Nonlinear Pricing without Single Crossing

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Outline

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GENERAL SETUP

- A continuum of consumer with type $\theta \in \Theta$.
- Consumer with type θ values quantity q by $v(q,\theta)$.
- Monopolist, without being able to observe consumers' types, charges nonlinear tariff $t(q)$.
- Monopolist's cost function $C(q)$.

With Revelation Principle, monopolist solve the following problem

$$\begin{aligned}
\texttt{maximize}_{q,t} \quad & \int_{\underline{\theta}}^{\overline{\theta}} t(\theta) - C(q(\theta))dF(\theta) \\
\texttt{subject to} \quad & v(q(\theta),\theta) - t(\theta) \geqslant 0 \quad \forall \theta \in \Theta \text{ (IR)} \\
& v(q(\theta),\theta) - t(\theta) \geqslant v(q(\theta'),\theta) - t(\theta') \quad \forall \theta, \theta' \in \Theta \text{ (IC)}
\end{aligned}$$

THE ROLE OF SINGLE-CROSSING

Definition: v_q monotonic in types θ. What does it mean?

- Ordering of demands.
- Incentive to lie “downwards”.
- Local incentive constraints imply global incentive constraints (F.O.C. is valid).

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An Example With Single Crossing

- Values: $v(q,\theta) = \theta\sqrt{q}$, with $\theta \sim U[2,3]$ and $q \geqslant 0$.
- Cost: $C(q) = cq$, $c > 0$.
- Tariff: $t \geqslant 0$.

$$\begin{array}{ll} \texttt{maximize}_{q,t} & \displaystyle\int_2^3 t(\theta) - C(q(\theta))dF(\theta) \\ \texttt{subject to} & v(q(\theta),\theta) - t(\theta) \geqslant 0 \quad \forall\theta \text{ (IR)} \\ & v(q(\theta),\theta) - t(\theta) \geqslant v(q(\theta'),\theta) - t(\theta') \quad \forall\theta,\theta' \text{ (IC)} \end{array}$$

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\end{array}
$$

NUMERICAL APPROACH

We solve this constrained maximization problem numerically.

1. Discretize type space with N grid points, $\theta \in \{\theta_1, \ldots, \theta_N\}$.
2. Reformulate the original problem to the discretized problem

$$
\begin{aligned}
\texttt{maximize}_{q,t} \quad & \frac{1}{N}\sum_{i=1}^{N} t(\theta_i) - C(q(\theta_i)) \\
\texttt{subject to} \quad & v(q(\theta_i), \theta_i) - t(\theta_i) \geqslant 0 \quad \forall i \text{ (IR)} \\
& v(q(\theta_i), \theta_i) - t(\theta_i) \geqslant v(q(\theta_j), \theta_i) - t(\theta_j) \quad \forall i, j \text{ (IC)}
\end{aligned}
$$

