Computing Equilibria of Repeated And Dynamic Games

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DYNAMIC GAMES

A specific example: Dynamic Oligopoly

Oligopoly game with endogenous productive capacity.

- Study the nature of dynamic competition and its evolution.
- Study the nature of cooperation and competition.
- Specifically:
    - Is ability to collude affected by state variables?
    - Do investment decisions increase gains from cooperation?
    - Does investment present opportunities to deviate from collusive agreements?

Existing Literature in Dynamic Oligopoly

Existing literature in IO

- Two stage games
  - Firms choose capacities in stage one, prices in stage two
  - Kreps-Scheinkman (1983), Davidson-Deneckere (1986)
- Dynamic games
  - Firms choose capacities and prices
  - Benoit-Krishna (1987), Davidson-Deneckere (1990)

Goals revisited

- Limiting assumptions in previous work
  - Capacity chosen at $t=0$, OR
  - No disinvestment, OR
  - Examine only equilibria supported by Nash reversion, OR
  - Restrictive functional forms for demand and cost functions
- **Our goal**: Examine full set of pure strategy Nash equilibria for dynamic games with arbitrary cost and demand functions.

Stage Game: Environment

- N infinitely lived agents.
- Individual state: $x_i \in X_i$
- Aggregate state: $x \in X = \times_{i=1}^{N} X_i$
- Finite action space for player i: A_i, $i = 1, ..., N$
- Action profiles: $A = \times_{i=1}^{N} A_i$
- Aggregate state evolution: $g : A \times X \to X$

Stage Game: Payoffs

- Per period payoff function $\Pi_i : A \to \Re$
- Minimal payoffs
$$\underline{\Pi}_{i,x} \equiv \min_{a \in A} \Pi_i(a,x)$$
- Maximal payoffs
$$\overline{\Pi}_{i,x} \equiv \max_{a \in A} \Pi_i(a,x)$$
- Equilibrium payoffs in state x contained in
$$W_x = \times_{i=1}^{N} [\underline{\Pi}_{i,x}, \overline{\Pi}_{i,x}].$$
- Payoff correspondence:
$$W : X \rightrightarrows \Re^N$$

Dynamic Game

- Action space: A^∞
- h_t: t-period history:

$$\{\{a_s, x_s\}_{s=0}^{t-1}, x_t\} \text{ with } x_s = g(x_{s-1}, a_{s-1}), a_s \in A$$

- Set of t-period histories: H_t
- Preferences:

$$w_i(a^\infty, x^\infty) = \frac{1-\delta}{\delta} E_0 \Sigma_{t=1}^{\infty} \delta^t \Pi_i(a_t, x_t).$$

- Strategies: $\{\sigma_{i,t}\}_{t=0}^{\infty}$ with $\sigma_{i,t} : H_t \to A_i$.

Equilibrium Payoff Correspondence

- SPE payoff correspondence: $V^* \equiv \{V_x^* | x \in X\}$
- $\mathcal{P}$: set of all correspondences $\mathcal{W} : X \rightrightarrows \Re^N$ s.t.
    - Graph of $\mathcal{W}$ is compact
    - Graph of $\mathcal{W}$ contained within Graph of $\mathcal{P}$.
    - V^* may be shown to be an element of $\mathcal{P}$.

Steps: Computing the Equilibrium Value Correspondence

1. Define an operator that maps today's equilibrium values to tomorrow's at each state.
2. Show that this operator is monotone and the equilibrium correspondence is its largest fixed point.
3. Define approximation for operator and correspondences that
   - Represents correspondence parsimoniously on computer
   - Preserves monotonicity of operator
4. Define an appropriately chosen initial correspondence, apply the monotone operator until convergence.

Step 1: Set Valued Dynamic Programming

- Recursive formulation
- Each SPE payoff vector is supported by
  - profile of actions consistent with Nash today
  - continuation payoffs that are SPE payoffs
- Construct self-generating correspondences to find V^*

Step 1: Operator

$B^* : \mathcal{P} \to \mathcal{P}$.

- Let $\mathcal{W} \in \mathcal{P}$.

$$B^*(\mathcal{W})_x = \cup_{(a,w)}\{(1-\delta)\Pi(a,x) + \delta w\}$$

subject to:

$$w \in \mathcal{W}_{g(a,x)}$$

and for each $\forall i \in N$, $\forall \tilde{a} \in A_i$

$$(1-\delta)\Pi_i(a,x) + \delta w_i \geq \Pi_i(\tilde{a}, a_{-i}, x) + \delta \mu_{i,g(\tilde{a},a_{-i},x)}\}$$

where $\mu_{i,x} = \min\{w_i | w \in \mathcal{W}_x\}$.

