Initiative for Computational Economics

Numerical Methods for Solving Auctions I

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Acknowledgements

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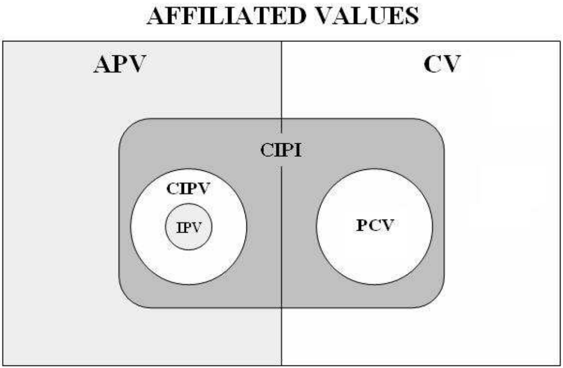
Simple Model of an Auction

- Each bidder (player/firm/buyer/agent) demands at most one unit of a good being sold at auction.
- Heterogeneity in how each bidder values the good is individual specific, private.
- Valuations are modelled as a continuous random variable V.
- Each bidder gets a draw from some distribution(s) which is (are) known to all bidders.
- The realization of each bidder's valuation is known only to that bidder.
- Conditional on a bidder's realization v, he acts purposefully by solving a clearly defined and known optimization problem.

Informational Paradigms

Adding structure to this model results in different informational paradigms:



We shall focus almost exclusivly on the IPV paradigm, although I shall discuss the APV paradigm, too.

Additional Structure

It will be clear that even within the IPVP, things soon get difficult.

Another common distinction concerns how the distributions of one player's valuations relate to those of another.

- When valuation distributions for all bidders are identically distributed ⇒ *symmetric* IPVP
- When valuations for at least two bidders are drawn from different distributions ⇒ *asymmetric* IPVP

Rules of the Auction

Finally, we shall focus on a particular auction format and pricing rule. The two common formats are oral and sealed-bid,

while the two common pricing rules are pay-your-bid (sometimes referred to as *first-price*) and second price. Thus, four types of auctions exist:

- first-price auctions: *first-price, sealed-bid* (FPSB) auctions and *Dutch* auctions;
- second-price auctions: *second-price, sealed-bid* (SPSB) or Vickrey auctions and *English* auctions.

We shall focus on FPSB auctions.

Bidder's Problem

In all that we do, consider the following:

1. bidders are risk neutral (not necessary), thus when bidder n submits bid (strategy) s_n he receives the following payoff:

$$U_n(V_n, \boldsymbol{s}) = \begin{cases} V_n - s_n & \text{if } s_n > s_m \text{ for all } n \neq m \\ 0 & \text{otherwise.} \end{cases}$$

2. Bidder n chooses his bid (strategy) s_n to maximize his expected profit.

$$\mathbb{E}(U_n | s_n) = (V_n - s_n) \Pr(\text{win} | s_n).$$

Some Standard Assumptions

Below, we assume the following:

1. the number of potential buyers N is known by all bidders;
2. the distribution functions $F_n(v)$ from which each bidder's valuations are drawn from are known by all bidders;
3. $F_n(v)$ are continuous with associated probability density functions $f_n(v)$ that are positive on the common compact interval $[\underline{v}, \bar{v}]$ where $\underline{v} \geq 0$.

Symmetric IPVP

For now, let us focus on the symmetric IPVP:

- $F_n(v)$ is the same for all bidders;
- denote this common distribution $F_0(v)$;
- thus, bidder n's valuation V_n is an independent draw from $F_0(v)$.

Under these assumptions, we

- can focus on representative bidder n;
- this assumption puts structure on $\Pr(\text{win}|s_n)$.

$\text{Pr}(\text{win}|s_n)$ in Symmetric IPVP

Suppose the opponents of bidder n are using a monotonically, increasing function $s = \sigma(v)$ to bid.

