Computation of Moral-Hazard Problems with Applications in Designing Executive Compensation Contracts

Che-Lin Su
The University of Chicago
Booth School of Business
Che-Lin.Su@ChicagoBooth.edu

Based on joint work with Kenneth L. Judd, Chris Armstrong, and David Larcker

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Agenda

● Static Moral-Hazard Model
  ○ With Kenneth Judd (Hoover Institution & NBER)
  ○ Deterministic contract
  ○ LP lottery approach
  ○ MPEC formulation
  ○ Hybrid method for a global solution
  ○ Numerical results on deterministic contract
  ○ Contract with action lotteries and numerical results

● Executive Compensation Design
  ○ With David Larcker and Chris Armstrong (Stanford GSB)
Complementarity

Complementarity in simplest form: for $x, y \in \mathbb{R}$

$$0 \leq x \perp y \geq 0 \iff x \geq 0, y \geq 0, x = 0 \text{ or } y = 0$$
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Complementarity between nonnegative vectors: $x, y \in \mathbb{R}^n_+$

\[ 0 \leq x \perp y \geq 0 \iff x_i \geq 0, y_i \geq 0, x_i = 0 \text{ or } y_i = 0, \text{ for all } i \]
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Complementarity Problem: Given a mapping $F : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$, find a vector $x \in \mathbb{R}^n_+$ satisfying

$$0 \leq x \perp F(x) \geq 0$$
Complementarity

Complementarity between nonnegative vectors: \( x, y \in \mathbb{R}_+^n \)

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0 \leq x \perp y \geq 0 \iff x_i \geq 0, y_i \geq 0, x_i = 0 \text{ or } y_i = 0, \text{ for all } i.
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0 \leq x \perp F(x) \geq 0
\]

- First-order optimality conditions: \( f \) smooth

\[
\text{argmin}\{f(x) : x \geq 0\} \subseteq \{x^* : 0 \leq x \perp \nabla_x f(x) \geq 0\}
\]
Complementarity

Complementarity between nonnegative vectors: \( x, y \in \mathbb{R}_n^+ \)

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\text{argmin}\{f(x) : x \geq 0\} \subseteq \{x^* : 0 \leq x \perp \nabla_x f(x) \geq 0\}
\]

- Nash equilibrium in game theory: \( f_1, f_2 \) smooth

\[
\left[ \begin{array}{c}
\text{argmin}\{f_1(x; y) : x \geq 0\} \\
\text{argmin}\{f_2(y; x) : y \geq 0\}
\end{array} \right] \subseteq \left\{ (x^*, y^*) : \begin{bmatrix}
0 & \leq x \perp \nabla_x f_1(x; y) \geq 0 \\
0 & \leq y \perp \nabla_y f_2(y; x) \geq 0
\end{bmatrix} \right\}
\]
Complementarity

Complementarity between nonnegative vectors: \( x, y \in \mathbb{R}_+^n \)

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0 \leq x \perp y \geq 0 \iff x_i \geq 0, y_i \geq 0, x_i = 0 \text{ or } y_i = 0, \text{ for all } i.
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Mathematical Programs with Equilibrium Constraints (MPEC):

\[
\text{minimize}_{(x)} \quad f(x) \\
\text{subject to} \quad g(x) \leq 0 \\
0 \leq x \perp F(x) \geq 0
\]
Static Moral-Hazard Model

expected utility $W(c, a)$:
$$\sum_i p(q_i | a) w(q_i - c_i)$$

compensation ($\$)$:
$$c = (c_i) \text{ with } c_i \in \mathbb{R}_+$$

outcomes ($\$)$: $q_i \in Q$

uncertainty: $p(q_i | a)$

action ($\$)$: $a \in A \subset \mathbb{R}_+$

expected utility $U(c, a)$:
$$\sum_i p(q_i | a) u(c_i, a)$$
Timeline of Moral-Hazard Model

