariate Taylor expansion in tensor notation:

$$f\left(x^0 + v\right) = f(x^0) + f_i(x^0)v^i + \frac{1}{2} f_{ij}(x^0)v^i v^j + \frac{1}{3!} f_{ijk}(x^0)v^i v^j v^k + \ldots,$$

where $f_i \equiv \frac{\partial f}{\partial x^i}(x^0)$, $f_{ij} \equiv \frac{\partial^2 f}{\partial x^i \partial x^j}(x^0)$, etc. More generally, if $f : \mathbb{R}^n \to \mathbb{R}^m$, then f^i_j will be the derivative of the i'th component of f with respect to x^j. We will make extensive use of the multivariate chain rule. If $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^\ell$ and $h(x) = g(f(x))$, then $h : \mathbb{R}^n \to \mathbb{R}^\ell$, and the Jacobian of h is

$$h^i_j \equiv \frac{\partial h^i}{\partial x^j} = g^i_\ell f^\ell_j.$$

Furthermore the 1-2 tensor of second-order derivatives is

$$h^i_{jk} \equiv \frac{\partial^2 h^i}{\partial x^j \partial x^k} = g^i_{\ell m} f^m_k f^\ell_j + g^i_\ell f^\ell_{jk}.$$

This can be continued to express arbitrary derivatives.

Equation (1) is based on

$$g(x, y, z, w, \varepsilon) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \mathbb{R}^{n_4} \times \mathbb{R} \to \mathbb{R}$$

which is a function of five groups of variables. Our perturbation analysis needs to distinguish among derivatives with respect to different subsets of the variables. We will use the standard device of letting different indices denote the differentiation with respect to different sets of variables. For example, in general relativity theory, one typically uses Latin letters, $a, b, c, ...$, to represent summation over the three space coordinates (x, y, z) and Greek letters, $\mu, \nu, \rho, ...$, to represent summation over space-time coordinates (x, y, z, t). We apply this practice here to distinguish among x, y, z, and w. Specifically, the derivatives of g with respect to x^i will be denoted by lower case Latin letters as in

$$g^\mu_i(x, y, z, w, \varepsilon) = \frac{\partial}{\partial x^i}\left(g^\mu(x, y, z, w, \varepsilon)\right)$$

This implies that

$$g^\mu_j(x, y, z, w, \varepsilon) = \frac{\partial}{\partial x^i}\left(g^\mu(x, y, z, w, \varepsilon)\right)$$

denotes the partial derivative of $g^{\mu}(x,y,z,w)$ w.r.t. the j'th component of x. In general, g with a subscript from $\{i,j,k,...\}$ denotes the vector of derivatives of $g^{\mu}(x,y,z,w,\varepsilon)$ with respect to the components of x. We use different indices to denote derivatives with respect to components of y. In particular, we will use lower case Greek letters to index components of y and define

$$\begin{aligned} g_{\alpha}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial y^{\alpha}}(g(x,y,z,w,\varepsilon)) \\ g_{\beta}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial y^{\beta}}(g(x,y,z,w,\varepsilon)) \end{aligned}$$

Derivatives of $g^{\mu}(x,y,z,w)$ w.r.t. components of z will use capitalized Latin letters, as in

$$\begin{aligned} g^{\mu}_{I}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial z^{I}}(g^{\mu}(x,y,z,w,\varepsilon)) \\ g^{\mu}_{J}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial z^{J}}(g^{\mu}(x,y,z,w,\varepsilon)) \end{aligned}$$

and derivatives of $g^{\mu}(x,y,z,w)$ w.r.t. components of w will use capitalized Greek letters, as in

$$\begin{aligned} g^{\mu}_{A}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial w^{A}}(g^{\mu}(x,y,z,w,\varepsilon)) \\ g^{\mu}_{\Gamma}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial w^{\Gamma}}(g^{\mu}(x,y,z,w,\varepsilon)) \\ g^{\mu}_{\Delta}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial w^{\Delta}}(g^{\mu}(x,y,z,w),\varepsilon) \end{aligned}$$

The distinction holds only for subscripts. For example, the notation y^{i} will denote the same vector as would y^{I}.

Since ε is a scaler, the derivatives w.r.t. ε will be denoted in the standard manner,

$$\begin{aligned} g^{\mu}_{\varepsilon}(x,y,z,w,\varepsilon) &= \frac{\partial}{\partial \varepsilon}(g^{\mu}(x,y,z,w,\varepsilon)) \\ g^{\mu}_{\varepsilon\varepsilon}(x,y,z,w,\varepsilon) &= \frac{\partial^{2}}{\partial \varepsilon^{2}}(g^{\mu}(x,y,z,w,\varepsilon)) \end{aligned}$$

Combinations of indices represent cross derivatives. For example,

$$g^{\mu}_{\alpha i}(x, y, z, w, \varepsilon) = \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^{\alpha}}\left(g^{\mu}(x, y, z, w, \varepsilon)\right)$$

We will often want the composite gradient of $g^{\mu}(x, y, z, w, \varepsilon)$ consisting of the derivatives with respect to (y, z, w). We let $\aleph = \{\alpha, I, A\}$ denote the set of all indices for (y, z, w) and use the notation $g^{\mu}_{\aleph}(x, y, z, w, \varepsilon)$ to represent the derivatives as in

$$g^{\mu}_{\aleph}(x, y, z, w, \varepsilon) = \left(\; g^{\mu}_{\alpha}(x, y, z, w, \varepsilon)\,,\;\; g^{\mu}_{I}(x, y, z, w, \varepsilon)\,,\;\; g^{\mu}_{A}(x, y, z, w, \varepsilon)\;\right)$$

2.2. Implicit Function Theorem and Solvability Conditions. The general implicit function theorem for $\mathbb{R}^n$ is the critical tool motivating our computations, even though it is not directly applicable to our problems. A review of the Implicit Function Theorem highlights the critical issues we will face. Let $H(x, y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and suppose $H(x_0, y_0) = 0$. Let

$$H^{\mu}(x, h(x)) = 0 \tag{8}$$

implicitly define $h : \mathbb{R}^n \to \mathbb{R}^m$ for x near x_0. In particular, we know $h(x_0) = y_0$. Taking the derivative of (8) with respect to x^i shows

$$H^{\mu}_{i}(x, h(x)) + H^{\mu}_{\alpha}(x, h(x))h^{\alpha}_{i}(x) = 0$$

which, at $x = x_0$, implies

$$\frac{\partial Y^{\alpha}}{\partial x^i}(x_0) = h^{\alpha}_{i}(x_0) = -\widetilde{H}^{\alpha}_{\mu}H^{\mu}_{i}$$

where $\widetilde{H}^{\alpha}_{\mu}$ is the tensor (matrix) satisfying

$$\widetilde{H}^{\alpha}_{\mu}\left(H^{\mu}_{\beta}(x_0, y_0)\right) = \delta^{\alpha}_{\beta}$$

The tensor h^{α}_{i} is the comparative static matrix which expresses how components of $h(x)$ change as we move x away from x_0. The solution h^{α}_{i} exists as long as $\widetilde{H}^{\alpha}_{q}$, the inverse matrix of $H^q_\alpha(x_0, y_0)$, exists. Therefore, the solvability condition for h^α_i is the nonsingularity of $H^q_\alpha(x_0, y_0)$. If $H^q_\alpha(x_0, y_0)$ is invertible, then the linear approximation of $h(x)$ based at $x = x_0$ is

$$h^\alpha(x_0 + v) \doteq h^\alpha(x_0) + h^\alpha_i v^i.$$

which is just the beginning of the multivariate Taylor series approximation of $h(x)$ based at x_0.

We often need to go beyond the first-order expression of and construct higher-order terms of the Taylor series expansion. To do this in a clean and compact fashion, we need tensor notation. Taking another derivative of (8) w.r.t. x^j implies

$$H_{ij}(x, h(x)) + H^\mu_{\alpha j}(x, h(x)) h^\alpha_i(x) h^\beta_j(x) + H^\mu_\alpha(x, h(x)) h^\alpha_{ij}(x) = 0$$

which, at $x = x_0$, implies

$$h^\alpha_{ij}(x_0) = -\widetilde{H}^\alpha_q \left(H_{ij} + H^q_{\beta j} h^\beta_i h^\alpha_j \right) \tag{9}$$

Again, we see that the second-order coefficients are given by (9) as long as $H^\mu_\alpha(x_0, y_0)$ is invertible. Further differentiation shows that at each stage the critical solvability condition is the same, the invertibility of $H^\mu_\alpha(x_0, y_0)$. Therefore, we can continue to solve for terms in a Taylor series expansion as long as H has the necessary derivatives. We will compute the solvability conditions for the dynamic perturbation problem, and find that they differ from this in that the k'th order solvability condition depends on k.

3. A Simple Example

We first illustrate regular perturbation in the context of the basic rational expectations equations in a simple optimal growth model. This will help us see through the complexity of the multidimensional case and also show why we use ε as a perturbation variable as opposed to the variance ε^2 which is the perturbation variable in continuous-time asymptotic methods, such as in Fleming, Judd and Guu (1993), and Gaspar and Judd (1997).