Step 2: Self-generation

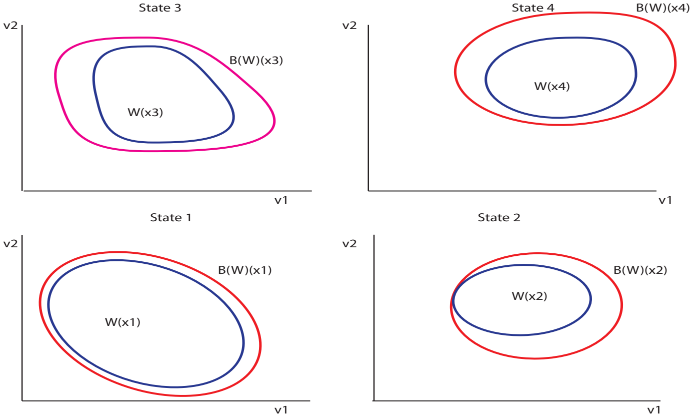
A correspondence W is self-generating if :

$$\mathcal{W} \subseteq B^*(\mathcal{W}).$$

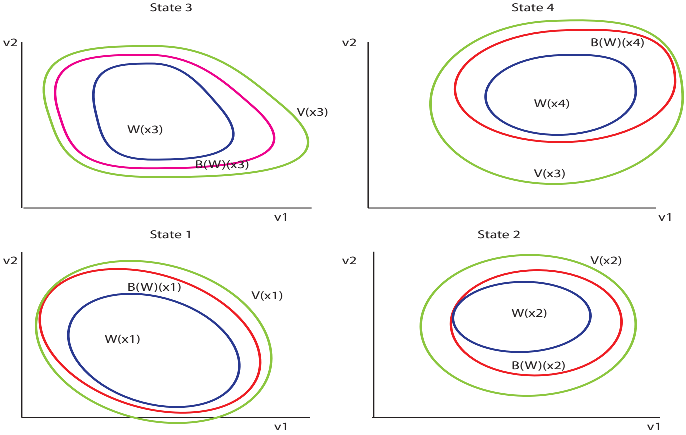
An extension of the arguments in APS establishes the following:

- Graph of any self-generating correspondence is contained within $Graph(V^*)$,
- V^* itself is self-generating.

Self-generation visually



Self-generation visually

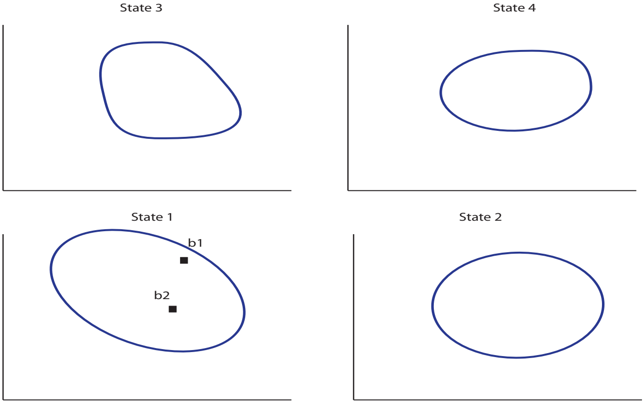


Step 2: Factorization

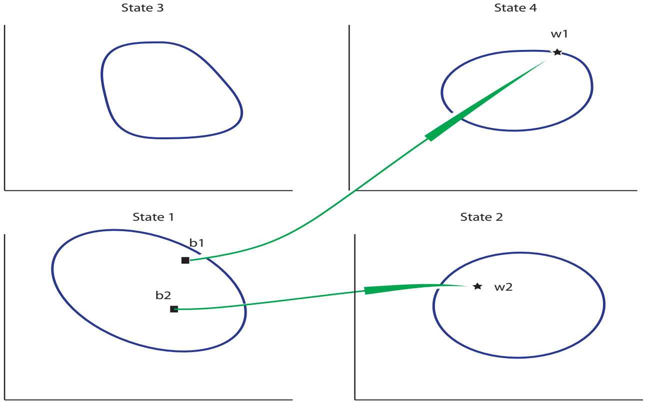
$b \in B^*(\mathcal{W})_x$ if there is an action profile a and continuation payoff $w \in \mathcal{W}_{g(a,x)}$, s.t

- b is value of playing a today in state x and receiving continuation value w ,
- for each i, player i will choose to play a_i
- $x\prime = g(a,x)$ if no defection
- $\tilde{x} = g(\tilde{a}_i, a_{-i}, x)$ if defection.
- punishment value drawn from set $\mathcal{W}_{\tilde{x}}$.