3. Use KNITRO Active Set Algorithm to solve the discretized problem.
4. Increase N to improve the approximation.

SOLUTIONS

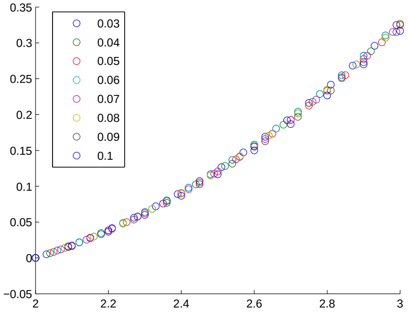

FIGURE: $v(q, \theta)$ under different discretization schemes

SOLUTIONS

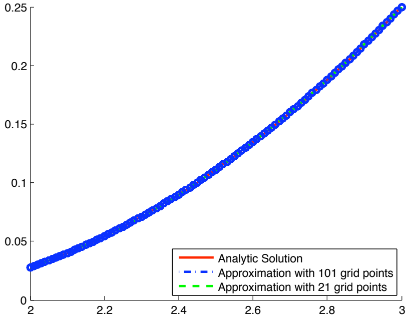

FIGURE: $q(\theta)$ under different discretization schemes

Solutions

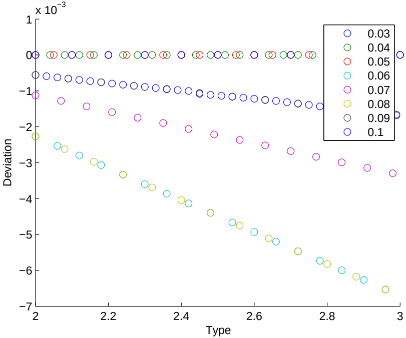

Figure: Approx. error for $q(\theta)$ under different discretization schemes

Outline

AN EXAMPLE WITHOUT SINGLE CROSSING

- Values: $v(q,\theta) = \theta q - \theta^2 q^2$, with $\theta \sim U[2,3]$ and $q \geqslant 0$.
- Cost: $C(q) = 3q^2$, $c > 0$.
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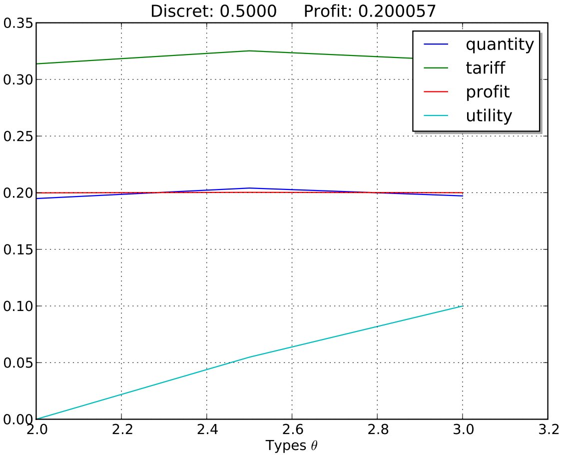
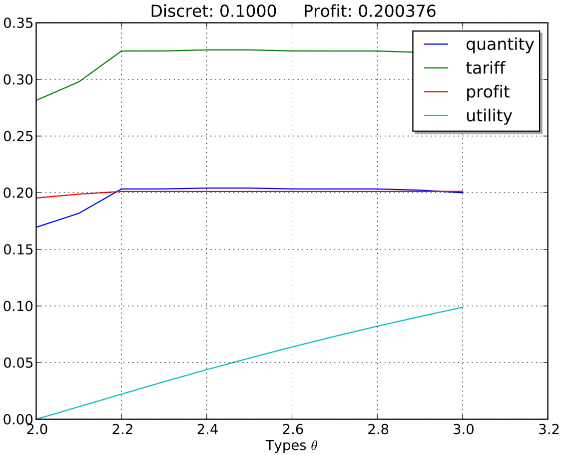
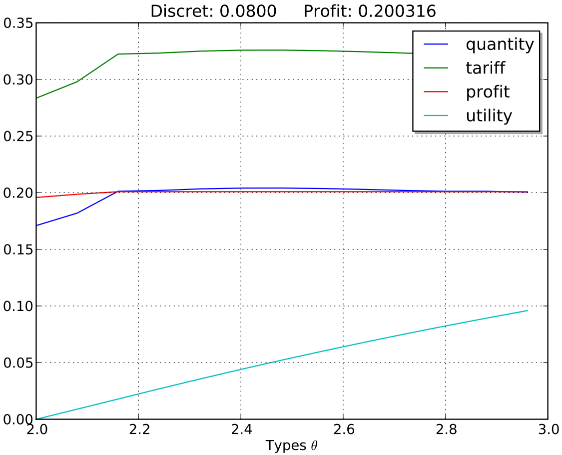

Discret: 0.5000 Profit: 0.200057
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
Types θ
quantity
tariff
profit
utility

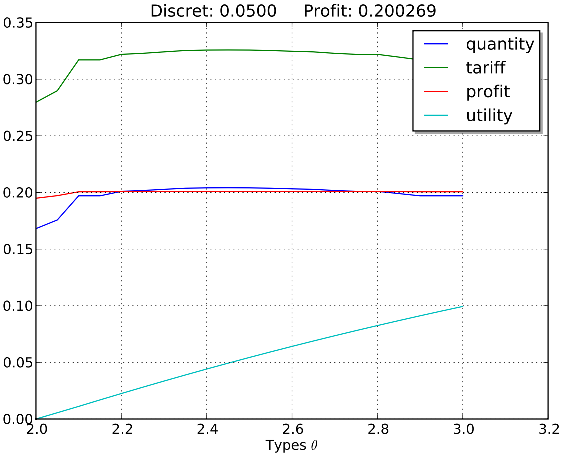
SOLUTIONS


Discret: 0.1000 Profit: 0.200376
quantity
tariff
profit
utility
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
Types θ

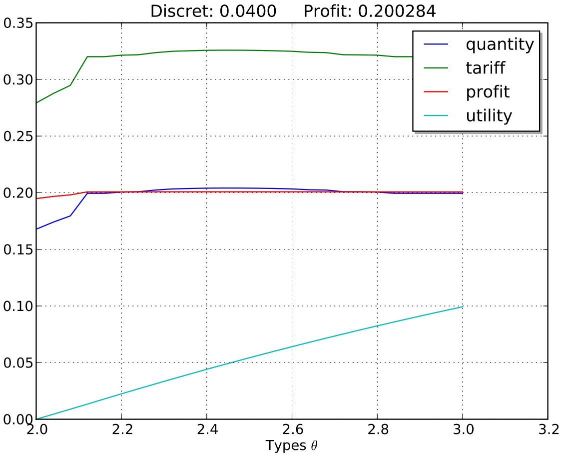
Solutions


Discret: 0.0800 Profit: 0.200316
quantity
tariff
profit
utility
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
Types θ