Consider the simple stochastic optimal growth problems indexed by ε

$$\begin{array}{ll} \max_{c_t} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} & k_{t+1} = F(k_t - c_t)(1+\varepsilon z_t) \end{array} \tag{10}$$

where the z_t are i.i.d. with unit variance, and ε is a parameter expressing the standard deviation. The solution of the deterministic case, $\varepsilon = 0$, can be expressed as a policy function, $C(k)$, satisfying the Euler equation

$$u'\left(C(k)\right) = \beta\, u'\left(C\left(F(k - C(k))\right)\right)\, F'\left(k - C(k)\right).$$

Standard linearization methods produce $C'(k)$. Successive differentiations of (10) produce higher-order derivatives of $C(k)$ at $k = k^*$. For example, the second derivative of (10) together with the steady-state condition $k = k^*$ implies that $C''(k^*)$ satisfies the linear equation

$$\begin{aligned} u''C'' + u'''C'C' = \;& \beta u''' \left(C'F'(1-C')\right)^2\, F' + \beta u''C'' \left(F'(1-C')\right)^2\, F' \\ & + 2\beta u''C'F'(1-C')^2\, F'' + \beta u'F'''(1-C')^2 \\ & + \beta u'F''(-C'') \end{aligned}$$

where the parentheses denote multiplication, not application of a function. All functions are evaluated at the steady state value of their arguments. This is a linear equation in the unknown $C''(k^*)$. Linear operations combined with successive differentiations of (10) produce all higher-order derivatives.

The solution in the general case is a policy function, $C\,(k, \varepsilon)$, which expresses consumption as a function of the capital stock k as well as the standard deviation ε. $C(k, \varepsilon)$ satisfies the Euler equation

$$\begin{aligned} u'\left(C(k,\varepsilon)\right) &= \beta\, E\left\{u'\left(g(\varepsilon,\, k,\, z)\right)\, R(\varepsilon,\, k, z)\right\} \\ g(\varepsilon, k, z) &\equiv C((1+\varepsilon z)F(k - C(k))) \\ R(\varepsilon,\, k,\, z) &\equiv (1+\varepsilon\, z)\, F'\left(k - C(k)\right) \end{aligned} \tag{11}$$

Differentiation of (11) with respect to ε produces a linear equation for $C_\varepsilon(k^*, 0)$, which has the solution $C_\varepsilon = 0$. This is a natural result since ε parameterizes the standard deviation of uncertainty, whereas basic economic intuition says that the economic response should be proportional to variance, which is ε^2. Furthermore, the perturbation variable in Fleming (1971) was instantaneous variance. Therefore, $C_\varepsilon = 0$ is a natural result.

Further differentiation with respect to ε produces a linear equation for $C_{\varepsilon\varepsilon}(k^*, 0)$ shows that

$$C_{\varepsilon\varepsilon}(k^*, 0) = \frac{u''' \, C'C'F^2 + 2u''C'F + u''C''F^2}{u''C'F' + \beta u'F''}$$

where all the derivatives of u and F are evaluated at the steady-state values of c and k. This can be continued to compute higher-order derivatives as long as u and F have the necessary derivatives.

It may initially appear more natural to use variance, ε^2, as the perturbation variable since $C_\varepsilon(k, 0) = 0$. However, using the variance would cause difficulty in a discrete-time problem. The ε^3 term $C_{\varepsilon\varepsilon\varepsilon}$ is nonzero in the deterministic case since skewness can be nonzero. This is not a problem in the continuous-time, Ito process case since all odd (instantaneous) moments are zero. A Taylor series in ε^2 in discrete-time stochastic problems would miss the ε^3 terms and would fail at or before the ε^3 term. In terms of asymptotic theory, the appropriate gauge for discrete-time stochastic problems is ε^k instead of ε^{2k}.

4. Perturbations of the Deterministic Model

We now begin applying perturbation methods to rational expectations models of the form in (1).We first describe the perturbation method for the deterministic problem. The deterministic problem has independent interest. Since the perturbations are with respect to the state variables x, we drop the ε parameter in this section to simplify the notation.

Suppose that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then $Y(x) : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$, and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Note that we assume that the number of equations in $g = 0$ equals the number of free variables, m. We express equilibrium in the more convenient form

$$0 = \mathcal{G}(x, Y(x), \mathcal{X}(x), \mathcal{Y}(x)) \equiv \begin{pmatrix} g^{\mu}(x, Y(x), \mathcal{X}(x), \mathcal{Y}(x)) \\ \mathcal{X}^I(x) - F^I(x, Y(x)) \\ \mathcal{Y}^A(x) - Y^A(\mathcal{X}(x)) \end{pmatrix} \tag{12}$$

where g^{μ} denotes the μ'th equation in the collection of equilibrium equations. This formulation uses expressions $Y(x)$ and $F(x, Y(x))$ as well as the composite expressions $\mathcal{X}(x) - F(x, Y(x))$ and $\mathcal{Y}(x) = Y(\mathcal{X}(x)) = Y(F(x, Y(x))$. The introduction of the intermediate terms $\mathcal{X}(x)$ and $\mathcal{Y}(x)$ helps us make clear the essentially linear structure of the problem. It also indicates a direction for efficient programming since it tells us that we should separately compute the derivatives of $\mathcal{X}(x)$ and $\mathcal{Y}(x)$ before we compute the derivatives of $g(x, Y(x), \mathcal{X}(x), \mathcal{Y}(x))$. This approach also distinguishes the cases where Y occurs as $Y(x)$ as opposed to $Y(F(x, Y(x))$. Y^{α} will refer to occurrences of $Y(x)$ and Y^A will refer to occurrences of $Y(F(x, Y(x))$. We used F^I, F^J, etc., to refer to components of $F(x, Y(x)) = \mathcal{X}(x)$. $\mathcal{X}^I$, $\mathcal{X}^J$, etc., will also refer to components of $\mathcal{X}(x)$, and components of $\mathcal{Y}(x) = Y(F(x, Y(x)))$ will be denoted $\mathcal{Y}^A$, $\mathcal{Y}^B$, $\mathcal{Y}^{\Delta}$, etc.

The objective is to find the derivatives of $Y(x)$ with respect to x at the deterministic steady state, and use that information to construct Taylor series approximations of $Y(x)$. In conventional notation, that Taylor series is expressed as

$$Y(x) \doteq y_* + Y_x(x_*, 0)(x - x_*) + (x - x_*)^{\top} Y_{xx}(x_*, 0)(x - x_*) + \ldots$$

but tensor notation expresses it as

$$\begin{aligned} Y(x) &\doteq y_* + Y_i(x_*, 0)\left(x^i - x_*^i\right) + Y_{ij}(x_*, 0)\left(x^i - x_*^i\right)\left(x^j - x_*^j\right) \\ &\quad + Y_{ijk}(x_*, 0)\left(x^i - x_*^i\right)\left(x^j - x_*^j\right)\left(x^k - x_*^k\right) + \ldots \end{aligned}$$

Deterministic Steady State and the Linear Approximation. Perturbation methods begin with the deterministic steady state, which is the solution to

$$\begin{aligned} 0 &= g\left(x_*, y_*, x_*, y_*\right) \\ x_* &= F(x_*, y_*) \end{aligned}$$

This is a nonlinear set of equations. The second step in perturbation methods is to compute the linear terms of the approximation $Y_x\left(x_*, 0\right)$. Standard linearization methods show that the coefficients $Y_x\left(x_*, 0\right)$ are the solution, $y = y_* + Y_x\left(x - x_*\right)$, to the linear rational expectations model

$$g_i^\mu x_t^i + g_\alpha^\mu y_t^\alpha + g_I^\mu x_{t+1}^I + g_A^\mu y_{t+1}^A = 0 \tag{13}$$

where all the gradients of g^μ in (13) are evaluated at the deterministic steady state. We assume locally unique and stable dynamics. This may not be true for all steady states; we confine our attention to only saddlepoint stable steady states. This linearization procedure is justified by the implicit function theorem applied to (6). The solution is

$$y_t - y_* = H\left(x_t - x_*\right) \tag{14}$$

Anderson et al. (1996) and Anderson, Evans, Sargent, and McGrattan (1996) survey methods for solving such models. This is the difficult step computationally, but can be handled by conventional methods.

Theorem 1. *If g and F are locally analytic in a neighborhood of (x_*, y_*, x_*, y_*), and (13) has a unique locally asymptotically stable solution, then for some $\varepsilon > 0$ there is a unique function $Y(x)$ such that $Y(x)$ solves (12) and is locally analytic for $\|x - x_*\| < \varepsilon$. In particular, $Y(x)$ is infinitely differentiable and its derivatives solve the equations derived by implicit differentiation (12).*

Proof. This follows from the standard application of the analytic implicit function theorem to the space of sequences that converge to the steady state at an exponential rate. ■

Before we move on to the second-order approximation, we need to formulate the first order problem in functional and tensor form. We start with equilibrium expressed in the form (12). Let $\mathcal{Z}(x) = (x, Y(x), \mathcal{X}(x), \mathcal{Y}(x))$. The first-order derivatives with respect to the x^i variables are

$$\begin{aligned} 0 &= g_i^\mu(\mathcal{Z}(x)) + g_\alpha^\mu(\mathcal{Z}(x))\, Y_i^\alpha(x) + g_I^\mu(\mathcal{Z}(x))\, \mathcal{X}_i^I(x) + g_A^\mu(\mathcal{Z}(x))\, \mathcal{Y}_i^A(x) \\ \mathcal{X}_i^I(x) &= F_i^I(x, Y(x)) + F_\alpha^I(x, Y(x))\, Y_i^\alpha \\ \mathcal{Y}_i^A(x) &= Y_I^A(\mathcal{X}(x))\mathcal{X}_i^I(x) \end{aligned} \tag{15}$$

At the steady state (we drop arguments here since they are all understood to have their steady state values)

$$\begin{aligned} 0 &= g_i^\mu + g_\alpha^\mu Y_i^\alpha + g_I^\mu \mathcal{X}_i^I + g_A^\mu \mathcal{Y}_i^A \\ \mathcal{X}_i^I &= F_i^I + F_\alpha^I Y_i^\alpha \\ \mathcal{Y}_i^A &= Y_I^A \mathcal{X}_i^I \end{aligned}$$

After substitution, we see that the tensor (matrix) Y_i^α is the solution to

$$0 = g_i^\mu + g_\alpha^\mu Y_i^\alpha + g_I^\mu \left(F_i^I + F_\alpha^I Y_i^\alpha\right) + g_A^\mu Y_I^A \left(F_i^I + F_\alpha^I Y_i^\alpha\right)$$

This is a matrix polynomial equation for the matrix $Y_i^\alpha(x_*)$ with n^2 equations in the n^2 unknown values Y_i^α. This is also the H matrix in (14) computed by standard methods.

