Computation of Moral-Hazard Problems with Applications in Designing Executive Compensation Contracts

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Based on joint work with Kenneth L. Judd, Chris Armstrong, and David Larcker

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Agenda

- Static Moral-Hazard Model
  - With Kenneth Judd (Hoover Institution & NBER)
  - Deterministic contract
  - LP lottery approach
  - MPEC formulation
  - Hybrid method for a global solution
  - Numerical results on deterministic contract
  - Contract with action lotteries and numerical results
- Executive Compensation Design
  - With David Larcker and Chris Armstrong (Stanford GSB)

Complementarity

Complementarity in simplest form: for $x, y \in \mathbb{R}$

$$0 \leq x \perp y \geq 0 \quad \Longleftrightarrow \quad x \geq 0,\ y \geq 0,\ x = 0 \text{ or } y = 0$$

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$$\text{argmin}\{f(x) : x \ge 0\} \quad \subseteq \quad \{x^* : 0 \le x \perp \nabla_x f(x) \ge 0\}$$

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- Nash equilibrium in game theory: f_1, f_2 smooth

$$\begin{bmatrix} \text{argmin}\{f_1(x; y) : x \ge 0\} \\ \text{argmin}\{f_2(y; x) : y \ge 0\} \end{bmatrix} \subseteq \left\{ (x^*, y^*) : \begin{bmatrix} 0 \le x \perp \nabla_x f_1(x; y) \ge 0 \\ 0 \le y \perp \nabla_y f_2(y; x) \ge 0 \end{bmatrix} \right\}$$

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Mathematical Programs with Equilibrium Constraints (MPEC):

$$\begin{array}{ll} \text{minimize}_{(x)} & f(x) \\ \text{subject to} & g(x) \leq 0 \\ & 0 \leq x \perp F(x) \geq 0 \end{array}$$

Static Moral-Hazard Model

expected utility W(c,a):

$\Sigma_i\, p(q_i|a)w(q_i - c_i)$

compensation ($):

$c = (c_i)$ with $c_i \in \mathbf{R}_+$

outcomes ($): $q_i \in \boldsymbol{Q}$

uncertainty: $p(q_i|a)$

action ($): $a \in \mathbf{A} \subset \mathbf{R}_+$

expected utility U(c, a):

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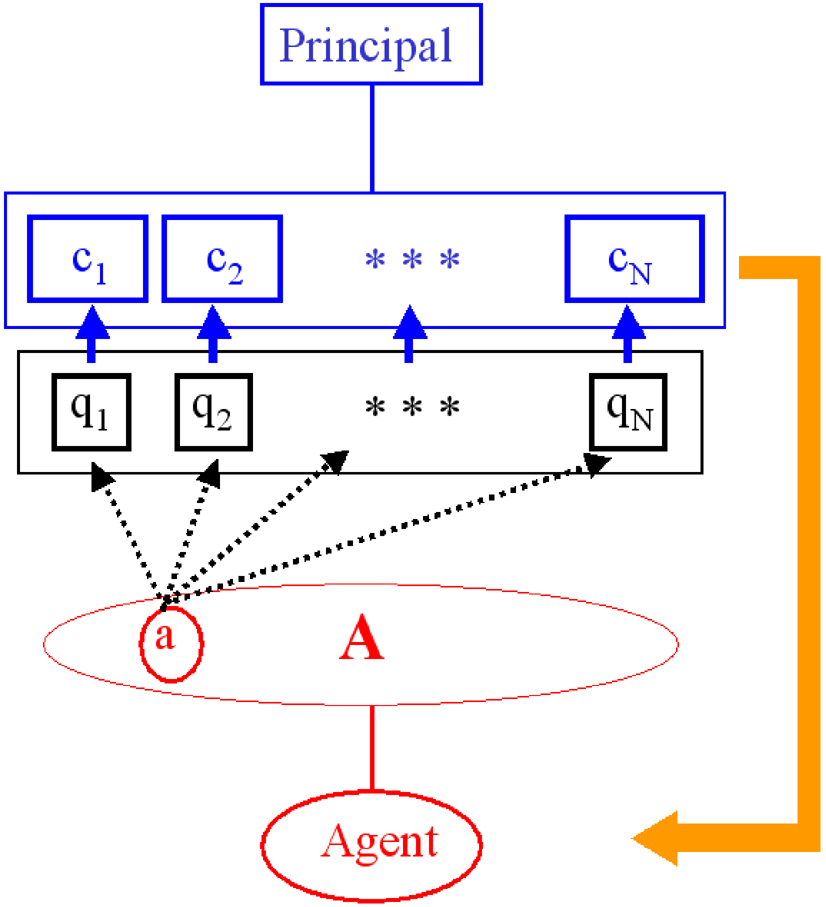