Factorization I



Factorization II



Step 2: Eqm Value Correspondence as Fixed Point

- Monotonicity: B^* is monotone in the set inclusion ordering:

$$\text{If } \mathcal{W}_1 \subseteq \mathcal{W}_2, \text{ then } \ B^*(\mathcal{W}_1) \subseteq B^*(\mathcal{W}_2)$$

- Compactness: B^* preserves compactness.
- Implications:
  1) V^* is the maximal fixed point of the mapping B^*;
  2) V^* can be obtained by repeatedly applying B^* to any set that contains graph of V^*.

Step 3: Approximating Value Correspondences

- Represent candidate value correspondences on computer
- Preserve monotonicity of operator
- Proceed in 2 steps
  1. Convexify underlying game.
  2. Develop method for approximating convex-valued correspondences.

Step A: Public randomization

- Public lottery with support contained in $\mathcal{W}_{g(a,x)}$.
- Public lottery specifies continuation values for the next period
  - Lottery dependent on current actions determines Nash equilibrium for next period.
  - Strategies now condition on histories of actions and lottery outcomes.
- Modified operator:

$$B(W) = co(B^*(co(\mathcal{W}))), \qquad \mathcal{W} \in \mathcal{P}.$$

- V equilibrium value correspondence of supergame with public randomization.
- B is monotone and V is the largest fixed point of B.

Environment: Dynamic Cournot with Capacity

- Firm i has sales of $q_i \in Q_i(k_i)$, and unit cost c_i.
- MC$=$ maintenance cost of machine
- SP$=$ resale/scrap value of machine
- FC $=$cost of a new machine
- Cost of capital maintenance and investment:

$$C(k_i, k_i') = \begin{cases} MC * (k_i - 1) + FC * (k_i' - k_i) & \text{if } k_i' \geq k_i \\ MC * (k_i - 1) - SP * (k_i - k_i') & \text{if } k_i' \leq k_i \end{cases}$$

Profit: Dynamic Cournot with Capacity

- Firm i's current profits:

$$\Pi_i(q_1, q_2, k_i, k_i') = q_i(p(q_1, q_2) - c_i) - C(k_i, k_i')$$

- Linear demand curve:

$$p(q_1, q_2) \ = \ \max\{a - b(q_1 + q_2), 0\}.$$

Stage Game: Dynamic Cournot with Capacity

- Action Space:
  - sets of outputs
  - sets of capital stocks
- State Space:
  - set of feasible capital stocks
- $A_i = Q_i \times K_i$
- $X = K_1 \times K_2$

Dynamic Strategies and Payoffs

- Strategies: collection of functions that map from histories of outputs and capital stocks into current output and capital choices.

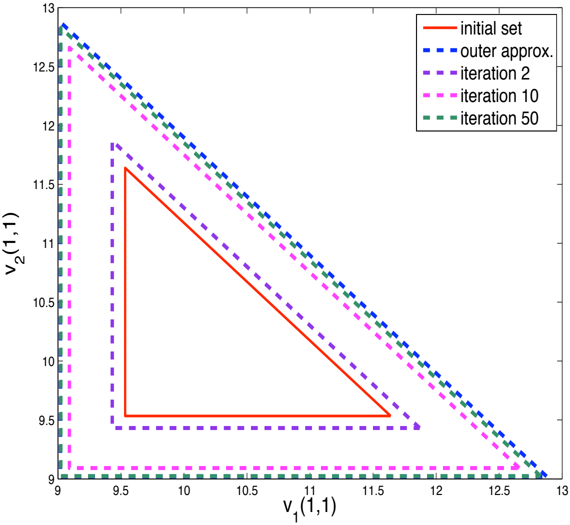
- Maximize average discounted profits.

$$\frac{(1-\delta)}{\delta}\sum_{t=0}^{t=\infty}\delta^t \Pi_{i,t}(q_1, q_2, k_i, k_i')$$

Dynamic Duopoly: Example 1

- Finite action version of the dynamic duopoly game.
- Discretize action space over q_i and k_i
- Full capacity: Actions from interval $[0, \bar{Q}]$
- Partial capacity: Actions from interval $[0, \bar{Q}/2]$
- Firms endowed with 1 machine each.
- 4 states: $(k_1, k_2) \in \{(1,1), (1,2), (2,1), (2,2)\}$
- 48 hyperplanes for the approximation.