Before we move to higher order approximations, we express the first order system in (15) in a convenient form. Let

$$\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}=\begin{pmatrix} g^{\mu}_{\alpha}\left(\mathcal{Z}\left(x\right)\right) & g^{\mu}_{I}\left(\mathcal{Z}\left(x\right)\right) & g^{\mu}_{A}\left(\mathcal{Z}\left(x\right)\right) \\ -F^{I}_{\alpha}\left(x,Y\left(x\right)\right) & \mathbf{1}_{n} & 0 \\ 0 & 0 & \mathbf{1}_{m} \end{pmatrix}$$

The (α, I, A) subscript notation represents the fact that $\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}$ is three columns of tensors, the first being derivatives with respect to α, etc. The $\mathbf{1}_n$ ($\mathbf{1}_m$) entry in $\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}$ represents the $n \times n$ ($m \times m$) identity map[2]. Then the system of equations in (15) defining the first-order coefficients can be expressed as

$$0=\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}\begin{pmatrix} Y^{\alpha}_{i}\left(x\right) \\ \mathcal{X}^{I}_{i}\left(x\right) \\ \mathcal{Y}^{A}_{i}\left(x\right) \end{pmatrix}-\begin{pmatrix} 0 \\ 0 \\ Y^{A}_{I}(\mathcal{X}\left(x\right))\mathcal{X}^{I}_{i}\left(x\right) \end{pmatrix}+\begin{pmatrix} g^{\mu}_{i}\left(\mathcal{Z}\left(x\right)\right) \\ F^{I}_{i}\left(x,Y\left(x\right)\right) \\ 0 \end{pmatrix} \tag{16}$$

and the steady state values $Y^{\alpha}_{i}\left(x_*\right)$ satisfy

$$0=\mathcal{G}(z_*)_{\aleph}\begin{pmatrix} Y^{\alpha}_{i}\left(x_*\right) \\ \mathcal{X}^{I}_{i}\left(x_*\right) \\ \mathcal{Y}^{A}_{i}\left(x_*\right) \end{pmatrix}-\begin{pmatrix} 0 \\ 0 \\ Y^{A}_{J}(x_*)\mathcal{X}^{J}_{i}\left(x_*\right) \end{pmatrix}+\begin{pmatrix} g^{\mu}_{i}\left(z_*\right) \\ F^{I}_{i}\left(x_*,y_*\right) \\ 0 \end{pmatrix}$$

where $z^*_* = \mathcal{Z}\left(x_*\right)$. We will use the form in (16) below.

Second-order Approximation. We next want to compute $Y^{\alpha}_{ij}(x_*)$, the Hessian of $Y^{\alpha}\left(x\right)$ at $x = x_*$. Let $D_j f(x)$ represent the total derivative of $f(x)$ w.r.t. x^j. Dif-

[2] We use the $\mathbf{1}_n$ notation instead of the Kronecker delta tensor δ^i_j notation since we do not want to change indices.

ferentiation of (16) with respect to x^j shows

$$\begin{aligned} 0 &= D_j\left(\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}\right)\begin{pmatrix} Y_i^{\alpha}\left(x\right) \\ \mathcal{X}_i^{I}\left(x\right) \\ \mathcal{Y}_i^{A}\left(x\right)\end{pmatrix} + \mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{(\alpha,I,A)}\begin{pmatrix} Y_{ij}^{\alpha}\left(x\right) \\ \mathcal{X}_{ij}^{I}\left(x\right) \\ \mathcal{Y}_{ij}^{A}\left(x\right)\end{pmatrix} \\ &\quad - \begin{pmatrix} 0 \\ 0 \\ Y_{IJ}^{A}(\mathcal{X}\left(x\right))\mathcal{X}_j^{J}\left(x\right)\mathcal{X}_i^{I}\left(x\right) + Y_I^{A}(\mathcal{X}\left(x\right))\mathcal{X}_{ij}^{I}\left(x\right)\end{pmatrix} + D_j\begin{pmatrix} g_i^{\mu}\left(\mathcal{Z}\left(x\right)\right) \\ F_i^{I}\left(x, Y\left(x\right)\right) \\ 0\end{pmatrix} \end{aligned} \tag{17}$$

which implies the steady state conditions

$$\begin{aligned} 0 &= D_i\left(\mathcal{G}(z_*)\right)\begin{pmatrix} Y_i^{\alpha}\left(x_*\right) \\ \mathcal{X}_i^{I}\left(x_*\right) \\ \mathcal{Y}_i^{A}\left(x_*\right)\end{pmatrix} + D_i\begin{pmatrix} g_i^{\mu}\left(z_*\right) \\ F_i^{I}\left(x_*, Y\left(x_*\right)\right) \\ 0\end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{X}_j^{J}\left(x_*\right)\mathcal{X}_i^{I}\left(x_*\right) & Y_I^{A}(x_*) & 0 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} Y_{IJ}^{A}(x_*) \\ \mathcal{X}_{ij}^{I}\left(x_*\right) \\ \mathcal{Y}_{ij}^{A}\left(x_*\right)\end{pmatrix} \\ &\quad + \mathcal{G}\left(z_*\right)_{\aleph}\begin{pmatrix} Y_{ij}^{\alpha}\left(x_*\right) \\ \mathcal{X}_{ij}^{I}\left(x_*\right) \\ \mathcal{Y}_{ij}^{A}\left(x_*\right)\end{pmatrix} \end{aligned} \tag{18}$$

which is a linear equation in the unknowns $\left(Y_{ij}^{\alpha}\left(x_*\right), \mathcal{X}_{ij}^{I}\left(x_*\right), \mathcal{Y}_{ij}^{A}\left(x_*\right)\right)$. Note here that the solvability condition is the nonsingularity of $\mathcal{G}(z_*^*)$ plus some other terms.

Theorem 2. *$\left(Y_{ij}^{\alpha}\left(x_*\right), \mathcal{X}_{ij}^{I}\left(x_*\right), \mathcal{Y}_{ij}^{A}\left(x_*\right)\right)$ satisfies the linear system (18). It is uniquely solved by (18) if and only if (18) is nonsingular*

Third-order approximation. The third-order terms $\mathcal{Z}^{\aleph}_{ijk}(x)$ are found by differentiating (17) with respect to x^j, producing

$$\begin{aligned} 0 &= D_k\left(D_j\left(\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}\right)\begin{pmatrix} Y^{\alpha}_i(x) \\ \mathcal{X}^I_i(x) \\ \mathcal{Y}^A_i(x)\end{pmatrix} + D_j\begin{pmatrix} g^{\mu}_i\left(\mathcal{Z}\left(x\right)\right) \\ F^I_i\left(x, Y\left(x\right)\right) \\ 0\end{pmatrix}\right) \\ &\quad + D_k\left(\mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}\right)\begin{pmatrix} Y^{\alpha}_{ij}(x) \\ \mathcal{X}^I_{ij}(x) \\ \mathcal{Y}^A_{ij}(x)\end{pmatrix} + \mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}\begin{pmatrix} Y^{\alpha}_{ijk}(x) \\ \mathcal{X}^I_{ijk}(x) \\ \mathcal{Y}^A_{ijk}(x)\end{pmatrix} \\ &\quad - \begin{pmatrix} 0 \\ D_{ijk}Y^A(\mathcal{X}\left(x\right)) \\ 0\end{pmatrix} \end{aligned}$$

The steady state is now given by

$$0 = (\text{Terms without } \mathcal{Z}_{ijk}) + \mathcal{G}\left(\mathcal{Z}\left(x\right)\right)_{\aleph}\begin{pmatrix} Y^{\alpha}_{ijk}(x_*) \\ \mathcal{X}^I_{ijk}(x_*) \\ \mathcal{Y}^A_{ijk}(x_*)\end{pmatrix} - \begin{pmatrix} 0 \\ D_{ijk}Y^A(\mathcal{X}\left(x_*\right)) \\ 0\end{pmatrix} \tag{19}$$

where

$$\begin{aligned} D_{ijk}Y^A(\mathcal{X}\left(x_*\right)) &= Y^A_{IJK}(x_*)\,\mathcal{X}^K_k(x_*)\,\mathcal{X}^J_j(x_*)\,\mathcal{X}^I_i(x_*) \\ &\quad + Y^A_{IJ}(x_*)D_{x_k}\left(\mathcal{X}^J_j(x_*)\,\mathcal{X}^I_i(x_*)\right) \\ &\quad + Y^A_{IK}(x_*)\mathcal{X}^K_k(x_*)\,\mathcal{X}^I_{ij}(x_*) + Y^A_I(x_*)\mathcal{X}^I_{ijk}(x_*) \end{aligned}$$

Theorem 3. $\left(Y^{\alpha}_{ijk}(x_*), \mathcal{X}^I_{ijk}(x_*), \mathcal{Y}^A_{ijk}(x_*)\right)$ *satisfies the linear system (19). It is uniquely solved by (19) if and only if (19) is nonsingular*

There are two items to note. First, $Y^{\alpha}_{ijk}(x_*)$ satisfies a linear system of equations. Second, the solvability matrix for $\mathcal{Z}_{ijk}(x_*)$ will be different than the solvability matrix for $\mathcal{Z}_{ij}(x_*)$. The fact that the solvability conditions change as we change order is a potential problem. However, one suspects that these matrices are generically determinate, but that remains an open issue.