- Notations:
  - $a \in A \subset \mathbb{R}$; $q_i \in Q = \{q_1, \ldots, q_N\}$; $c = (c_1, \ldots, c_N) \in \mathbb{R}^N_+$
  - Uncertainty: $p(q_i|a)$
  - Utility: principal $w(q_i - c_i)$; agent $u(c_i, a)$
  - Expected utility: principal $W(c, a) = \sum_i p(q_i|a)w(q_i - c_i)$;
    
    agent $U(c, a) = \sum_i p(q_i|a)u(c_i, a)$
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- The principal and the agent agree to the contract with a suggested action $a$ and compensation schedule $c = (c_1, \ldots, c_N)$
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- The principal and the agent agree to the contract with a suggested action $a$ and compensation schedule $c = (c_1, \ldots, c_N)$
- The agent implements the suggested but unobservable action $a$
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- Output $q_i$ is realized
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- The principal and the agent agree to the contract with a suggested action $a$ and compensation schedule $c = (c_1, \ldots, c_N)$
- The agent implements the suggested but unobservable action $a$
- Output $q_i$ is realized
- Compensation $c_i$ is paid according to the realized output $q_i$
Optimal Deterministic Contract

maximize \(W(c, a)\)

subject to \(U(c, a) \geq U_0\) (Reservation Utility)

\(a \in \text{argmax}\{U(c, \tilde{a}) : \tilde{a} \in A\}\)

\(c \in \mathbb{R}^N_+\)
Optimal Deterministic Contract

maximize \((c, a)\) \(W(c, a)\)

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\(a \in \arg\max\{U(c, \tilde{a}) : \tilde{a} \in \mathcal{A}\}\)

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- Both \(c\) and \(a\) are continuous variables
- Need global optimality at both levels
Optimal Deterministic Contract

\[
\begin{align*}
\text{maximize}_{(c,a)} & \quad W(c, a) \\
\text{subject to} & \quad U(c, a) \geq U_0 \quad \text{(Reservation Utility)} \\
& \quad a \in \arg\max\{U(c, \tilde{a}) : \tilde{a} \in \mathcal{A}\} \\
& \quad c \in \mathbb{R}_+^N
\end{align*}
\]

- Both \( c \) and \( a \) are continuous variables
- Need global optimality at both levels
- First-order approach [Mirrlees ’75] [Rogerson ’85]
  - Replace \( \{\max U(c, a) : a \in \mathcal{A}\} \) by first-order conditions, but usually, \( U(c, a) \) is not concave in \( a \) for all \( c \)
LP Lotteries Approach

- [Myerson '82] [Prescott & Townsend '84a, b] [(Ned) Prescott '04]
- Consider finite action set $\mathcal{A}$ and finite compensation set $\mathcal{C}$ with element $\xi$
LP Lotteries Approach

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- Timeline
  - The principal and the agent agree to the contract with lotteries $\pi(a)$ and $\pi(\xi|q, a)$
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  - Output $q$ is realized
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• Timeline
  
  ◦ The principal and the agent agree to the contract with lotteries $\pi(a)$ and $\pi(\xi|q, a)$

  ◦ The action lottery is done according to the agreed probabilities $\pi(a)$

  ◦ Output $q$ is realized

  ◦ Compensation is paid according to the agreed lottery conditional on the action $a$ and the realized output $q$: $\pi(\xi|q, a)$
LP Lotteries Approach

- **Ideas:** Consider action and compensation lotteries
  - Construct an action grid \( A = \{a_1, \ldots, a_M\} \) and a compensation grid \( C = \{\xi_1, \ldots, \xi_L\} \)
  - Introduce probability measures \( \pi(a) \) and \( \pi(\xi|q, a) \)
  - Transform ODC into an LP with \( \pi = (\pi(\xi, q, a))_{\xi \in C, q \in Q, a \in A} \)
LP Lotteries Approach

- **Ideas:** Consider action and compensation lotteries
  - Construct an action grid $\mathcal{A} = \{a_1, \ldots, a_M\}$ and a compensation grid $\mathcal{C} = \{\xi_1, \ldots, \xi_L\}$
  - Introduce probability measures $\pi(a)$ and $\pi(\xi|q,a)$
  - Transform ODC into an LP with $\pi = (\pi(\xi,q,a))_{\xi \in \mathcal{C}, q \in \mathcal{Q}, a \in \mathcal{A}}$