Timeline of Moral-Hazard Model

- Notations:
  - $a \in \mathcal{A} \subset \mathbb{R}$; $q_i \in \mathcal{Q} = \{q_1, \ldots, q_N\}$; $c = (c_1, \ldots, c_N) \in \mathbb{R}_+^N$
  - Uncertainty: $p(q_i|a)$
  - Utility: principal $w(q_i - c_i)$; agent $u(c_i, a)$
  - Expected utility: principal $W(c, a) = \sum_i p(q_i|a) w(q_i - c_i)$;

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- Output q_i is realized
- Compensation c_i is paid according to the realized output q_i

Optimal Deterministic Contract

$$\begin{aligned} \text{maximize}_{(c,a)} \quad & W(c,a) \\ \text{subject to} \quad & U(c,a) \geq U_0 \text{ (Reservation Utility)} \\ & a \in \text{argmax}\{U(c,\tilde{a}) : \tilde{a} \in \mathcal{A}\} \\ & c \in \mathbb{R}_+^N \end{aligned}$$

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- First-order approach [Mirrlees '75] [Rogerson '85]
  - Replace $\{\max U(c,a) : a \in \mathcal{A}\}$ by first-order conditions, but usually, $U(c,a)$ is not concave in a for all c

LP Lotteries Approach

- [Myerson '82] [Prescott & Townsend '84a, b] [(Ned) Prescott '04]
- Consider finite action set $\mathcal{A}$ and finite compensation set $\mathcal{C}$ with element ξ

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  - The action lottery is done according to the agreed probabilities $\pi(a)$
  - Output q is realized
  - Compensation is paid according to the agreed lottery conditional on the action a and the realized output q: $\pi(\xi|q,a)$

LP Lotteries Approach

- Ideas: Consider action and compensation lotteries
  - Construct an action grid $\mathcal{A} = \{a_1, \ldots, a_M\}$ and a compensation grid $\mathcal{C} = \{\xi_1, \ldots, \xi_L\}$
  - Introduce probability measures $\pi(a)$ and $\pi(\xi | q, a)$
  - Transform ODC into an **LP** with $\pi = (\pi(\xi, q, a))_{\xi \in \mathcal{C}, q \in \mathcal{Q}, a \in \mathcal{A}}$

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  - Transform ODC into an **LP** with $\pi = (\pi(\xi, q, a))_{\xi \in \mathcal{C}, q \in \mathcal{Q}, a \in \mathcal{A}}$
- Disadvantages:
  - The resulting LP is large:

    $L * M * N$ variables and $M * (N + M - 1) + 2$ constraints.