Then,

$$
\begin{aligned}
\text{Pr}(\text{win}|s_n) &= \Pr(S_1 < s_n, S_2 < s_n, \ldots, S_{n-1} < s_n, S_{n+1} < s_n, \ldots, S_N < s_n) \\
&= \prod_{m \neq n} \Pr\left(S_m < s_n\right) \\
&= \prod_{m \neq n} \Pr\left[\sigma(V_m) < s_n\right] \\
&= \prod_{m \neq n} \Pr\left[V_m < \sigma^{-1}(s_n)\right] \\
&= F_0\left[\sigma^{-1}(s_n)\right]^{N-1} \\
&\equiv F_0\left[\varphi(s_n)\right]^{N-1}.
\end{aligned}
$$

Bidder's Problem within the Symmetric IPVP

Bidder n maximizes

$$\mathbb{E}(U_n|s_n) = (V_n - s_n)\,\Pr(\mathsf{win}|s_n) = (V_n - s_n)F_0\left[\varphi(s_n)\right]^{N-1}$$

which yields the following FOC:

$$\frac{\mathrm{d}\mathbb{E}(U_n|s_n)}{\mathrm{d}s_n} = 0 \;\Rightarrow$$

$$-F_0\left[\varphi(s_n)\right]^{N-1} + (V_n - s_n)\,(N-1)F_0\left[\varphi(s_n)\right]^{N-2} f_0\left[\varphi(s_n)\right]\frac{\mathrm{d}\varphi(s_n)}{\mathrm{d}s_n} = 0.$$

First-Order Condition is an Ordinary Differential Equation

In a Bayes–Nash equilibrium, $\varphi(s) = v$

Monotonicity $\Rightarrow$ $\sigma'(v) = \mathrm{d}s/\mathrm{d}\varphi(s)$, so the first-order condition can be written as

$$\sigma'(v) + \sigma(v)\frac{(N-1)f_0(v)}{F_0(v)} = \frac{(N-1)v f_0(v)}{F_0(v)}.$$

$\sigma'(v)$ is a linear function of $\sigma(v)$ so this is a linear, first-order ordinary differential equation.

We Live a Charmed Life . . .

This is among the few differential equations that have a closed-form solution.

To solve this ODE, we need an initial condition, such as

- with no reserve price $\sigma(\underline{v}) = \underline{v}$;
- with positive reserve price r_0, $\sigma(r_0) = r_0$.

The appropriate initial condition, together with the differential equation, constitute an initial-value problem which has the following unique solution:

$$\sigma(v) = v - \frac{\int_{r_0}^{v} F_0(u)^{N-1}\, \mathrm{d}u}{F_0(v)^{N-1}}.$$

Asymmetric IPVP

Let us now focus on the asymmetric IPVP:

- $F_n(v)$ is now bidder-specific;
- thus, bidder n's valuation V_n is an independent draw from $F_n(v)$;
- all distributions have common support $[\underline{v}, \bar{v}]$ and strictly positive densities $f_n(v)$ over this support.

Consider how this assumption changes structure of $\Pr(\text{win}|s_n)$

Let us start by assuming $N = 2$.

Pr(win|s_n) in Asymmetric IPVP with $N = 2$

Assuming each potential buyer n is using a bid $\sigma_n(v_n)$ that is monotonically increasing in his value v_n, we can write the probability of winning the auction as

$$\begin{aligned} \Pr(\text{win}|s_n) &= \Pr(S_m < s_n) \\ &= \Pr[\sigma_m(V_m) < s_n] \\ &= \Pr[V_m < \sigma_m^{-1}(s_n)] \\ &= \Pr[V_m < \varphi_m(s_n)] \\ &= F_m[\varphi_m(s_n)]. \end{aligned}$$

Bidder's Problem in Asymmetric IPVP with $N = 2$

Thus, the expected profit function for bidder 1 is

$$\mathbb{E}(U_1|s_1) = (V_1 - s_1)F_2[\varphi_2(s_1)],$$

while the expected profit function for bidder 2 is

$$\mathbb{E}(U_2|s_2) = (V_2 - s_2)F_1[\varphi_1(s_2)].$$

Maximizing these by choosing s_1 or s_2, respectively, yields the FOCs

$$\frac{\mathrm{d}\mathbb{E}(U_1|s_1)}{\mathrm{d}s_1} = -F_2[\varphi_2(s_1)] + (V_1 - s_1)f_2[\varphi_2(s_1)]\frac{\mathrm{d}\varphi_2(s_1)}{\mathrm{d}s_1} = 0$$

$$\frac{\mathrm{d}\mathbb{E}(U_2|s_2)}{\mathrm{d}s_2} = -F_1[\varphi_1(s_2)] + (V_2 - s_2)f_1[\varphi_1(s_2)]\frac{\mathrm{d}\varphi_1(s_2)}{\mathrm{d}s_2} = 0.$$

Where Did the Luck Go?