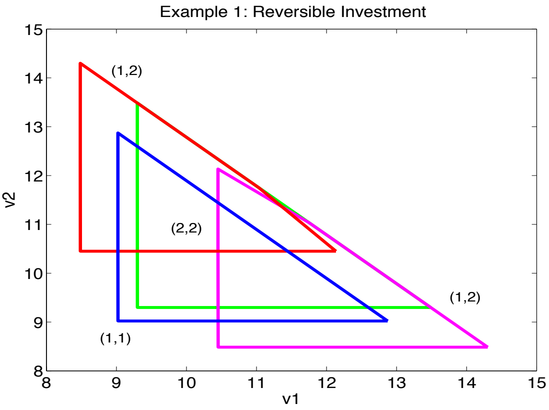
Monotone Operator and Convergence



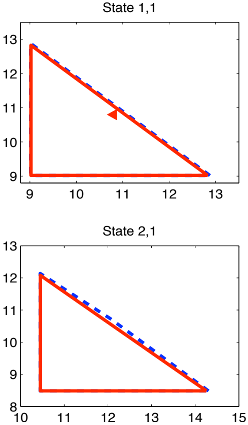
Fluctuation Market Power

Parameters: MC =SP=1.5, FC =2.5, $\delta = 0.8$, $\bar{Q} = 6.0$ $c = 0.6$, $b = 0.3$, $a = 6.0$

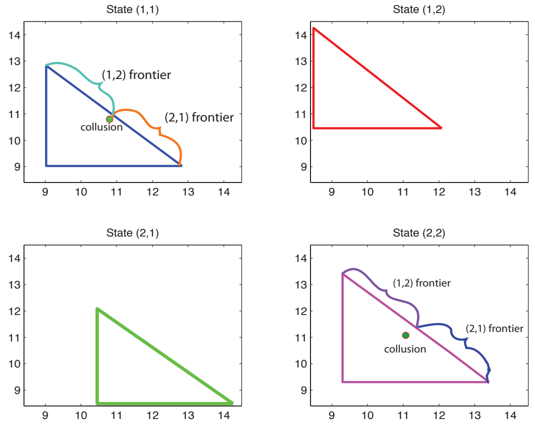
$$p(q_1, q_2) \;=\; \max\,\{a - b(q_1 + q_2), 0\}.$$