    If $M = 50, N = 40, L = 500$, $\Rightarrow$ an LP with one million variables and 4452 constraints
  - Economic choice variables are continuous in nature
  - The curse of dimensionality for multidimensional problems

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- Agent's mixed strategy problem is an LP

$$a^* \in \operatorname{argmax} \{U(c, a) : a \in \{a_1, \ldots, a_M\}\}$$
$$\Updownarrow$$
$$\delta^* \in \operatorname{argmax} \left\{ \sum_{k=1}^{M} U(c, a_k)\delta_k \, : \, e^{\mathrm{T}} \delta = 1, \, \delta \geq 0 \right\}$$

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Optimality + strong duality $\Updownarrow$ Let $U(c) := (U(c, a_k))_{k=1}^{M}$

$$\delta^* \text{ solves } \begin{cases} 0 \leq \delta \perp \left(U(c)^{\mathrm{T}}\delta\right) e - U(c) \geq 0 \\ e^{\mathrm{T}}\delta = 1 \end{cases}$$

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Lemma:

(i) $(c^*, \delta^*) \in$ **SOL**(MPEC) $\Longrightarrow (c^*, a_i^*)$ an ODC with $a_i^* \in \mathcal{A}$ finite, and $i \in \{j : \delta_j^* > 0\}$.

(ii) (c^*, a_i^*) an ODC with $a_i^* \in \mathcal{A}$ finite $\Longrightarrow (c^*, e_i) \in$ **SOL**(MPEC), where e_i is the i-th column of an identity matrix

A Hybrid Method [Su-Judd '05]

Observations:

- MPEC: Allows continuous compensation $c \in \mathbb{R}^N_+$ but may stop at local maximum
- LP: Produce global optimal but the grid $\mathcal{C}$ may be too coarse

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Step 0: Construct a coarse grid $\mathcal{C}$

Step 1: Solve the **LP** for the given grid $\mathcal{C}$

Step 2: Setup and solve the **MPEC** :

$$\begin{cases} (2.1): & \text{construct } (c^0, \delta^0) \text{ using } \textbf{LP} \text{ solution} \\ (2.2): & \text{Solve the } \textbf{MPEC} \text{ with the feasible starting pt } (c^0, \delta^0) \end{cases}$$

Step 3: Refine the grid and repeat Step 1 and Step 2

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Result: A hybrid solution is always better than an LP solution

An Example in [Karaivanov '01]

- Risk-neutral principal: $w(q_i - c_i) = q_i - c_i$
- Risk-averse agent: $u(c_i, a) = \dfrac{c_i^{1-\gamma}}{1-\gamma} + \kappa \dfrac{(1-a)^{1-\delta}}{1-\delta}$
- Two outcomes: $q_H = \$3$ and $q_L = \$1$
- Action set: $|\mathcal{A}| = 10$ with equally-spaced effort level within $[0.01, 0.99]$
- The production technology $p(q = q_H | a) = a^\alpha$ with $0 < \alpha < 1$

γ	κ	δ	α	U_0
0.5	1	0.5	0.7	1

- Both LP and MPEC are coded in AMPL and solved by SNOPT on NEOS server (host: `prado.iems.northwestern.edu`)

LP Solutions

LP Solutions with 8 different grids (# of constraints = 112)

$\|\mathcal{C}\|$	# of Variables	Read Time (in sec.)	Solve Time (in sec.)	# of Iterations	Objective Value
21	420	0.01	0.04	37	1.875882746
41	820	0.02	0.07	46	1.877252910
81	1620	0.03	0.12	46	1.877259193
161	3220	0.06	0.25	46	1.877262265
321	6420	0.13	0.58	69	1.877263785
641	12820	0.26	1.12	52	1.877259905
1281	25620	0.53	2.67	101	1.877262221
2561	51220	1.09	4.81	73	1.877262201
5121	102420	2.46	11.70	101	1.877263113

- [(Ned) Prescott '04]: Dantzig-Wolfe decomposition.
- Warm Start: Simplex method.