The following is an obvious continuation of our method.

Theorem 4. *Given the solution to all lower order derivatives, the degree m derivatives $\left(Y^{\alpha}_{i_1..i_m}(x_*), \mathcal{X}^{I}_{i_1..i_m}(x_*), \mathcal{Y}^{A}_{i_1...i_m}(x_*)\right)$ satisfy a linear equation that is solvable if the linear map*

$$\mathcal{G}(z_*)_{\aleph}\begin{pmatrix} Y^{\alpha}_{i_1..i_m}(x_*) \\ \\ \mathcal{X}^{I}_{i_1..i_m}(x_*) \\ \\ \mathcal{Y}^{A}_{i_1...i_m}(x_*) \end{pmatrix} - \begin{pmatrix} 0 \\ \\ Y^{A}_{I_1..I_m}(x_*)\,\mathcal{X}^{I_1}_{i_1}(x_*)\cdots\mathcal{X}^{I_m}_{i_m}(x_*) + Y^{A}_{I}(x_*)\mathcal{X}^{I}_{i_1..i_m}(x_*) \\ \\ 0 \end{pmatrix}$$

is invertible.

4.1. Algorithm. We have established that Taylor expansions of the equilibrium equations produce a series of linear equations in the derivatives of Y once we get past the first-order terms. This implies that a fairly simple algorithm can be applied once we have the linear systems. Define

$$G^{\mu}(x) = g^{\mu}(x, Y(x), \mathcal{X}(x), \mathcal{Y}(x))$$

where

$$\begin{aligned} \mathcal{X}^{I}(x) &= F^{I}(x, Y(x)) \\ \mathcal{Y}^{A}(x) &= Y^{A}(\mathcal{X}(x)) \end{aligned}$$

The steady state definition tells us that $G(x_*) = 0$. Furthermore, if $Y^{\alpha}_{i}(x_*)$ is fixed at its true linear solution, $G^{\mu}_{i}(x_*) = 0$. The preceding calculations show that the

Taylor expansion continues with the form

$$\begin{aligned} G^{\mu}(x_* + \varepsilon v) &= G^{\mu}(x_*) + \varepsilon G_i^{\mu}(x_*) v^i \\ &+\varepsilon^2 \left(M_{\alpha}^{\mu,2}(x_*) + N_{\alpha}^{\mu,2}(x_*) Y_{ij}^{\alpha}(x_*) \right) v^i v^j \\ &+\varepsilon^3 \left(M_{\alpha}^{\mu,3}(x_*) + N_{\alpha}^{\mu,3}(x_*) Y_{ijk}^{\alpha}(x_*) \right) v^i v^j v^k \\ &+ ... \end{aligned} \tag{20}$$

where each $M_{\alpha}^{\mu,\ell}(x_*)$ and $N_{\alpha}^{\mu,\ell}(x_*)$ term involves no derivative of $Y^{\alpha}(x)$ of order ℓ or higher. Therefore, we take a specific problem, have the computer produce the Taylor series expansion in (20) where the $Y_i^{\alpha}(x_*)$, $Y_{ij}^{\alpha}(x_*)$, etc., terms are left free, and then compute them in a sequentially linear fashion by solving in sequence the linear systems

$$\begin{aligned} 0 &= M_{\alpha}^{\mu,2}(x_*) + N_{\alpha}^{\mu,2}(x_*) Y_{ij}^{\alpha}(x_*) \\ 0 &= M_{\alpha}^{\mu,3}(x_*) + N_{\alpha}^{\mu,3}(x_*) Y_{ijk}^{\alpha}(x_*) \\ &... \end{aligned}$$

where at each stage we use the solutions in the previous stage.

5. A General Stochastic Problem

We next compute the stochastic portion of our approximation. The stochastic rational expectations problem is

$$\begin{aligned} 0 &= E\left\{ g^{\mu}(x_t, y_t, x_{t+1}, y_{t+1}, \varepsilon z) | x_t \right\},\ \mu = 1, ..., m \\ x_{t+1}^i &= F^i(x_t, y_t, \varepsilon z_t),\ i = 1, ..., n \end{aligned}$$

The objective is to find some equilibrium rule, $Y(x, \varepsilon)$, such that

$$E\left\{ g\left(x, Y(x,\varepsilon), F(x, Y(x,\varepsilon), \varepsilon z), Y\left(F(x, Y(x,\varepsilon), \varepsilon z), \varepsilon\right), \varepsilon z\right) | x \right\} = 0 \tag{21}$$

We have constructed the derivatives of $Y(x,0)$ with respect to the components of x. We now compute the ε derivatives.

Before we describe our method, we first present an example which highlights the pitfalls of pursuing a Taylor series expansion procedure in a casual fashion ignoring the relevant mathematics. We will posit a simple example of a rational expectations model and follow the approach taken in Kydland and Prescott (1982), and elsewhere. This casual application of the standard approach will produce a nonsensical result and highlight a potential problem. We will then proceed to develop an approach consistent with the implicit function theorem.

5.1. A Cautionary Example. We now present a simple example which highlights the dangers of a casual approach to computing Taylor series expansions. Suppose that an investor receives K endowment of wealth and that there are two possible investments, I_1 and I_2; therefore, $I_{1,t}+I_{2,t}=K$. We allow negative values for $I_{1,t}$ and $I_{2,t}$, making this example like a portfolio problem. Assume a gross "adjustment cost" for deviations of type 1 investment from some target $\bar{I}$ equal to $\alpha\left(I_{1,t}-\bar{I}\right)^2/2$ units of utility. In period $t+1$ the investments produce $f\left(I_{1,t},I_{2,t},Z_{t+1}\right)$ of the consumption good. Assume $f\left(I_{1,t},I_{2,t},Z_{t+1}\right)=I_{1,t}+I_{2,t}Z_{t+1}$ where Z_t is log Normal with mean $1+\mu$ and variance σ^2. Assume that all of the second-period gross return is consumed and that the utility function is $u(c)=c^{1-\gamma}/(1-\gamma)$. The complete utility function is

$$U\left(I_{1,t},I_{2,t},Z_{t+1}\right)=-\alpha\left(I_{1,t}-\bar{I}\right)^2/2+E_t\left\{u\left(I_{1,t}+I_{2,t}Z_t\right)\right\} \tag{22}$$

This is a rational expectations model where the endogenous variables are defined by the equations

$$\begin{aligned} 0 &= -\alpha\left(I_{1,t}-\bar{I}\right)+E_t\left\{u'\left(I_{1,t}+I_{2,t}Z_t\right)Z_t\right\} \\ \overline{K} &= I_{1,t}+I_{2,t} \end{aligned} \tag{23}$$

This is a trivial rational expectations model[3], but if a method cannot reliably approximate the solution to this problem then we would not have any confidence in its general validity.

The deterministic "steady state" is $I_{1,t} = \bar{I}, I_{2,t} = K - \bar{I}$; that is, we set type 1 investment equal to the type 1 target $\bar{I}$ and put the rest of the capital in type 2 investments.[4]. This is obvious since the investments have the same return and there is an adjustment cost for any deviations of $I_{1,t}$ from zero. We now want to know how the investment policy is altered if we add some noise to the risky type 2 investment. If we were to take a certainty equivalent approximation then the investment rules are unchanged by an increase in variance. However, this will produce nonsensical answers if $\bar{I} < 0$ and $I_{2,t} > K$ since this would imply that there is some chance that $I_{1,t} + I_{2,t}Z_t < 0$, implying negative consumption. A more sophisticated application of the key idea in Kydland and Prescott (1982)[5] tells us to replace the utility function in (22) with a quadratic approximation around the deterministic consumption K and solve the resulting linear equation for $I_{1,t}$. For any $\bar{I}$, μ, and σ^2, as α goes to zero we converge to the limiting investment rule

$$I_{2,t} = K \frac{\mu/\gamma}{\sigma^2 + \mu^2} \tag{24}$$

Consider the situation when $\mu/\gamma > \sigma^2 + \mu^2$, as is the case when $\mu = .06$ and $\sigma^2 = .04$, the standard calibration for equity investment, and $\gamma = 1$, which is log utility. In this case $I_{1,t}$, the safe investment, would be negative, implying that one borrows

[3]This example is also a bit silly with the utility adjustment costs, but it is an attempt to construct an example which looks like a portfolio problem (think of I_1 and I_2 as investments in alternative securities) but avoids the complications examined in Judd and Guu (2001).

[4]Note that if $\bar{I} < 0$ then $I_{2,t} > K$.

[5]Strictly speaking, Kydland and Prescott assume additive noise in the law of motion for the state, in which case their approximation is certainty equivalent. However, the application of their key linear-quadratic approximation idea to this example with nonadditive disturbances is clear.

($I_{1,t} < 0$) to buy more than K dollars of the risky investment. This, however, is nonsensical since the support of log Normal Z is all of the nonnegative real line, implying that there is some chance of negative consumption, which is inconsistent with CRRA utility. The possibility of a negative consumption makes expected utility undefined and can be avoided[6] by choosing any nonnegative $I_{1,t}$. Therefore, the ad hoc approximation in (24) cannot be a useful approximation to the the solution of (23).