- **Disadvantages:**
  - The resulting LP is large:
    - $L \times M \times N$ variables and $M \times (N + M - 1) + 2$ constraints.
    - If $M = 50$, $N = 40$, $L = 500$, ⇒ an LP with one million variables and 4452 constraints.
  - Economic choice variables are continuous in nature
  - The curse of dimensionality for multidimensional problems
MPEC Approach for finite $A$

- Goal of [Su-Judd '05]: Allow continuous compensation $c \in \mathbb{R}_+^N$ (and keep finite $A$)
MPEC Approach for finite $\mathcal{A}$

- Goal of [Su-Judd ’05]: Allow continuous compensation $c \in \mathbb{R}_+^N$ (and keep finite $\mathcal{A}$)

- Idea: Introduce mixed-strategy profile $\delta = (\delta_1, \ldots, \delta_M)$ for agent's action choices $\mathcal{A} = \{a_1, \ldots, a_M\}$:

$$\delta_k \iff a_k \text{ for } k = 1, \ldots, M$$
MPEC Approach for finite $\mathcal{A}$

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$$\delta_k \iff a_k \quad \text{for } k = 1, \ldots, M$$

- Agent's mixed strategy problem is an LP

$$a^* \in \arg\max \left\{ U(c, a) : a \in \{a_1, \ldots, a_M\} \right\}$$

$$\Leftrightarrow$$

$$\delta^* \in \arg\max \left\{ \sum_{k=1}^M U(c, a_k)\delta_k : e^T\delta = 1, \delta \geq 0 \right\}$$
MPEC Approach for finite $\mathcal{A}$

- **Goal of [Su-Judd ’05]:** Allow continuous compensation $c \in \mathbb{R}^N_+$ (and keep finite $\mathcal{A}$)

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  $$
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  $$

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  $$
  a^* \in \arg\max \ \{ U(c, a) : a \in \{a_1, \ldots, a_M\} \}
  \quad \iff \\
  \delta^* \in \arg\max \ \left\{ \sum_{k=1}^{M} U(c, a_k)\delta_k : e^T\delta = 1, \delta \geq 0 \right\}
  $$

  Optimality + strong duality \quad \iff \quad \text{Let } U(c) := (U(c, a_k))_{k=1}^{M}

  $\delta^*$ solves

  $$
  \begin{cases}
  0 \leq \delta \perp (U(c)^T\delta) \ e - U(c) \geq 0 \\
  e^T\delta = 1
  \end{cases}
  $$
MPEC Approach for finite $\mathcal{A}$

maximize \( (c, a) \) \[ W(c, a) \]

subject to \( U(c, a) \geq U_0 \)

\( a \in \text{argmax} \{U(c, \tilde{a}) : \tilde{a} \in \mathcal{A} = \{a_1, \ldots, a_M\}\} \)

\( c \in \mathbb{R}^N_+ \)
MPEC Approach for finite $A$

maximize$_{(c, \delta)}$ $W(c)^T \delta$

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This is a Mathematical Program with Equilibrium Constraints (MPEC)!
MPEC Approach for finite $\mathcal{A}$

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\begin{align*}
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\]

This is a Mathematical Program with Equilibrium Constraints (MPEC)!

Problem Size: $(N + M)$ variables, $2M + 2$ constraints
MPEC Approach for finite $\mathcal{A}$

maximize $(c, \delta)$ $W(c)^T \delta$

subject to $U(c)^T \delta \geq U_0$

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This is a Mathematical Program with Equilibrium Constraints (MPEC)!

Problem Size: $(N + M)$ variables, $2M + 2$ constraints

**Lemma:**

(i) $(c^*, \delta^*) \in \text{SOL}(\text{MPEC}) \implies (c^*, a^*_i) \text{ an ODC with } a^*_i \in \mathcal{A} \text{ finite, and } i \in \{j : \delta^*_j > 0\}$.