Now, a Bayes–Nash equilibrium is characterized by the following pair of differential equations:

$$\varphi_2'(s_1) = \frac{F_2[\varphi_2(s_1)]}{[\varphi_1(s_1) - s_1]f_2[\varphi_2(s_1)]}$$
$$\varphi_1'(s_2) = \frac{F_1[\varphi_1(s_2)]}{[\varphi_2(s_2) - s_2]f_1[\varphi_1(s_2)]}.$$

As you may realize, there is more we need before we can consider solving these, but first . . .

General Model

We can extend this to the asymmetric IPVP with N bidders:

- $F_n(v)$ is now bidder-specific;
- thus, bidder n's valuation V_n is an independent draw from $F_n(v)$;
- all distributions have common support $[\underline{v}, \bar{v}]$ and strictly positive densities $f_n(v)$ over this support.

Consider how this assumption changes structure of $\Pr(\mathsf{win}|s_n)$.

Pr(win|s_n) in Asymmetric IPVP

Assuming each potential buyer n is using a bid $\sigma_n(v_n)$ that is monotonically increasing in his value v_n, we can write the probability of winning the auction as

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\begin{aligned}
\Pr(\text{win}|s_n) &= \Pr(S_1 < s_n, S_2 < s_n, \ldots, S_{n-1} < s_n, S_{n+1} < s_n, \ldots, S_N < s_n) \\
&= \prod_{m \neq n} \Pr\left(S_m < s_n\right) \\
&= \prod_{m \neq n} \Pr\left[\sigma_m(V_m) < s_n\right] \\
&= \prod_{m \neq n} \Pr\left[V_m < \sigma_m^{-1}(s_n)\right] \\
&= \prod_{m \neq n} F_m\left[\sigma_m^{-1}(s_n)\right] \\
&= \prod_{m \neq n} F_m\left[\varphi_m(s_n)\right].
\end{aligned}
$$

Bidder's Problem in Asymmetric IPVP

Bidder n maximizes

$$\mathbb{E}(U_n|s_n) = (V_n - s_n)\prod_{m\neq n} F_m[\varphi_m(s_n)]$$

which results in the following FOC:

$$\frac{\partial \mathbb{E}(U_n|s_n)}{\partial s_n} = 0 \;\Rightarrow$$

$$-\prod_{m\neq n} F_m\left[\varphi_m(s_n)\right] + (V_n - s_n)\sum_{m\neq n} f_m[\varphi_m(s_n)]\frac{\mathrm{d}\varphi_m(s_n)}{\mathrm{d}s_n}\prod_{\ell\neq m} F_\ell[\varphi_\ell(s_n)] = 0.$$

Asymmetric IPVP Yields System of Differential Equations

Can rewrite this a couple of ways. First, divide through by the isolated product to get

$$\frac{1}{\varphi_n(s) - s} = \sum_{m \neq n} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi'_m(s),$$

which can be summed over N, so after some algebra, this can be written as

$$\varphi'_n(s) = \frac{F_n[\varphi_n(s)]}{f_n[\varphi_n(s)]} \left\{ \left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{\varphi_m(s) - s} \right] - \frac{1}{\varphi_n(s) - s} \right\}.$$

Boundary Conditions—Plural!

In contrast to the symmetric case, in the asymmetric case there are two types of boundary conditions that must hold:

Left-Boundary Condition on Bid Functions: $\sigma_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$;

or, said another way,

Left-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$.

Boundary Conditions—Plural!