What went wrong? There is an implicit constraint on consumption being positive in (23) since the utility function is not defined for negative consumption. However, that constraint is not binding in the deterministic problem since consumption is surely positive. Since this constraint is not present in the deterministic case, it is not present when one executes a purely local procedure. If there are no restrictions placed on the random disturbances, the Kydland and Prescott (1982) procedure implicitly replaces a nonlinear problem with a linear-quadratic problem *globally*. The approximate quadratic utility function is defined for negative consumption and is not an appropriate *global* approximation since the true utility function may not be defined for negative consumption. Therefore, without restrictions on the disturbances, the Kydland and Prescott approach is not a local analysis based on some implicit function theorem.

This is not a problem unique to the strategy recommended by Kydland and Prescott. The second-order procedure of Sims (2000) is a natural extension of Kydland and Prescott and would fail in this case for similar reasons. In fact, any scheme using purely local information and moments and ignoring global considerations can fail on problems like this since local information cannot alone model global considerations. The approximation procedure advocated in Campbell (2002) is different

[6]The reason for choosing log Normal z is obvious; if we had made z Normal than no nonzero choice for I_2 is consistent with the nonnegativity constraint on consumption.

but also fails on this point, often producing approximations with positive probabilities of negative consumption. Campbell's approach begins with the observation that one can analytically compute expected utility if utility is CRRA and consumption is log Normal. Therefore, it would be nice if our portfolio problems always reduced to computing the expected CRRA utility of log Normal consumption. However, if asset returns are log Normal, consumption will generally not be log Normal since nontrivial linear combinations of log Normal returns are not log Normal. Campbell approximates the original portfolio problem (our problem in (23) is a portfolio problem if $\alpha = 0$) by replacing the ex post distribution of consumption with an "approximating" log Normal distribution, and then finds that portfolio which maximizes the approximate expression for expected utility. Of course, the support of the log Normal approximation will not include any negative values and if mean returns are sufficiently large relative to the variance and risk aversion, the Campbell "approximation" will short bonds and cause a positive probability of negative consumption.

These problems do not go away even if we take the time period to zero. If we were to embed (23) in a sequence of problems with ever shorter time periods, we would want to maintain the Sharpe ratio, μ/σ^2, above some positive limit. Then, for many γ, the approximation in (24) would always be greater than K, implying shorting in each discrete-time problem and a positive probability of negative consumption.

The mathematical economics literature has long been aware of this problem and has recognized the importance of proceeding locally. In particular, Samuelson (1970) noted this problem and assumed that the disturbance Z has bounded support. Judd and Guu (2001) generalized this to a local analyticity condition. Judd (1998) (see chapter 16) also points out the importance of controlling the distribution of Z. In this paper, we will proceed with a bounded support assumption since that is the most general way to proceed. We will display the critical conditions which need to be checked before one can proceed with a perturbation approximation of rational expectations models.

5.2. Local Perturbation Approach for Stochastic Models. We have already computed the Taylor series approximation of the deterministic problem

$$\begin{aligned} Y^{\alpha}(x,\varepsilon) &\doteq y_* + Y_i^{\alpha}(x_*,0)\left(x^i - x_*^i\right) \\ &+ Y_{ij}^{\alpha}(x_*,0)\left(x^i - x_*^i\right)\left(x^j - x_*^j\right) \\ &+ Y_{ijk}^{\alpha}(x_*,0)\left(x^i - x_*^i\right)\left(x^j - x_*^j\right)\left(x^k - x_*^k\right) \\ &+ \ldots \end{aligned}$$

We next move to the stochastic terms, $Y_{\varepsilon}(x_*,0)$, $Y_{\varepsilon\varepsilon}(x_*,0)$, etc. The general stochastic system is

$$\begin{aligned} 0 &= E\left\{g\left(x_t, y_t, x_{t+1}, y_{t+1}, \varepsilon z\right)|x_t\right\} \\ x_{t+1} &= F\left(x_t, y_t, \varepsilon z_t\right) \end{aligned}$$

which implies the functional equation

$$\begin{aligned} 0 &= E\left\{g^{\mu}\left(x, Y(x,\varepsilon), F(x, Y(x,\varepsilon), \varepsilon z), Y(F(x, Y(x,\varepsilon), \varepsilon z)), \varepsilon z\right)\right\} \\ &= \mathcal{N}(Y)(x,\varepsilon) \end{aligned} \tag{25}$$

holds at all (x,ε). Differentiation with respect to ε shows

$$\begin{aligned} 0 &= E\left\{g_{\alpha}^{\mu}Y_{\varepsilon}^{\alpha} + g_I^{\mu}\left(F_{\alpha}^I Y_{\varepsilon}^{\alpha} + F_{\varepsilon}^I z\right) + g_A^{\mu}\left(Y_I^A\left(F_{\alpha}^I Y_{\varepsilon}^{\alpha} + F_{\varepsilon}^I z\right) + Y_{\varepsilon}^A\right) + z g_{\varepsilon}^{\mu}\right\} \\ &= E\left\{g_{\alpha}^{\mu}Y_{\varepsilon}^{\alpha}\right\} + E\left\{g_I^{\mu}F_{\alpha}^I Y_{\varepsilon}^{\alpha}\right\} + E\left\{g_I^{\mu}F_{\varepsilon}^I z\right\} + E\left\{g_A^{\mu}Y_I^A F_{\alpha}^I Y_{\varepsilon}^{\alpha}\right\} \\ &+ E\left\{g_A^{\mu}Y_I^A F_{\alpha}^I F_{\varepsilon}^I z\right\} + E\left\{g_A^{\mu}Y_{\varepsilon}^A\right\} + E\left\{z g_{\varepsilon}^{\mu}\right\} \\ &= Y_{\varepsilon}^{\alpha}\left(E\left\{g_{\alpha}^{\mu}\right\} + E\left\{g_I^{\mu}F_{\alpha}^I\right\} + E\left\{g_A^{\mu}Y_I^A F_{\alpha}^I\right\} + \delta_{\alpha}^A E\left\{g_A^{\mu}\right\}\right) \\ &+ E\left\{g_I^{\mu}F_{\varepsilon}^I z\right\} + E\left\{g_A^{\mu}Y_I^A F_{\alpha}^I F_{\varepsilon}^I z\right\} + E\left\{z g_{\varepsilon}^{\mu}\right\} \end{aligned} \tag{26}$$

holds at all (x,ε)[7]. Note that in the last step we use the identity $Y^A_\varepsilon = \delta^A_\alpha Y^\alpha_\varepsilon$. At the steady state of the deterministic case, $(x,\varepsilon) = (x_*,0)$, the εz terms collapse to zero, the derivatives g^μ_α, g^μ_I, g^μ_A, F^I_α, F^I_ε, and g^μ_ε become deterministic, and (26) reduces to

$$0 = Y^\alpha_\varepsilon \left(g^\mu_\alpha + g^\mu_I F^I_\alpha + g^\mu_A Y^A_I F^I_\alpha + \delta^A_\alpha g^\mu_A\right) + \left(g^\mu_\varepsilon + g^\mu_I F^I_\varepsilon + g^\mu_A Y^A_I F^I_\alpha F^I_\varepsilon\right) E\left\{z\right\} \tag{27}$$

which has a unique solution for Y^α_ε iff all the terms in (26) exists at $\varepsilon = 0$ and

$$\mathcal{N}_Y = g^\mu_\alpha + g^\mu_I F^I_\alpha + g^\mu_A Y^A_I F^I_\alpha + \delta^A_\alpha g^\mu_A$$

is an invertible matrix. The problem of existence arises with the terms g^μ_ε and F^I_ε which only arise here. As we saw above, there are examples where these terms do not exist.

We can take higher-order derivatives of (25) with respect to ε to arrive at equations for $Y^\alpha_{\varepsilon\varepsilon}$, $Y^\alpha_{\varepsilon\varepsilon}$, etc. The formulas are complex, however, the pattern is clear. For example, the derivative of (26) with respect to ε implies

$$\begin{aligned} 0 &= Y^\alpha_{\varepsilon\varepsilon}\left(E\left\{g^\mu_\alpha\right\} + E\left\{g^\mu_I F^I_\alpha\right\} + E\left\{g^\mu_A Y^A_I F^I_\alpha\right\} + \delta^A_\alpha E\left\{g^\mu_A\right\}\right) \\ &\quad Y^\alpha_\varepsilon \frac{d}{d\varepsilon}\left(E\left\{g^\mu_\alpha\right\} + E\left\{g^\mu_I F^I_\alpha\right\} + E\left\{g^\mu_A Y^A_I F^I_\alpha\right\} + \delta^A_\alpha E\left\{g^\mu_A\right\}\right) \\ &\quad + \frac{d}{d\varepsilon}\left(E\left\{g^\mu_I F^I_\varepsilon z\right\} + E\left\{g^\mu_A Y^A_I F^I_\alpha F^I_\varepsilon z\right\} + E\left\{z g^\mu_\varepsilon\right\}\right) \end{aligned}$$

which in turn implies that $Y^\alpha_{\varepsilon\varepsilon}(x_*,0)$ solve an equation of the form

$$0 = Y^\alpha_{\varepsilon\varepsilon}\mathcal{N}_Y + \mathcal{M}$$

where $\mathcal{M}$ contains only derivatives of g and F and moments of z. This will determine $Y^\alpha_{\varepsilon\varepsilon}(x_*,0)$ if the terms in $\mathcal{M}$ exists and $\mathcal{N}_Y$ is invertible. Notice that the

[7] We are slightly abusing notation by writing

$$\frac{d}{d\varepsilon} g\left(x,y,\hat{x},\hat{y},\varepsilon z\right) = g_\varepsilon\left(x,y,\hat{x},\hat{y},\varepsilon z\right) z$$

More formally, this should be $g_5\left(x,y,\hat{x},\hat{y},\varepsilon z\right) z$ but we use the $g_\varepsilon(\ldots)$ notation for its mnemonic advantage. The same comment applies to our notation $F_\varepsilon(\ldots)$.

solvability condition, the invertibility of $\mathcal{N}_Y$, is the same as the solvability condition for $Y_\varepsilon^\alpha(x_*, 0)$. Continuing this process shows that the ε derivatives of Y exist as long as g and F have the necessary derivatives and the moments of z exist. The next theorem summarizes our argument under the assumption that g is analytic.