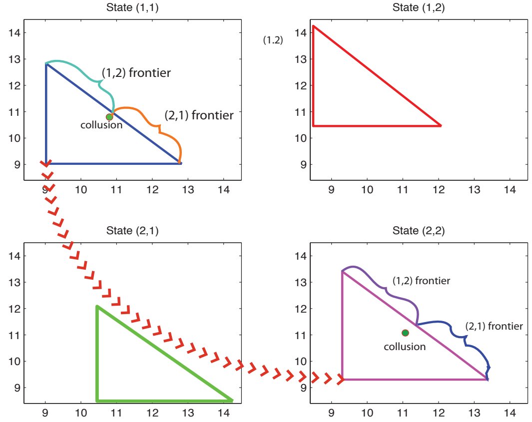
Error Bounds



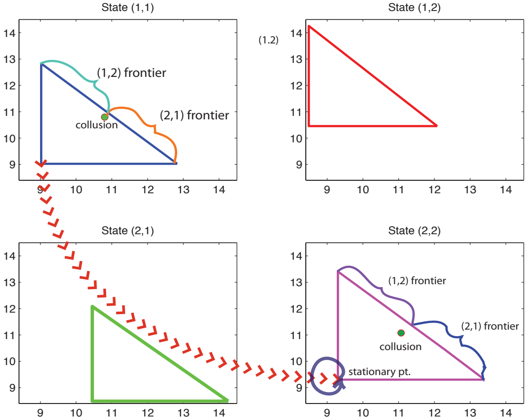
Striving for Market Power I



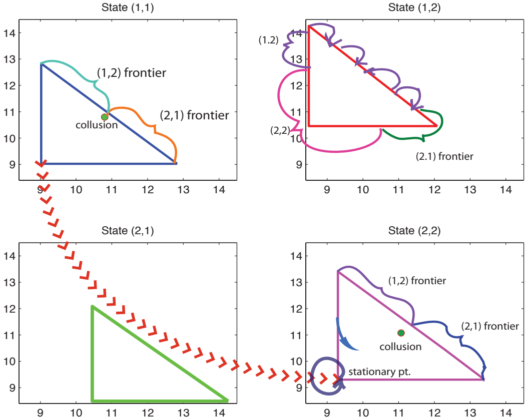
Striving for Market Power II



Striving for Market Power III



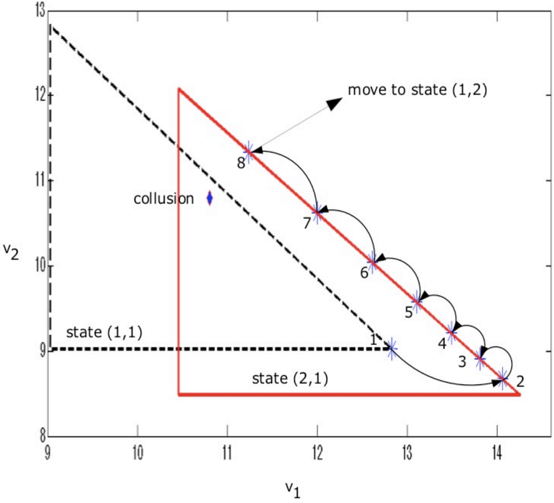
Striving for Market Power : Strategies



Strategies: Fluctuating Market Power

- Firms can do better than *symmetric* Nash collusion.
- Frontier of equilibrium value sets supported by
  - continuation play where firms alternate having market power.
- Worst equilibrium payoffs
  - firms produce at full capacity in current period
  - over-investment and over-production thereafter (symmetric cases).

Striving for Market Power : Strategies

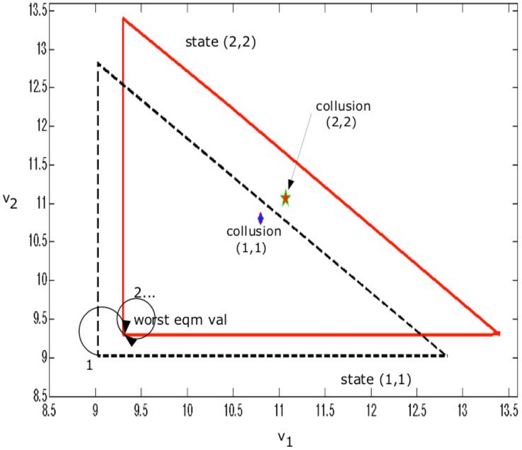


Striving for Market Power : Strategies

Table: Equilibrium Path

$Node$	v_1	v_2	k_1	k_2	q_1	q_2
1	12.8289	9.0232	1	1	3.0	3.0
2	14.0571	8.6750	2	1	6.0	3.0
3	13.8064	8.9118	2	1	6.0	3.0
4	13.4930	9.2078	2	1	6.0	3.0
5	13.1012	9.5777	2	1	6.0	3.0
6	12.6115	10.0401	2	1	6.0	3.0
7	11.9994	10.6181	2	1	6.0	3.0
8	11.2342	11.3407	2	1	6.0	3.0

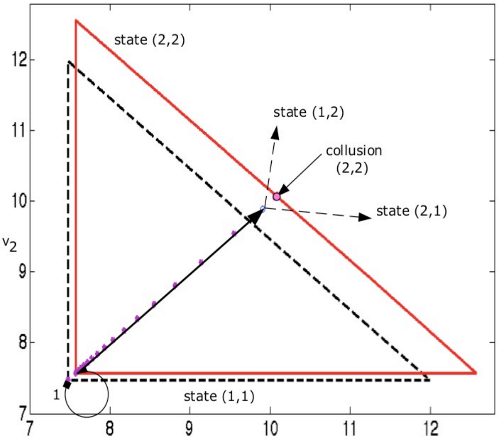
Worst Equilibrium



Worst Equilibrium c=0.9

- With higher per unit cost (c=0.9), playing uncooperatively too costly.
- Following one period of over investment and over production
  - Firms move towards Pareto frontier.
  - Continuation values increasing over time
  - Followed by alternating market power and high profits
- Nature of cooperation depends on state and on history.
- Markov perfect eqm. cannot capture this.