(ii) $(c^*, a^*_i)$ an ODC with $a^*_i \in \mathcal{A} \text{ finite} \implies (c^*, e_i) \in \text{SOL}(\text{MPEC})$, where $e_i$ is the $i$-th column of an identity matrix
A Hybrid Method [Su-Judd ’05]

Observations:

- MPEC: Allows continuous compensation $c \in \mathbb{R}_+^N$ but may stop at local maximum
- LP: Produce global optimal but the grid $C$ may be too coarse
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Goal: Combine the best features of both to find a global solution with continuous compensation $c \in \mathbb{R}_+^N$
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Goal: Combine the best features of both to find a global solution with continuous compensation \( c \in \mathbb{R}_+^N \)

Step 0: Construct a coarse grid \( C \)

Step 1: Solve the LP for the given grid \( C \)

Step 2: Setup and solve the MPEC:

\[
\begin{cases}
(2.1) : \text{ construct } (c^0, \delta^0) \text{ using LP solution} \\
(2.2) : \text{ Solve the MPEC with the feasible starting pt } (c^0, \delta^0)
\end{cases}
\]

Step 3: Refine the grid and repeat Step 1 and Step 2
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\end{align*}
\]

Step 3: Refine the grid and repeat Step 1 and Step 2

Result: A hybrid solution is always better than an LP solution
An Example in [Karaivanov ’01]

- Risk-neutral principal: \( w(q_i - c_i) = q_i - c_i \)
- Risk-averse agent: \( u(c_i, a) = \frac{c_i^{1-\gamma}}{1-\gamma} + \kappa \frac{(1-a)^{1-\delta}}{1-\delta} \)
- Two outcomes: \( q_H = $3 \) and \( q_L = $1 \)
- Action set: \( |A| = 10 \) with equally-spaced effort level within \([0.01, 0.99]\)
- The production technology \( p(q = q_H | a) = a^\alpha \) with \( 0 < \alpha < 1 \)

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \kappa )</th>
<th>( \delta )</th>
<th>( \alpha )</th>
<th>( U_0 )</th>
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<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
<td>0.7</td>
<td>1</td>
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</table>

- Both LP and MPEC are coded in AMPL and solved by SNOPT on NEOS server (host: prado.iems.northwestern.edu)
### LP Solutions

LP Solutions with 8 different grids (# of constraints = 112)

| $|C|$ | # of Variables | Read Time (in sec.) | Solve Time (in sec.) | # of Iterations | Objective Value |
|-----|----------------|---------------------|----------------------|-----------------|-----------------|
| 21  | 420            | 0.01                | 0.04                 | 37              | 1.875882746     |
| 41  | 820            | 0.02                | 0.07                 | 46              | 1.877252910     |
| 81  | 1620           | 0.03                | 0.12                 | 46              | 1.877259193     |
| 161 | 3220           | 0.06                | 0.25                 | 46              | 1.877262265     |
| 321 | 6420           | 0.13                | 0.58                 | 69              | 1.877263785     |
| 641 | 12820          | 0.26                | 1.12                 | 52              | 1.877259905     |
| 1281| 25620          | 0.53                | 2.67                 | 101             | 1.877262221     |
| 2561| 51220          | 1.09                | 4.81                 | 73              | 1.877262201     |
| 5121| 102420         | 2.46                | 11.70                | 101             | 1.877263113     |

- Warm Start: Simplex method.
# Hybrid Solution

<table>
<thead>
<tr>
<th></th>
<th>Read Time</th>
<th>Solve Time</th>
<th># of Iterations</th>
<th>Objective Value</th>
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<tbody>
<tr>
<td><strong>LP</strong></td>
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<td>$</td>
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<td><strong>MPEC</strong></td>
<td></td>
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<tr>
<td>Starting Point $\delta_6 = 1, \delta_i(\neq 6) = 0$</td>
<td>0.02</td>
<td>0.01</td>
<td>13</td>
<td>1.877265298</td>
</tr>
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</table>
# Hybrid Solution