Theorem 5. *If (i) $g^\mu(x, y, \hat{x}, \hat{y}, \varepsilon z)$ exists and is analytic for all z in some neighborhood of $(x, y, \hat{x}, \hat{y}, \varepsilon) = (x_*, y_*, x_*, y_*, 0)$, (ii) there exists a unique deterministic solution $Y(x, 0)$ locally analytic in x, and (iii) $\mathcal{N}_Y$ is invertible, then there is an $r > 0$ such that for all $(x, \varepsilon) \in B_r(x_*, 0)$, there exists a unique solution $Y(x, \varepsilon)$ to (25). Furthermore, all derivatives of $Y(x, \varepsilon)$ exist in a neighborhood of $(x_*, 0)$ and can be solved by implicit differentiation.*

Proof. Application of the implicit function theorem for analytic functions. See Zeidler (1986). ■

This theorem sounds obvious, but the conditions are important ones and should be checked. In particular, our example in (23) fails to satisfy condition *(i)*. One natural and general way to insure *(i)* is to assume z has bounded support. This may sound limiting since many popular processes in dynamic analyses, such as linear processes with Normal or log Normal. Also, continuous time Ito processes often imply finite-horizon distributions with infinite support. However, assuming z has bounded support is a suitable assumption for discrete-time models, and, in fact, may be a superior way to approximate continuous-time problems. Consider, for example, Merton's (1972) portfolio analysis. For some parameters, investors will short the bond market and use the proceeds to buy stocks even for log utility. Our example above shows that this would be unwise in a discrete-time model where returns have log Normal returns since there would be some chance that wealth goes negative. Therefore, assuming log Normal returns in a discrete-time model is a poor way to approximate the continuous-time model. Why is there such a difference? In the continuous-time model with an Ito process driving the returns, an investor hit with a series of negative return shocks can reduce his exposure to risk before his wealth goes negative, but this is impossible in the discrete-time model and log Normal returns. The finite-support assumption for z similarly allows an investor facing a series of negative shocks to adjust his position before hitting ruin.

The key point is that there are regularity conditions which must be satisfied if we are to use the implicit function theorem to justify asymptotic expansions as approximations. Our example shows that it may not be possible to combine popular utility functions with popular stochastic processes. We take the view that it is more important to accurately approximate the economic process than to stay with popular stochastic processes. Therefore, we assume Assumption 1:

Assumption 1: *The support of z is finite and $E\{z\} = 0$.*

The following theorem follows directly from $E\{z\} = 0$ and successive differentiation of (25). It just ratifies common sense that a first-order change in standard deviation affects nothing, and that the dominant order effect is variance, which here equals ε^2.

Theorem 6. *If $E\{z\} = 0$, then $0 = Y_\varepsilon^\alpha = Y_{\varepsilon i}^\alpha = Y_{\varepsilon ij}^\alpha = ...$*

Proof. $0 = Y_\varepsilon^\alpha$ follows from (27) and $E\{z\} = 0$. The other results follow from the fact that derivatives of (26) with respect to state variables in x will always reduce to expressions of the form $0 = Y_{\varepsilon i_1 .. i_\ell}^\alpha (...) + (...) E\{z\}$. ■

6. Nonlocal Accuracy Tests

Perturbation methods produce the best possible asymptotically valid local approximations to a problem. However, we often want to use them for nonlocal approximations. The existing literature shows that Taylor series approximations are often quite good even for states not close to the deterministic steady state. Judd and Guu (1993, 1997) investigated this issue in simple growth models. They find that, for empirically reasonable choices of tastes and technology, linear approximations do well for small but nontrivial neighborhoods of the deterministic steady state. However, the region of validity will be too small if the stochastic shocks cause the state to wander substantially from the deterministic steady state. Fortunately, they find that the quality of the approximations improve substantially as the higher–order terms are added. They also find that the certainty nonequivalence terms are important to achieve high quality approximations for stochastic approximations. More precisely, they substitute the computed Taylor series into the defining equations and evaluate the resulting error. The resulting error for capital stocks near the steady state is often the order of machine zero, an accomplishment which few other methods can claim. While their investigations have been limited to relatively small models, there is no reason to suspect that the performance of this approach will decay drastically as we move to larger models. In any case, any user of these methods should use some diagnostics to estimate the region where the constructed series is a good approximation.

It is tempting to compute higher-order approximations and then just assume that they are better than the certainty equivalent linear approximation. This approach is dangerous since it ignores the essential fact of Taylor series expansions – their range of validity is limited. Some elementary analysis shows the importance of this fact. Suppose that $f(x) = \left(9802 - 198x + x^2\right)^{-1}$ and we wanted to compute its power series around $x = 100$. The fourth-order Taylor series is

$$\widehat{f}(x) = 99990001 - 3999800x + 59999x^2 - 400x^3 + x^4$$

To measure the accuracy of this Taylor series, we computed the relative error in logarithm terms,

$$E\left(x\right) = \log_{10}\left|\frac{\widehat{f}(x)}{f(x)} - 1\right|.$$

The results are displayed in Table 1.

Table 1: Relative errors in Taylor series expansion of $(9802 - 198x + x^2)^{-1}$

$x - 100$:	.1	.2	.3	.5	1	1.5
$E(x)$:	-13	-9.7	-7.2	-4.2	0	2.4

Table 1 says that the fourth-order Taylor series approximation has very small errors for x within 0.5% of the central value of $x = 100$, but that it falls apart when $|x - 100| > 1$. We also computed the order 5, 10, and 20 Taylor series expansions and found the approximations to get better for $|x - 100| < 1$ but worse for $|x - 100| > 1$. It appears that we cannot construct a Taylor series based at $x = 100$ which is valid for any value of x more than 1% away from $x = 100$. The key fact is that the radius of convergence for the power series expansion of $f(x)$ around $x = 100$ is 1. This follows directly from the theory of power series in the complex plane. The polynomial $9802 - 198x + x^2$ has two zeroes of $x = 100 \pm \sqrt[2]{-1}$, both of which are distance 1 away from $x = 100$. Therefore, the infinite-order Taylor series based at $x = 100$ has a radius of convergence equal to 1. Radii of convergence for power series can be small; in fact, they can be arbitrarily small even when the function is simple and has no singularities on the real line. We have no idea about the radius of convergence for the Taylor series constructed by our methods. This is particularly problematic for us in stochastic models where, in reasonably calibrated cases, we do expect the state variables to roam more than 1% away from the deterministic steady state.

This cautionary example and the portfolio example above both show that we need ways to determine if our solutions are good, and these evaluations should be performed for any application before a computed approximation is accepted and used to make some economically substantive point. Therefore we need to develop diagnostic tools which can be applied to any problem.

To measure the accuracy of our approximations we evaluate

$$E\left(x,\varepsilon\right)=\max_{\mu}\widetilde{g}^{\mu}\left(x,\widehat{Y}\left(x,\varepsilon\right),F(x,\widehat{Y}\left(x,\varepsilon\right),\varepsilon z),\widehat{Y}(F(x,\widehat{Y}\left(x,\varepsilon\right),\varepsilon z),\varepsilon),\varepsilon z\right)$$

where $\widetilde{g}^{\mu}$ is a normalized, unit-free, version of g^{μ}. The specific details of the normalization depends on the equation. For example, Euler equations should be normalized so that the errors are in terms of percentage of consumption. Specifically, the unnormalized Euler equation is often expressed as

$$0=u'\left(c_t\right)-\beta E\left\{u'\left(c_{t+1}\right)R_{t+1}\right\}$$

which has units "utils per unit of consumption". We need to get rid of both the utility and consumption units. A unit-free version is

$$0=1-\beta\frac{E\left\{u'\left(c_{t+1}\right)R_{t+1}\right\}}{u'\left(c_t\right)}$$

a form often used in empirical studies. If an equation is a market-clearing condition with form form $0=S-D$ for supply S and demand D, then a natural unit-free form would be $0=1-D/S$ where any deviation expresses the excess supply as a fraction of total supply. In general, we need to use some set of equations $\widetilde{g}^{\mu}$ where deviations from zero represent a unit-free measure of irrationality of the agents, lack of market-clearing, mistakes in predictions, and whatever else is involved in describing equilibrium, all of which we want to make small in any approximation.