Striving for cooperation



Striving for cooperation

$Node$	v_1	v_2	k_1	k_2	q_1	q_2
1	7.47280025107915	7.47280025107915	1	1	3.0	3.0
2	7.57200031384894	7.57200031384894	2	2	6.0	6.0
3	7.59000039231118	7.59000039231118	2	2	6.0	6.0
4	7.61250049038897	7.61250049038897	2	2	6.0	6.0
5	7.64062561298621	7.64062561298621	2	2	6.0	6.0
6	7.67578201623276	7.67578201623276	2	2	6.0	6.0
7	7.71972752029095	7.71972752029095	2	2	6.0	6.0
8	7.77465940036369	7.77465940036369	2	2	6.0	6.0
9	7.84332425045461	7.84332425045461	2	2	6.0	6.0
10	7.92915531306827	7.92915531306827	2	2	6.0	6.0
11	8.03644414133533	8.03644414133533	2	2	6.0	6.0
12	8.17055517666916	8.17055517666916	2	2	6.0	6.0
13	8.33819397083645	8.33819397083645	2	2	6.0	6.0
14	8.54774246354557	8.54774246354557	2	2	6.0	6.0
15	8.80967807943196	8.80967807943196	2	2	6.0	6.0
16	9.13709759928994	9.13709759928994	2	2	6.0	6.0
17	9.54754361279848	9.54754361279842	2	2	6.0	6.0
18	10.0594295159981	10.0594295159980	2	1		
			1	2		

Summary

- Computation of equilibrium value correspondence reveals
  - dynamic interaction and competition missed by simplifying assumptions
  - rich set of equilibrium outcomes that involve
    - fluctuating market power
    - over-investment and over-production when cooperation breaks down
    - worst equilibrium resembles prisoner's dilemma
    - best equilibria resemble battle of the sexes.
    - equilibria with current profit of leading firm less than smaller firm

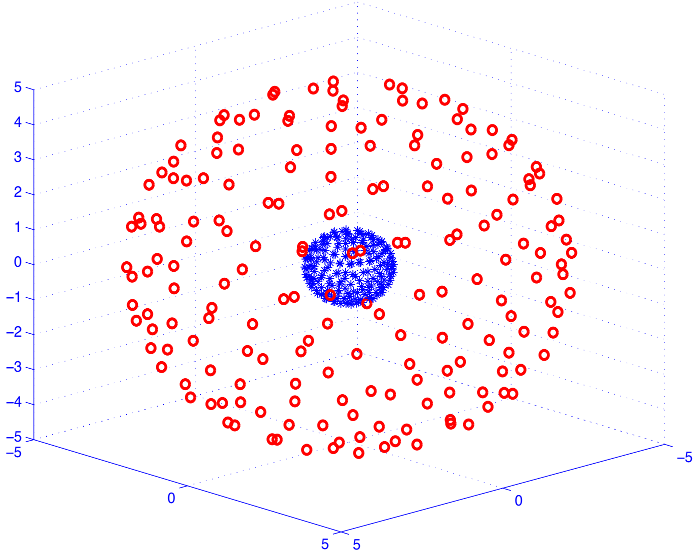
Supergames with Continuous States

- Approximation substantially more complicated than discrete states.
- Goal: Find an approximation scheme with right properties that preserves outer/inner bounds.
- Use set-valued step functions.
- See unpublished mimeo: Sleet and Yeltekin (1999); "On the approximation of value correspondences".

Number of players

- So far examples have $N = 2$.
- Algorithm applicable to $N > 2$
- Some computational issues.
  - Computational power. No of optimizations rise exponentially.
  - Choice of hyperplanes non-trivial. [Sampling on a sphere.]
  - Harder to define/calculate error bounds.

Sampling surface of sphere



Continuous Actions

- Optimizations are LP problems.
- LP has nearly negligible approximation error.
- Using LP ensures outer and inner approx. do not have optimization error.
- NLP methods can introduce optimization errors that distort the inner/outer structure.
- My advice: Stick to discrete actions.

Dynamic Games in Macro

- Credible policy designed as dynamic game between planner +continuum of agents with capital.
- One large strategic player + continuum of non-strategic players.
- How does one apply a variant of APS ?
- Use planner's value and tomorrow's marginal utility of capital.
- Example: Phelan and Stacchetti (Econometrica, 2001): Ramsey tax model w/ capital and no govt commitment.