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<tr>
<th>LP</th>
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<th>Objective Value</th>
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<tr>
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<tr>
<th>MPEC</th>
<th>Read Time (in sec.)</th>
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<th># of Major Iterations</th>
<th>Objective Value</th>
</tr>
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## LP solution

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Contract with Action Lotteries

• A probability distribution $\pi(a)$ and a compensation schedule $c(a) \in R^N$ for every action $a \in A$
Contract with Action Lotteries

- A probability distribution \( \pi(a) \) and a compensation schedule \( c(a) \in R^N \) for every action \( a \in A \)
- Modifications:
  - Objective function: \( \sum_a \pi(a)W(c(a), a) \)
  - Participation constraint: \( \sum_a \pi(a)U(c(a), a) \geq U_0 \)
  - Incentive compatibility constraints:
    \[
    \begin{align*}
    &\pi(a) \geq 0 \\
    &U(c(a), a) \geq U(c(a), a_1), \\
    &\vdots \\
    &U(c(a), a) \geq U(c(a), a_M).
    \end{align*}
    \]
    If \( \pi(a) > 0 \), then
    \[
    \begin{align*}
    &\pi(a) \geq 0 \\
    &\pi(a)(U(c(a), a) - U(c(a), \tilde{a})) \geq 0, \quad \forall (a, \tilde{a}) \in A \times A
    \end{align*}
    \]
Contract with Action Lotteries

- A probability distribution $\pi(a)$ and a compensation schedule $c(a) \in \mathbb{R}^N$ for every action $a \in \mathcal{A}$

maximize $\sum_{a \in \mathcal{A}} \pi(a) W(c(a), a)$

subject to $\sum_{a \in \mathcal{A}} \pi(a) U(c(a), a) \geq U_0$,

$\sum_{a \in \mathcal{A}} \pi(a) = 1$,

$\pi(a) (U(c(a), a) - U(c(a), \tilde{a})) \geq 0, \quad \forall (a, \tilde{a}(\neq a)) \in \mathcal{A} \times \mathcal{A}$,

$\pi(a) \geq 0, \quad \forall a \in \mathcal{A}$.
Star-Shaped Feasible Region

maximize \( f(x, y) \)

s.t. \( x \geq 0 \)

\( xy \geq 0 \)

\((-2, 0)\) \quad \(0, -2)\) 

\((1, 0)\)
Star-Shaped Feasible Region

- Nonconvex feasible region due to switch-off constraints [Scholtes '04]
- LICQ fails on $y$-axis
MPEC for Action Lotteries

\[ \pi(a) \geq 0, \quad \text{if } \pi(a) > 0, \quad \text{then} \quad \begin{cases} U(c(a), a) \geq U(c(a), a_1), \\ \vdots \\ U(c(a), a) \geq U(c(a), a_M). \end{cases} \]

\[ \forall a \in A: \begin{cases} \pi(a) (U(c(a), a) - U(c(a), \tilde{a})) \geq 0, \quad \forall \tilde{a} \neq a \in A, \\ \pi(a) \geq 0. \end{cases} \]

\[ \forall a \in A: \begin{cases} U(c(a), a) - U(c(a), \tilde{a}) + s(a, \tilde{a}) \geq 0, \quad \forall \tilde{a} \neq a \in A, \\ 0 \leq \pi(a) \perp s(a, \tilde{a}) \geq 0. \end{cases} \]
MPEC for Action Lotteries

maximize \( \sum_{a \in A} \pi(a)W(c(a), a) \)

subject to \( \sum_{a \in A} \pi(a)U(c(a), a) \geq U_0, \)

\( \sum_{a \in A} \pi(a) = 1, \)

\( \forall a \in \mathcal{A} : \) \[ \begin{cases} 
U(c(a), a) - U(c(a), \tilde{a}) + s(a, \tilde{a}) \geq 0, & \forall \tilde{a}(\neq a) \in \mathcal{A}, \\
0 \leq \pi(a) \perp s(a, \tilde{a}) \geq 0. 
\end{cases} \]
The A-L Example in [Prescott ’04]