In contrast to the symmetric case, in the asymmetric case there are two types of boundary conditions that must hold:

Right-Boundary Condition on Bid Functions: $\sigma_n(\bar{v}) = \bar{s}$ for all $n = 1, 2, \ldots, N$;

or, said another way,

Right-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\bar{s}) = \bar{v}$ for all $n = 1, 2, \ldots, N$.

Comparing Symmetric IPVP and Asymmetric IPVP

The asymmetric problem has the following differences:

- we need to solve for the inverse-bid functions in the asymmetric case;
- a system of differential equations obtain;
- each equation in the system is still a first-order differential equation, but it is no longer linear;
- no longer an initial value problem, but now a two-point boundary value problem;
- $\bar{s}$ is unkown *a priori* and determines domain of solutions;
- boundary value problem is *overidentified*;
- we know some characteristics that the solutions must respect (rationality and monotonicity).

Oh, But There is Just One More Thing . . .

A function $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies the Lipschitz condition on a d-dimensional interval I if there exists a Lipschitz constant $\lambda > 0$ such that

$$\|g(\mathbf{y}) - g(\mathbf{x})\| \leq \lambda \|\mathbf{y} - \mathbf{x}\|$$

for a given vector norm $\|\cdot\|$ and for all $\mathbf{x} \in I$ and $\mathbf{y} \in I$.

Bad News for the System . . .

The system of differential equations do not satisfy the Lipschitz condition in a neighborhood of $\underline{v}$ because a singularity obtains at $\underline{v}$.

Let's revisit $N = 2$ case where

$$\varphi_2'(s_1) = \frac{F_2[\varphi_2(s_1)]}{[\varphi_1(s_1) - s_1]f_2[\varphi_2(s_1)]}$$
$$\varphi_1'(s_2) = \frac{F_1[\varphi_1(s_2)]}{[\varphi_2(s_2) - s_2]f_1[\varphi_1(s_2)]}.$$

The denominator terms in the right-hand side of these equations which involve $[\varphi_n(s) - s]$ vanish. Along with this $F_n(\underline{v}) = 0$ for all bidders.

Consequently, . . .

Now Nearly Everything Goes Out the Window

The Lipschitz condition is critical for standard results concerning existence and uniqueness of a solution, and forms the basis of most approaches to solving systems of differential equations numerically.

This makes the problem interesting to

- theorists;
- computational economists;
- applied researchers.

Existence and Uniqueness

Before considering *how* to solve asymmetric auctions, it is important to know that solutions exist, and ideally are unique.

Fortunately, there are some very talented theorists out there

- Lebrun (1999) proved that the inverse-bid functions are differentiable on $(\underline{v}, \bar{s}]$ and that a unique Bayes–Nash equilibrium exists when all valuation distributions have a mass point at $\underline{v}$ and the value distributions have a common support (as we have assumed above).
- Existence was also demonstrated by Maskin and Riley (2000b), while Maskin and Riley (2000a) investigated some equilibrium properties of asymmetric first-price auctions.
- See also, Lebrun (1996), Reny (1999), Lizzeri and Persico (2000), Athey (2001) as well as Krishna's (2002) book.

Complicating Factors

The issues that obtain in an asymmetric IPVP setting show up if other assumptions are relaxed as well. For example, in first-price auctions with

- risk averse bidders with bidder-specific Arrow–Pratt coefficient of relative risk aversion;
- bidder collusion (coalition formation);
- bid preference policies.

In addition, the results everything we have done extends naturally to one of the most important auction settings (in terms of dollar amounts): low-price, sealed-bid (LPSB) or procurement auctions.

What to Do?

Because of the complications we have discussed, reasearchers must employ numerical methods to solve these asymmetric first-price auctions.

I can partition the approaches researchers have used into three types:

1. shooting methods;
2. projection methods;
3. fixed-point/Newton iterative methods.

Shooting Methods: An Analogy

One way to solve boundary-value problems is to treat them like initial-value problems, solving them repeatedly, until the solution satisfies both boundary conditions.

Consider firing an object at a target some distance away:

- suppose that you do not hit the target successfully on your first try;
- if hitting the target is important, then you will learn from your first miss, make appropriate adjustments, and fire again;
- you will continue to repeat this process until you successfully hit your target.