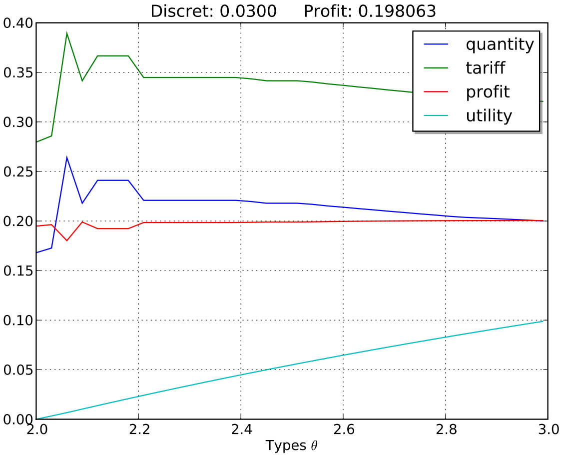
SOLUTIONS


Discret: 0.0500 Profit: 0.200269
quantity
tariff
profit
utility
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
Types θ

SOLUTIONS

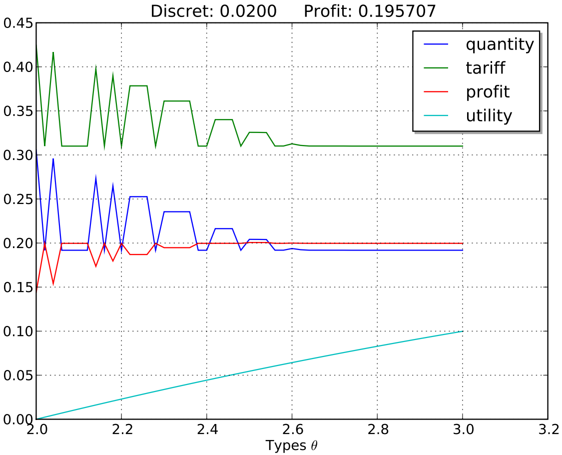

Discret: 0.0400 Profit: 0.200284
quantity
tariff
profit
utility
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
Types θ

SOLUTIONS

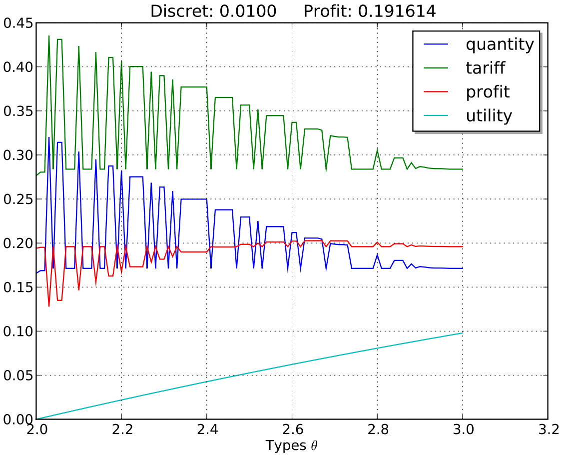


Discret: 0.0300 Profit: 0.198063
quantity
tariff
profit
utility
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
Types θ

SOLUTIONS


Discret: 0.0200 Profit: 0.195707
quantity
tariff
profit
utility
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
Types θ

SOLUTIONS


Discret: 0.0100 Profit: 0.191614
quantity
tariff
profit
utility
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
2.0
2.2
2.4
2.6
2.8
3.0
3.2
Types θ

SOLUTIONS

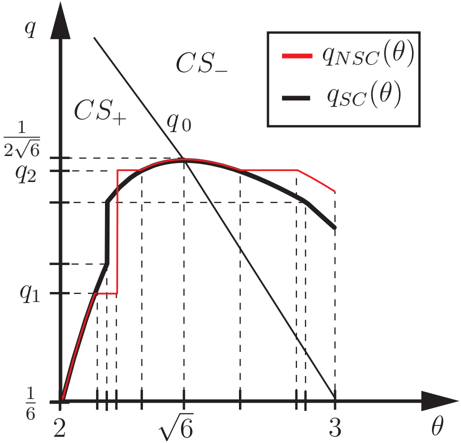


FIGURE: Vieira's Isoperimetric Approach

Outline

NON-UNIFORM DISTRIBUTION OF TYPES

Experiments: Variations of Uniform Discretization

- More mass (grid points) near the beginning.
- More mass near the end.
- More mass at both ends.
- More mass in the middle.