Hybrid Solution

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MPEC Starting Point	Read Time (in sec.)	Solve Time (in sec.)	# of Major Iterations	Objective Value
$\delta_6 = 1, \delta_{i(\neq 6)} = 0$	0.02	0.01	13	1.877265298

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- Modifications:
  - Objective function: $\sum_a \pi(a) W(c(a), a)$
  - Participation constraint: $\sum_a \pi(a) U(c(a), a) \geq U_0$
  - Incentive compatibility constraints:

$$\pi(a) \geq 0$$

$$\text{if } \pi(a) > 0, \text{ then } \begin{cases} U(c(a), a) \geq U(c(a), a_1), \\ \vdots \\ U(c(a), a) \geq U(c(a), a_M). \end{cases}$$

$$\Updownarrow$$

$$\pi(a) \geq 0$$

$$\pi(a)(U(c(a), a) - U(c(a), \tilde{a})) \geq 0, \quad \forall (a, \tilde{a}) \in \mathcal{A} \times \mathcal{A}$$

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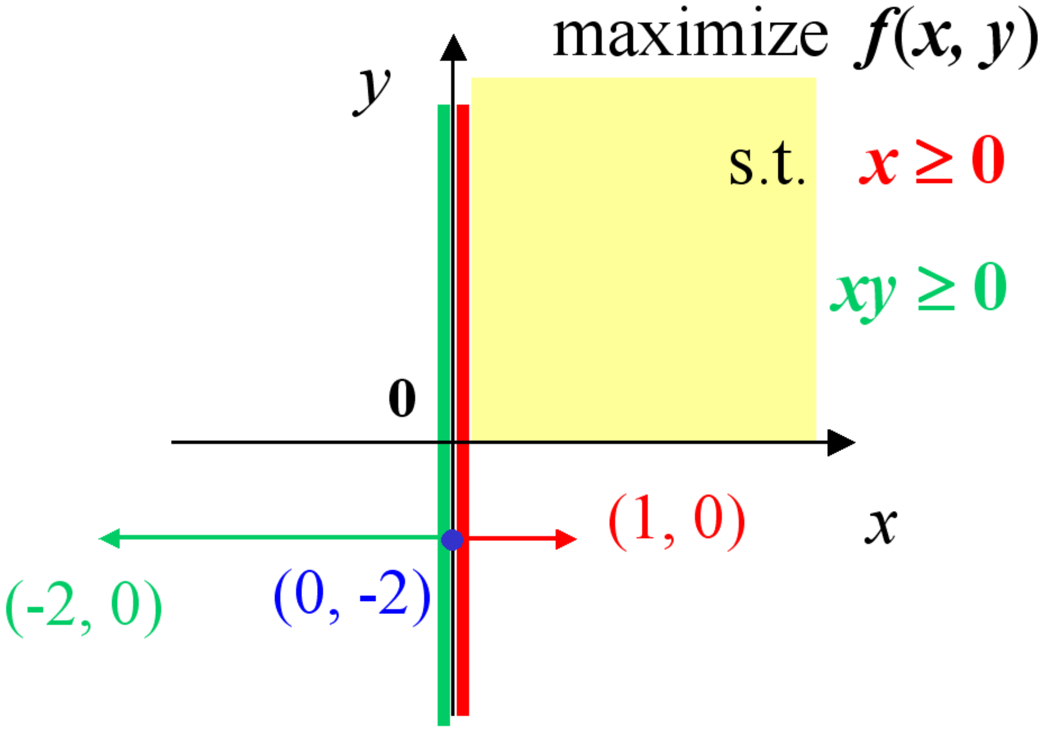
\text{subject to} \quad & \sum_{a\in\mathcal{A}} \pi(a) U(c(a), a) \geq U_0, \\
& \sum_{a\in\mathcal{A}} \pi(a) = 1, \\
& \pi(a) \left(U(c(a), a) - U(c(a), \tilde{a}) \right) \geq 0, \quad \forall\, (a, \tilde{a}(\neq a)) \in \mathcal{A} \times \mathcal{A}, \\
& \pi(a) \geq 0, \quad \forall\, a \in \mathcal{A}.
\end{aligned}$$