Shooting Methods: An Analogy

The key characteristics:

- you understand *which point* you are shooting from and which is the target;
- *how to fire* an object using whatever mechanism is used to send the object at the target;
- you need to recognize the *type of adjustments* that need to be made so that your successive shots at the target improve.

This story provides an analogy for the procedure used in a shooting algorithm to solve boundary-value problems.

More Formally

The shooting algorithm treats one of the boundaries like an initial value.

Given that initial value, there are well known ways to solve a system of differential equations . Today, I shall discuss two simple finite difference ones:

- Euler's method;
- Runge–Kutta method(s).

After solving the system and arriving at the other boundary, we check to see whether the other (target) condition is satisfied and (if not) understand how to adjust our initial condition.

Where to Begin?

In our asymmetric FPSB auction model we have two boundary conditions:

$$\varphi_n(\underline{v}) = \underline{v},\ n = 1, \ldots, N,$$

and

$$\varphi_n(\bar{s}) = \bar{v},\ n = 1, \ldots, N.$$

Which condition should serve as our initial condition and which should serve as a terminal condition?

Choice of Initial Condition

Note the difference between the two conditions:

- for the left-boundary, we know both the bid as well as the valuation *a priori*;
- for the right-boundary we know only the valuation $\bar{v}$, but not the common high bid $\bar{s}$ for which we must solve.

Since $\bar{s}$ is unknown it makes for a poor target: how will we know whether we hit it? Thus, we use it as an "initial" condition $\Rightarrow$ *backwards* (*reverse*) shooting.

The left-boundary condition makes for a good target (ignoring the Lipschitz issue): we know the bid as well as its corresponding valuation for all players.

Finite Difference Methods

See any book on differential equations for this; here, quickly are two explicit approaches.

Consider the following first-order ordinary differential equation for σ as a function of v:

$$\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v} = D(v, \sigma).$$

Euler's method

$$\sigma(v_t + h) \approx \sigma(v_t) + h \left.\frac{\mathrm{d}\sigma(v)}{\mathrm{d}v}\right|_{v=v_t} = \sigma(v_t) + hD[v_t, \sigma(v_t)]$$

for given *step size* h and initial condition $\sigma(v_0) = s_0$.

Finite Difference Methods

Runge–Kutta method (of order four)

$$\sigma_{k+1} = \sigma_k + h\frac{1}{6}(d_1 + 2d_2 + 2d_3 + d_4)$$

where

$$\begin{aligned}
d_1 &= D\left(v_k, \sigma_k\right) \\
d_2 &= D\left(v_k + \frac{1}{2}h, \sigma_k + \frac{1}{2}hd_1\right) \\
d_3 &= D\left(v_k + \frac{1}{2}h, \sigma_k + \frac{1}{2}hd_2\right) \\
d_4 &= D\left(v_k + h, \sigma_k + hd_3\right).
\end{aligned}$$

For a system of differential equations, the individual equations are just stacked.

Adjusting $\bar{s}$

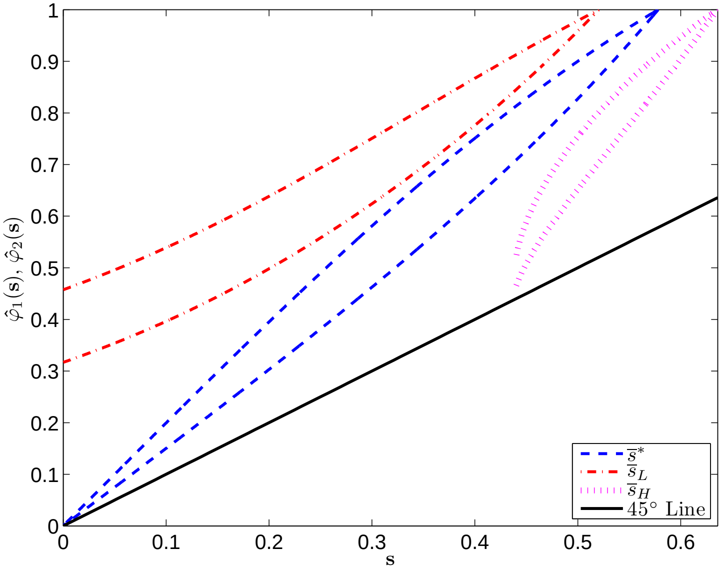
There are two types of failures that may obtain which inform us concerning how to change $\bar{s}$:

1. one value at terminal condition is "too far" from true (known) value $\underline{v}$, that is, $[\hat{\varphi}_n(\underline{v}) - \underline{v}]$ is too large;
2. the solution "blows up" or diverges, specifically, the solutions explode toward minus infinity as the bids approach. $\underline{v}$

The first case means $\bar{s}$ was too low $\Rightarrow$ need to increase it.

The second case means $\bar{s}$ was too high $\Rightarrow$ need to decrease it.

Intuition



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s
$\hat{\varphi}_1(\mathbf{s}), \hat{\varphi}_2(\mathbf{s})$
$\overline{s}^*$
$\overline{s}_L$
$\overline{s}_H$
45° Line

An Algorithm

Consider the following algorithm:

1. make guess for $\bar{s}_i \in [\underline{v}, \bar{v}]$;
2. solve the system of differential equations backwards on over the interval $[\underline{v}, \bar{s}_i]$;
3. determine whether to increase or to decrease the guess $\bar{s}_i$;
   - if the solution blows up, then set $\bar{s}_{i+1} < \bar{s}_i$ in step 1 and try again;
   - if the approximated solution at $\underline{v}$ is in $[\underline{v}, \bar{v}]$, but does not satisfy $[\varphi_n(\underline{v}) - \underline{v}] > \varepsilon$ for some bidder n, then set $\bar{s}_{i+1} > \bar{s}_i$ in step 1 and try again;
4. stop when

$$\|\hat{\varphi}_n(\underline{v}) - \underline{v}\| \leq \varepsilon \text{ for all } n = 1, \ldots, N$$

for some pre-specified norm $\|\cdot\|$ and pre-specified tolerance level ε.

Marshall, Meurer, Richard, and Stromquist (1994, GEB)

MMRS (1994, GEB) were the first to try and solve asymmetric auctions numerically and they used a shooting method.

Others have refined the shooting approach: Bajari (2001, ET), Li and Riley (2007, IJIO), Gayle and Richard (2008, CE).

They inspired much theoretical work. Some recent examples include:

- Cantillon (2008, GEB) "rationalized" and generalized many of their results in investigating the effect asymmetries have on the auctioneer's expected revenue;
- Marshall and Marx (2007, JET) considered bidder collusion in an asymmetric setting (whether it obtains endogenously or not)

MMRS (1994, GEB) Example

Consider a symmetric IPVP model in which N bidders draw independent valuations from the same distribution $F_0(v)$, having an associated positive probability density function $f_0(v)$ that has compact support $[\underline{v}, \bar{v}]$

The N potential bidders form K coalitions with a representative coalition k having size n_k with

$$n_k \geq 1, \text{ for } k = 1, \ldots, K$$

and

$$\sum_{k=1}^{K} n_k = N$$

where K is less than or equal to N.
Let us consider $K = 2$.

MMRS (1994, GEB) Example Continued ...

Let us consider $K = 2$ and assume $n_1 \neq n_2$

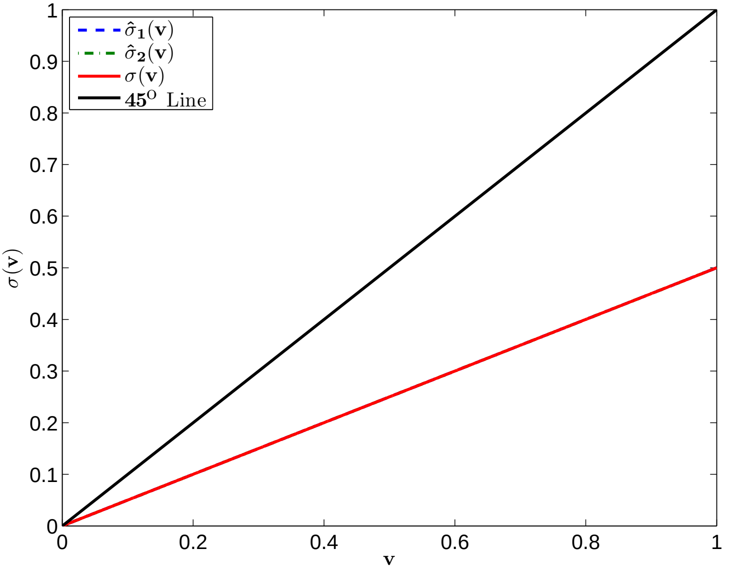
Assume $F_0(v)$ is a uniform distribution.