7. Specific Examples

We have developed the full perturbation method for stochastic models and proposed diagnostic tests to ascertain accuracy. We next apply this approach to dynamic programming problems of the form

$$\begin{aligned}
\max_{\{L_t, I_{it}\}} \quad & E_t \sum_{\tau=t}^{\infty} \beta^t u(c_t, l_t) \\
& c_t = F(K_t^1, K_t^2, \cdots, K_t^n, L_t, \theta_t) - \sum_{i=1}^{n} I_t^i \\
& K_{t+1}^i = K_t^i + \varphi_i\left(\frac{I_t^i}{K_t^i}\right) I_{it}, \quad i = 1, \cdots, n \\
& \theta_{t+1} = \lambda\theta_t + \sigma\xi_{t+1}
\end{aligned}$$

where K_t^i, $i = 1, \cdots, n$, is the stock of type i capital goods at the beginning of period t, I_t^i is gross investment of type i capital in period t, θ_t is the productivity level in period t, and ξ_t is the i.i.d. productivity shock with mean zero and unit variance. We assume that ξ_t is truncated Normal with truncation at 3 standard deviations. The function $\varphi_i(I_t^i/K_t^i)$ represents net investment of type i capital after deducting adjustment costs. We assume that $\varphi_i(\cdot)$ has the following form

$$\varphi_i(s) = 1 - \frac{\delta_i}{2} s \tag{28}$$

Denote $K_t = (K_t^1, K_t^2, \cdots, K_t^n)$, $s_t^i = I_t^i/K_t^i$. Let $V(K_t)$ be the value function. The Bellman equation for the above problem can be written as:

$$V(K_t) = \max_{L_t, s_t^i} \; u(c_t, l_t) + \beta E_t V(K_{t+1})$$

subject to

$$\begin{aligned}
c_t &= F(K_t, l_t) - \textstyle\sum_{i=1}^n s_t^i K_t^i \\
K_{t+1}^i &= (1 + \varphi_i(s_t^i)s_t^i)K_t^i, \quad i = 1, \cdots, n \\
\theta_{t+1} &= \lambda\theta_t + \varepsilon\sigma\xi_{t+1}
\end{aligned} \tag{29}$$

The law of motion equation in (29) displays the position and of the perturbation parameter ε. The ξ_t shocks are i.i.d. random variables and are unchanged by the perturbation. The deterministic case is when $\varepsilon = 0$ since then the ξ_t shocks have no effect on production. When $\varepsilon = 1$, the variance of the productivity shocks is σ^2.

The first order conditions are:

$$
\begin{aligned}
0 &= u_c(c_t,l_t)F_L(K_t,l_t)+u_l(c_t,l_t) &(30)\\
0 &= -u_c(c_t,l_t)+\beta E_t\left\{V_i(K_{t+1})\right\}\left(\varphi_i'(s_t^i)s_t^i+\varphi(s_t^i)\right) &(31)\\
V_i(K_t) &= u_c(c_t,l_t)\left(F_i(K_t,l_t)-s_t^i\right)+\beta E_t\left\{V_i(K_{t+1})\right\}\left(1+\varphi_i(s_t^i)\right) &(32)
\end{aligned}
$$

while $F_i = \partial F(K,L)/\partial K^i$, $V_i = \partial V(K)/\partial K^i$. Note that the Euler equations (30), (31) and (32) are functional equations, which implicitly defines the policy functions $l_t = L(K_t)$ and $s_t^i = S^i(K_t)$, and the gradient functions $V_i(K)$. We are going to solve these functions by the perturbation method as described by Judd (1998), that is, to compute Taylor expansions around the steady state and use them as approximations.

7.1. Error Bounds. At each capital stock K_t, the error bound of our solution, $E(K_t)$, is defined as the maximum of absolute Euler equation errors at this point.

$$E(K_t)=\max\{||E_L||,||E_{K_i}||,||E_{V_i}||,i=1,\cdots,n\}$$

where E_l, E_{K_i}, E_{V_i} are normalized errors, as given by:

$$
\begin{aligned}
E_l &= (u_cF_l+u_l)/u_l\\
E_{K^i} &= \beta E_t\left\{V_i(K_{t+1})\right\}\left(\varphi_i'(s_t^i)s_t^i+\varphi(s_t^i)\right)/u_C-1\\
E_{V_i} &= 1-u_c(F_i-s_t^i)-\beta E_t\left\{V_i(K_{t+1})\right\}\left(1+\varphi_i(s_t^i)\right)/V_i(K_t)
\end{aligned}
$$

Normally we expect the error bounds become bigger as we move away from the steady state. To see how the errors grow, we introduce an overall measure of error bounds as a function of relative distance from the steady state. Formally, for every $r \geq 0$, we can define:

$$E(r)=\sup\left\{E(K)\left|\sum_{i=1}^{n}\left(\frac{K^i-\bar{K}^i}{\bar{K}^i}\right)^2\leq r\right.\right\}$$

where $\bar{K}^i$ denotes the steady state value of K^i. The $E(r)$ is the error bound function we are seeking.

In practice, however, we cannot check all points on the surface of a sphere. We must confine to some finite sets. Let $D = \{-1, 0, 1\}^n$. Define

$$X = \left\{ \left(x^1, \cdots, x^n\right) \middle| x^i = \frac{d^i}{\sqrt{\sum_{j=1}^n \left(d^j\right)^2}},\ d \in D,\ d \neq 0 \right\}$$

Thus all points in X are on the surface of a unit sphere. We define our error bound function as

$$E(r) = \max\{\, E(K) \mid K_i = (1 + r x^i)\bar{K}_i,\ x \in X \,\}$$

7.2. Computational Results. Our examples use the following functional forms:

$$\begin{aligned} u(c,l) &= \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\eta}}{1+\eta}, \quad \gamma > 0,\ \eta > 0 \\ F(K,l) &= \left(K^1\right)^{\alpha_1} \left(K^2\right)^{\alpha_2} \cdots \left(K^n\right)^{\alpha_n} l^{1-\sum_i \alpha_i},\ \ \alpha_i > 0,\ \sum_i \alpha_i < 1 \end{aligned}$$

We compute the model for the cases of 2,3 and 4 capital goods. In all the cases we choose $\beta = 0.95$, and $\delta_i = 0.1$, all i. We test the algorithm on several parameter values displayed in Table 1. For each combination of parameters in Table 1, we compute the first through fourth, and sometimes fifth, order Taylor series expansion. For each case, we compute $E\,(r)$ for various values of radii r, dimensions n, and expansion orders k. We then find the worst case for each scenario. That is, for radius r, dimension n, and order k, we find that case which had the worst $E\,(r)$. We report the worst cases in Table 2. For example, in the two capital good cases, the worst Euler equation error for the linear approximation at radius r was $10^{-3.2}$. That worst case may be different for the $r = .05$ case and for the $k = 2$ case. Therefore, every solution for the cases in Table 1 was better than the errors reported in Table 2.

Table 1: Parameter Values

(n, α_i):	(2,.15), (3,.1), (4, .075)
γ:	0.5, 2, 5, 10
η:	10, 3, 1
(λ, σ):	(0, 0), (0.05, 0), (0.10, 0) (0.01, 0.90), (0.01, 0.95)

The results in Table 2 show that the higher order approximations are very valuable and follow our intuition. For a fixed order k, the errors of the k'th order approximation increase as we move away from the steady state. The linear approximation is acceptable for $r < .01$, but is of questionable value for $r > .05$. None of the approximations have acceptable Euler equation errors for $r = .5$.

For any fixed radius r we see that there is substantial payoff to using higher-order approximations. In particular, at $r = 0.10$, the linear approximation has Euler equation errors up to 10% of consumption but the fifth-order approximation has normalized errors on the order of 10^{-5}, an improvement of four orders of magnitude.

Table 2: Error bounds $\log_{10} E(r)$

r	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
		2 capital good cases			
0.01	-3.2	-3.7	-4.1	-5.5	-5.7
0.05	-1.8	-2.8	-4.1	-5.1	-5.7
0.10	-1.1	-2.1	-3.1	-4.1	-5.1
0.20	-0.5	-1.2	-1.9	-2.6	-3.4
0.30	-0.1	-0.6	-1.3	-1.8	-2.3
0.40	0.2	-0.2	-0.7	-1.1	-1.6
0.50	0.6	0.2	-0.2	-0.6	-1.0
		3 capital good cases			
0.01	-3.2	-3.8	-4.0	-5.5	
0.05	-1.8	-2.9	-4.0	-5.2	
0.10	-1.2	-2.2	-3.3	-4.3	
0.20	-0.6	-1.3	-2.1	-2.8	
0.30	-0.2	-0.7	-1.3	-1.9	
0.40	0.2	-0.3	-0.8	-1.3	
0.50	0.5	0.1	-0.4	-0.8	
		4 capital good cases			
0.01	-3.3	-3.9	-4.1	-5.6	
0.05	-1.9	-3.0	-4.1	-5.6	
0.10	-1.3	-2.3	-3.4	-4.4	
0.20	-0.7	-1.4	-2.2	-2.9	
0.30	-0.3	-0.8	-1.5	-2.1	
0.40	0.1	-0.4	-0.9	-1.5	
0.50	0.4	-0.1	-0.5	-1.0	

Table 2 examined the quality of the approximations near the deterministic steady state. While this information is useful and exactly the kind of information which is related to the implicit function theorem, it does not tell us what we need to know about how good the approximation is for a stochastic problem because we do not know the range of the states. For example, the linear approximation looks good for only k within 5% of the steady state. If the capital stocks stay within that range, the linear approximation may be acceptable, but we would not be so accepting if the stochastic shocks pushes some capital stocks to levels more than 10% away from the steady state.