Star-Shaped Feasible Region



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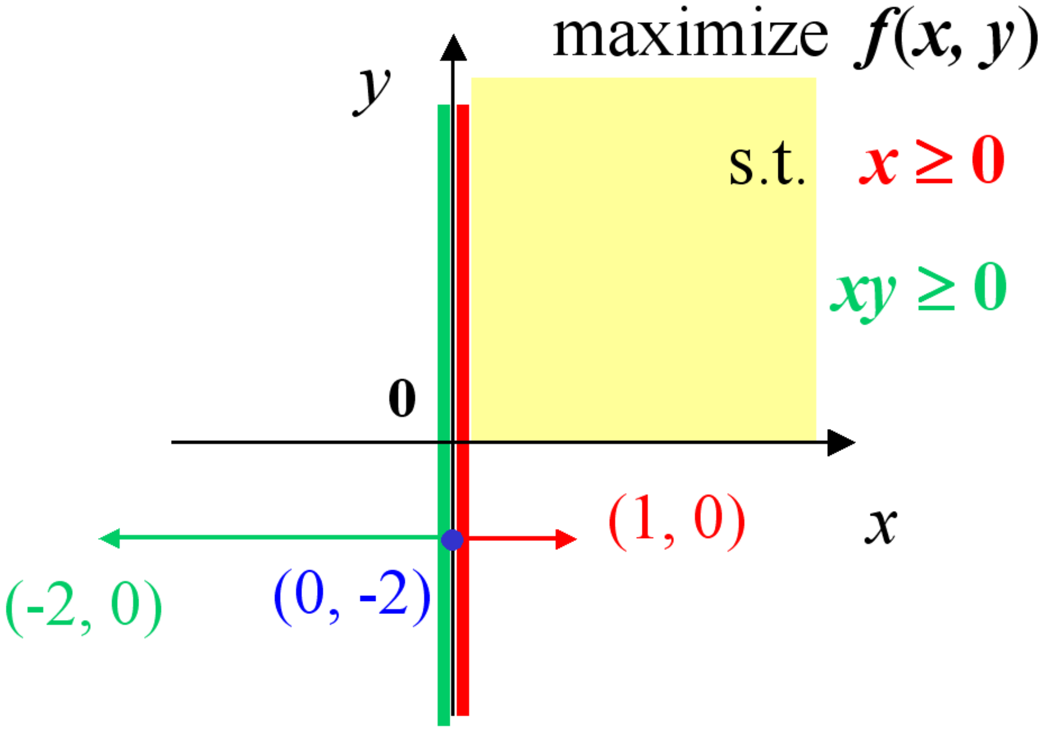

- Nonconvex feasible region due to switch-off constraints [Scholtes '04]
- LICQ fails on y-axis

MPEC for Action Lotteries

$$\pi(a) \geq 0, \qquad \text{if } \pi(a) > 0, \text{ then } \begin{cases} U(c(a), a) \geq U(c(a), a_1), \\ \vdots \\ U(c(a), a) \geq U(c(a), a_M). \end{cases}$$

$$\Updownarrow$$

$$\forall\, a \in \mathcal{A}: \quad \begin{cases} \pi(a)\left(U(c(a), a) - U(c(a), \tilde{a})\right) \geq 0, & \forall\, \tilde{a}(\neq a) \in \mathcal{A}, \\ \pi(a) \geq 0. \end{cases}$$

$$\Updownarrow$$

$$\forall\, a \in \mathcal{A}: \quad \begin{cases} U(c(a), a) - U(c(a), \tilde{a}) + s(a, \tilde{a}) \geq 0, & \forall\, \tilde{a}(\neq a) \in \mathcal{A}, \\ 0 \leq \pi(a) \perp s(a, \tilde{a}) \geq 0. \end{cases}$$