What is relevant to the coalition is the bidder with the highest valuation?

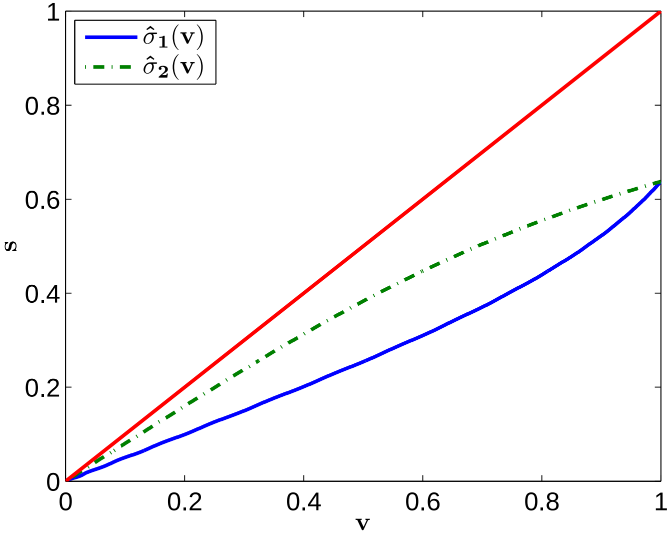
Implication: each coalition is using its highest valuation to compete against the maximum of the other coalition's valuations which is distributed as $F_0(v)^{n_k}$; i.e., it is like facing a bidder from a power distribution.

Consider how coalition behavior changes as the number of coalition members changes.

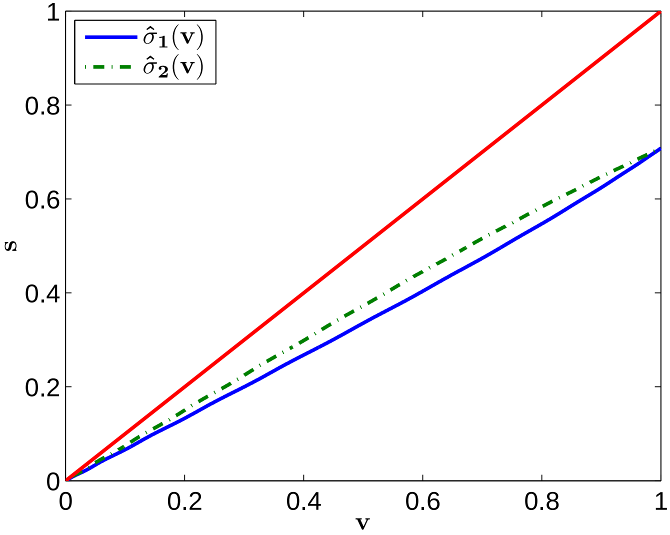
Coalition of 1 versus Coalition of 1



Coalition of 4 versus Coalition of 1



Coalition of 3 versus Coalition of 2



Raining on the Shooting Parade

I was very careful about the example I chose—with uniform $F_0(\cdot)$ the maximum valuations from each coalition imply asymmetric power distributions, one of the only cases with closed-form solutions.

Nearly all researchers who used shooting methods noted that the algorithm was very sensitive and instable.

Recently, Fibich and Gavish (2011, GEB) have proven analytically that the inherent instability is not a "technical issue," but rather an analytic property of backward integration in this setting.

Furthermore, shooting methods are very costly (in terms of time), require more advanced programming techniques, and typically involve a lot of "fiddling."

We shall continue tomorrow with projection methods . . .