Tables 3 and 4 address these issues with stochastic simulation for a particular case. Tables 3 and 4 takes a degree k approximation and uses it to simulate the economy for 10^5 periods. We compute the deviation, $(K_t - \bar{K})/\bar{K}$, from the steady state and the magnitude of the Euler equation error at each realized state. Table 3 reports the mean deviation from the steady state of the capital stock, the standard deviation, and the maximum deviation. The mean deviation for $k = 1$ is nearly zero, as it should be since the linear approximation is a certainty equivalent approximation. Higher order approximations indicate that the mean capital stock is about 2% from the deterministic steady state, a fact not possible to approximate with the linear approximation. The other moments are largely unaffected by the higher orders of approximation.

Table 3: $(K-\bar{K})/\bar{K}$ along the simulation path

$\gamma = 10$, $\eta = 10$, $(\sigma, \lambda) = (0.1, .95)$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
mean	-0.002	0.019	0.018	0.019	0.019
std. dev.	0.087	0.090	0.089	0.089	0.089
maximum	0.257	0.304	0.291	0.294	0.293
minimum	-0.249	-0.229	-0.227	-0.226	-0.226

Table 4 reports the mean of the absolute value of the errors, their standard deviation, and the maximum Euler equation error over the 10,000 period simulation. Since the true distribution of the states is not centered at the deterministic steady state, the results in Table 4 are not as impressive as in Table 2, but they again indicate the great value of higher-order approximations.

Table 4: Error bounds $\log_{10} E(r)$ along simulation paths

$\gamma = 10$, $\eta = 10$, $(\sigma, \lambda) = (0.1, .95)$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
mean	2.6×10^{-2}	1.2×10^{-3}	1.1×10^{-4}	6.2×10^{-5}	3.4×10^{-7}
std. dev.	3.7×10^{-2}	2.5×10^{-3}	1.5×10^{-3}	1.6×10^{-4}	1.4×10^{-6}
maximum	4.4×10^{-1}	4.0×10^{-2}	9.8×10^{-3}	3.0×10^{-3}	2.9×10^{-5}

We need to be clear about these error results. We do not present them to indicate that higher-order perturbation methods are good approximations and that the reader should feel free to apply them to his problems. Our point is that these error analyses need to be done as part of any application of perturbation methods. It is the critical fifth step in the perturbation method. The statistics displayed in Tables 2 and 4 should be reported in any application of the perturbation method just as t

statistics and confidence intervals are reported in any application of regression and other statistical methods.

Table 5 displays the computational costs associated with the higher-order approximations. We see that the number of derivatives to compute rise substantially as we increase approximation order and dimensions. There is a similar increase in time and space needed to compute the approximations. We include statistics on space since the space necessary to store all the necessary derivatives may be a limitation for perturbation methods. While the computational costs are substantial, they are not a serious problem. With increasing speed of computers and the fall in memory prices, perturbation methods are clearly competitive with alternatives for multidimensional problems.

Table 5: Computation costs

no. of capitals	state vars.	endog. vars.	order 1	2	3	4	5	6	7
			number of derivatives to compute						
2^a	2	5	10	25	45	70	100	135	175
2^b	3	5	20	70	170	345	625	1045	
3^a	3	7	21	63	133	238	385		
3^b	4	7	35	140	385	875	1757		
4^a	4	9	36	126	306	621			
4^b	5	9	54	243	747	1881			
			computing time in seconds						
2^a	2	5	0	0.03	0.37	3.38	23.6	148	923
2^b	3	5	0	0.06	0.83	15.5	199	907	
3^a	3	7	0	0.13	2.34	35.0	415	308	
3^b	4	7	0	0.22	6.58	127	1424		
4^a	4	9	0	0.40	9.89	202			
4^b	5	9	0	0.66	22.8	640			
			memory used in megabytes						
2^a	2	5	2.5	2.5	2.8	4.2	12.0	48.0	200
2^b	3	5	2.6	2.6	3.1	7.2	51.0	440	
3^a	3	7	2.6	2.6	3.8	17.0	132		
3^b	4	7	2.7	2.7	4.8	33.0	386		
4^a	4	9	2.7	2.8	6.9	74.0			
4^b	5	9	2.8	2.9	9.5	135			

Note: a is the riskless case and b is the risky case

Table 6 summarizes the steps of the perturbation method.

Table 6: Perturbation Method for Rational Expectations Models

Step 1:	Compute the deterministic steady state with nonlinear equation solver
Step 2:	Compute linear approximation with some rational expectations solution method
Step 3:	Compute higher-order terms of deterministic problem through differentiation and linear equation solving
Step 4:	Compute stochastic deviation terms through differentiation and linear equation solving
Step 5:	Compute normalized errors in ergodic set of states through deterministic sampling and stochastic simulation

8. Conclusion

This paper has shown that it is feasible to apply perturbation methods to numerically solve rational expectations models substantially more complex than the usual representative agent, single good model. However, theory shows that the perturbation approach faces some limitations related to the range of the stochastic shocks and the local validity of the approximations. In response, we develop diagnostic methods to evaluate the approximations.

References

[1] Bensoussan, Alain. *Perturbation Methods in Optimal Control*, John Wiley and Sons, 1988.

[2] Bishop, Richard L., and Samuel I. Goldberg, Tensor Analysis on Manifolds, Dover Publications, 1981.

[3] Blanchard, Olivier Jean and Charles M. Kahn.1980. "The Solution of Linear Difference Models under Rational Expectations" *Econometrica* **48**:1305–1311.

[4] Brock, William A., and L. J. Mirman. 1972. "Optimal Economic Growth and Uncertainty: The Discounted Case," *Journal of Economic Theory* **4**: 479–513.

[5] Campbell, John Y., and Luis M. Viceira. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*, Oxford University Press, 2002.

[6] Chen, Baoline and Peter Zadrozny, "Higher Moments in Perturbation Solution of the Linear-Quadratic Exponential Gaussian Optimal Control Problem," Rutgers University, *Computational Economics*, forthcoming.

[7] Christiano, Lawrence J. 1990. "Solving the Stochastic Growth Model by Linear-Quadratic Approximation and by Value-Function Iteration," *Journal of Business and Economic Statistics*, **8**:23–26.

[8] Collard, Fabrice, and Michel Juillard, "Accuracy of Stochastic Perturbation Methods: The Case of Asset Pricing Models," *Journal of Economic Dynamics and Control* 25, 2001, 979-999.

[9] Collard, Fabrice, and Michel Juillard, "Perturbation Methods for Rational Expectations Models," Manuscript. Paris: CEPREMAP, February 2001.

[10] Fleming, Wendell. 1971. "Stochastic Control for Small Noise Intensities," *SIAM Journal of Control* **9** (No. 3, August, 1971).

[11] Fleming, W. H, and Souganidis, P. E. "Asymptotic series and the method of vanishing viscosity," *Indiana University Mathematics* Journal 35 (1986), no. 2, 425-447

[12] Judd, Kenneth L. *Numerical Methods in Economics* , MIT Press, 1998.

[13] Judd, Kenneth L. 1996. "Approximation, Perturbation, and Projection Methods in Economic Analysis," in *Handbook of Computational Economics*, edited by Hans Amman, David Kendrick, and John Rust, North Holland.

[14] Judd, Kenneth L., and Sy-Ming Guu. 1993. "Perturbation Solution Methods for Economic Growth Models," in *Economic and Financial Modeling with Mathematica*, Hal Varian, ed., Springer-Verlag: New York.

[15] Judd, Kenneth L., and Sy-Ming Guu. "Asymptotic Methods for Aggregate Growth Models," *Journal of Economic Dynamics and Control* **21** (No. 6, June 1997): 1025-42.

[16] Kim, Jinill and Sunghyun Henry Kim, "SpuriousWelfare Reversals in International Business Cycle Models," *Journal of International Economics*, forthcoming.

[17] Kydland, Finn E., and Edward C. Prescott. 1982. "Time to Build and Aggregate Fluctuations," *Econometrica* **50**: 1345-1370.

[18] Laitner, John. 1981. "The Stability of Steady States in Perfect Foresight Models," *Econometrica* **49**:319-333.

[19] Magill, J. P. Michael. "A Local Analysis of N-Sector Capital Accumulation under Uncertainty," *Journal of Economic Theory*, **15** (1977): 211–219.

[20] Marcet, Albert. "Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation," in C. Sims (ed.) *Advances in Econometrics Sixth World Congress*, Vol II., 91–118, Cambridge University Press.

[21] McCullagh, P. (1987). Tensor Methods in Statistics. Chapman & Hall, London.

[22] McGrattan, Ellen R. 1990. "Solving The Stochastic Growth Model by Linear-Quadratic Approximation." *Journal of Business and Economic Statistics* **8**: 41–43.

[23] Misner, Charles W., Kip S. Thorne, and John A. Wheeler, *Gravitation*, W.H. Freeman & Co, 1973.

[24] Samuelson, Paul A. "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances and Higher Moments," *Review of Economic Studies* **37** (1970): 537–542.

[25] Schmitt-Grohe, S, and M. Uribe. "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation of the Policy Function, Working Paper, Rutgers University and University of Pennsylvania, 2002.

[26] Sims, Christopher. "Second Order Accurate Solution Of Discrete-time Dynamic Equilibrium Models", 2000, mimeo.

[27] Taylor, John B., and Harald Uhlig. 1990. "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods," *Journal of Business and Economic Statistics*, **8**:1–18.

[28] Tesar, Linda. "Evaluating the Gains from International Risksharing," *Carnegie-Rochester Conference Series on Public Policy*, **42** (1995): 95-143.**5** (1995): 43-78.

[29] Zeidler, E. *Nonlinear Functional Analysis and Its Applications: Volume I,* Springer-Verlag: New York, 1986.