**Hybrid method:** LP with 11 compensation grid points and then switch to MPEC.

Hybrid Solutions with 10 different action grids, $|C| = 11$, $|Q| = 50$

| $|A|$ | $t_{LP}$ (in sec.) | LP Obj. Val. | $t_{MPEC}$ (in sec.) | MPEC Obj. Val. | $t_{Total}$ (in sec.) |
|-----|-------------------|--------------|----------------------|----------------|----------------------|
| 6   | 2                 | 1.75868508   | 1                    | 1.76234445     | 3                    |
| 11  | 4                 | 1.75868508   | 2                    | 1.76234448     | 6                    |
| 16  | 9                 | 1.76265085   | 3                    | 1.76565622     | 12                   |
| 21  | 12                | 1.76351860   | 11                   | 1.76630661     | 23                   |
| 26  | 21                | 1.75924445   | 6                    | 1.76298273     | 27                   |
| 31  | 38                | 1.76265085   | 11                   | 1.76565620     | 49                   |
| 36  | 64                | 1.75606325   | 16                   | 1.76051776     | 80                   |
| 51  | 102               | 1.75924445   | 22                   | 1.76298271     | 124                  |
| 76  | 266               | 1.75778364   | 127                  | 1.76298273     | 393                  |
| 101 | 575               | 1.75572408   | 1108                 | 1.76234445     | 1683                 |
| 151 | 4203              | 1.75534000   | 14018                | 1.75996707     | 18221                |
Extensions

- MPEC formulations are also given for:
  - Contracts with compensation lottery (randomized payment)
  - Contracts with action and compensation lotteries
  - Multidimensional action choices - little economic theory, make special assumptions
  - Multidimensional compensation choices - infeasible for LP
Extensions

• MPEC formulations are also given for:
  ◦ Contracts with compensation lottery (randomized payment)
  ◦ Contracts with action and compensation lotteries
  ◦ Multidimensional action choices - little economic theory, make special assumptions
  ◦ Multidimensional compensation choices - infeasible for LP

• Future research
  ◦ Tournament (single-principal multi-agent) problem
    [Lazear & Rosen ’81]
  ◦ Incentive problem with both hidden information and moral-hazard
  ◦ Dynamic contracts (multi-period moral-hazard problem)
    [Phelan & Townsend ’91]
  ◦ Executive compensation (with D. Larcker and C. Armstrong)
Executive Compensation

• Components of Compensation
  o Fixed Salary $s$
  o Stocks $\beta_0$
  o Options $\beta_1$ (with Strike Price $K$ to be Determined)
  o Payment $c_i = s + \beta_0 \cdot p_i + \beta_1 \cdot \max(q_i - K, 0)$

• Action Choices for the CEO (Agent)
  1. mean ($a$) and/or variance ($\sigma$) of the performance of business operations
An Example

- Risk-neutral principal: \( w(q_i - c_i) = q_i - c_i \)
- Risk-averse agent: \( u(c_i, a) = \frac{c_i^{1-\gamma}}{1-\gamma} - \mu a^2 \)
- Outcomes \( q_i \): equally-spaced stock price level within \([0, 160]\)
- Action set: \(|A| = 50\) with equally-spaced effort level within \([0, 50]\)
- The production technology

\[
p(q|a) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{(q - (C + M \times a))^2}{2\sigma^2} \right)
\]

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \mu )</th>
<th>( C )</th>
<th>( M )</th>
<th>( U_0 )</th>
</tr>
</thead>
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<td>0.0025</td>
<td>60</td>
<td>((0.25, 0.5, 0.75))</td>
<td>1</td>
</tr>
</tbody>
</table>
Flexible Modeling Framework

(1) agent effort has a positive, but decreasing impact on the mean of the distribution
(2) agent effort has positive and increasing impact on volatility
(3) multiple options with different exercise prices
(4) multiple agent effort: the agent can “diversify” some of his holdings into the risk-free asset
(5) more realistic participation constraint – min utility is a function of estimated agent effort
(6) multiple period variations of these settings
(7) “robust” contracts – very simple compensation plans are almost as good as the very complicated plans that are observed in the real world?