MPEC for Action Lotteries

$$\text{maximize} \quad \sum_{a \in \mathcal{A}} \pi(a) W(c(a), a)$$

$$\text{subject to} \quad \sum_{a \in \mathcal{A}} \pi(a) U(c(a), a) \geq U_0,$$

$$\sum_{a \in \mathcal{A}} \pi(a) = 1,$$

$$\forall\, a \in \mathcal{A}: \quad \begin{cases} U(c(a), a) - U(c(a), \tilde{a}) + s(a, \tilde{a}) \geq 0, & \forall\, \tilde{a}(\neq a) \in \mathcal{A}, \\ 0 \leq \pi(a) \perp s(a, \tilde{a}) \geq 0. \end{cases}$$

The A-L Example in [Prescott '04]

Hybrid method: LP with 11 compensation grid points and then switch to MPEC.

Hybrid Solutions with 10 different action grids, $|\mathcal{C}| = 11$, $|\mathcal{Q}| = 50$

$\|\mathcal{A}\|$	t_{LP} (in sec.)	LP Obj. Val.	t_{MPEC} (in sec.)	MPEC Obj. Val.	t_{Total} (in sec.)
6	2	1.75868508	1	1.76234445	3
11	4	1.75868508	2	1.76234448	6
16	9	1.76265085	3	1.76565622	12
21	12	1.76351860	11	1.76630661	23
26	21	1.75924445	6	1.76298273	27
31	38	1.76265085	11	1.76565620	49
36	64	1.75606325	16	1.76051776	80
51	102	1.75924445	22	1.76298271	124
76	266	1.75778364	127	1.76298273	393
101	575	1.75572408	1108	1.76234445	1683
151	4203	1.75534000	14018	1.75996707	18221

Extensions

- MPEC formulations are also given for :
  - Contracts with compensation lottery (randomized payment)
  - Contracts with action and compensation lotteries
  - Multidimensional action choices - little economic theory, make special assumptions
  - Multidimensional compensation choices - infeasible for LP

Extensions

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  - Contracts with compensation lottery (randomized payment)
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  - Multidimensional action choices - little economic theory, make special assumptions
  - Multidimensional compensation choices - infeasible for LP
- Future research
  - Tournament (single-principal multi-agent) problem [Lazear & Rosen '81]
  - Incentive problem with both hidden information and moral-hazard
  - Dynamic contracts (multi-period moral-hazard problem) [Phelan & Townsend '91]
  - Executive compensation (with D. Larcker and C. Armstrong)

Executive Compensation

- Components of Compensation
  - Fixed Salary s
  - Stocks β_0
  - Options β_1 (with Strike Price K to be Determined)
  - Payment $c_i = s + \beta_0 * p_i + \beta_1 * \max(q_i - K, 0)$
- Action Choices for the CEO (Agent)
  1. mean (a) and/or variance (σ) of the performance of business operations

An Example

- Risk-neutral principal: $w(q_i - c_i) = q_i - c_i$
- Risk-averse agent: $u(c_i, a) = \frac{c_i^{1-\gamma}}{1-\gamma} - \mu a^2$
- Outcomes q_i: equally-spaced stock price level within $[0, 160]$
- Action set: $|\mathcal{A}| = 50$ with equally-spaced effort level within $[0, 50]$
- The production technology

$$p(q|a) = \frac{1}{\sqrt{2\pi}\sigma} * exp\left(-\frac{(q-(C+M*a))^2}{2\sigma^2}\right)$$

γ	μ	C	M	U_0
0.5	0.0025	60	(0.25, 0.5, 0.75)	1

Flexible Modeling Framework

(1) agent effort has a positive, but decreasing impact on the mean of the distribution

(2) agent effort has positive and increasing impact on volatility

(3) multiple options with different exercise prices

(4) multiple agent effort: the agent can "diversify" some of his holdings into the risk-free asset

(5) more realistic participation constraint – min utility is a function of estimated agent effort

(6) multiple period variations of these settings

(7) "robust" contracts – very simple compensation plans are almost as good as the very complicated plans that are observed in the real world?