Implementation

- coarse grid distance 0.04, fine grid distance 0.02.
- 64 -76 variables, 1024 - 1444 constraints
- Multistart option - 100 runs
- KNITRO with Active Set algorithm
- Best solution chosen

General findings

- Similar total profit.
- Similar pooling equilibrium near the end.
- Non-monotonic $q - \theta$ relationship near the beginning.
- Monotonic $u - \theta$ relationship.

Non-Uniform Discretization of Types

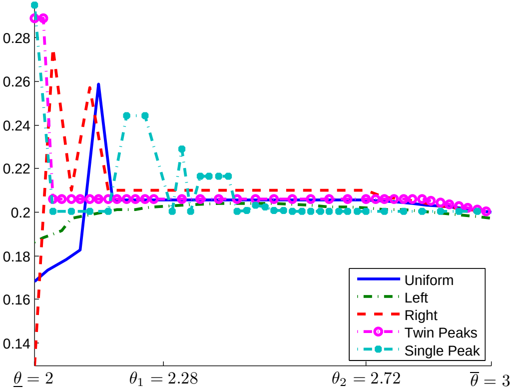


Figure: $q(\theta)$ for different distributions of types

Non-Uniform Discretization of Types

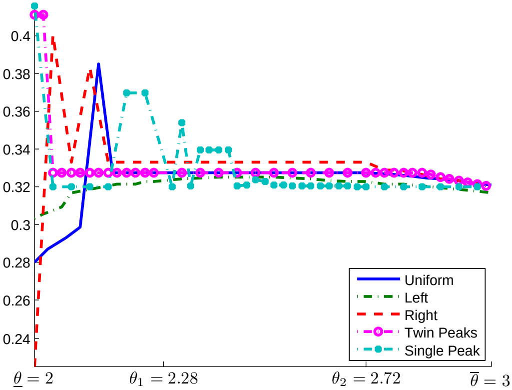


Figure: $T(q(\theta))$ for different distributions of types

NON-UNIFORM DISCRETIZATION OF TYPES

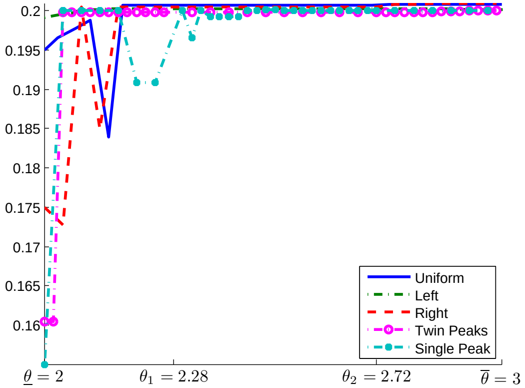


FIGURE: $T(q(\theta)) - C(q(\theta))$ for different distributions of types

NON-UNIFORM DISCRETIZATION OF TYPES

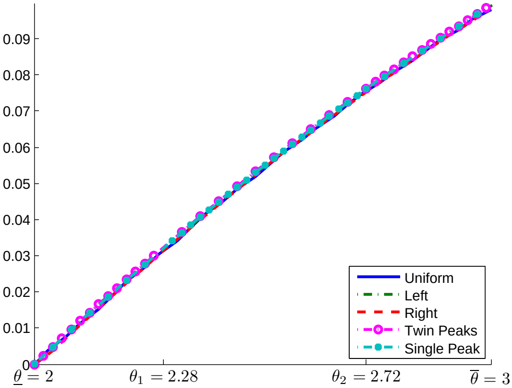


FIGURE: $v(q(\theta), \theta) - T(q(\theta))$ for different distributions of types

TWO DIMENSIONAL TYPES

- Similar setup
- Values: $v(q, \theta) = \theta q - \theta^2 q^2$, with $\theta \sim U[2,3]$ and $q \geqslant 0$.
- But now: $v(q, a, b) = aq - bq^2$ with $(a, b) \sim U[2,3]^2$
- Implications:
  - No single crossing again
  - Utility not monotone in types
  - Representation results do not hold - no analytical solution

UTILITY FUNCTIONS

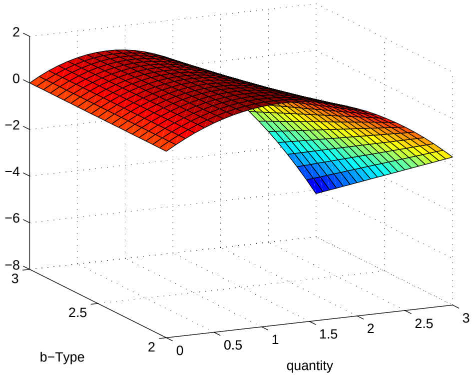


FIGURE: Utility

BENCHMARK: FULL INFORMATION

- Principal maximizes social surplus
- ...and takes it all.

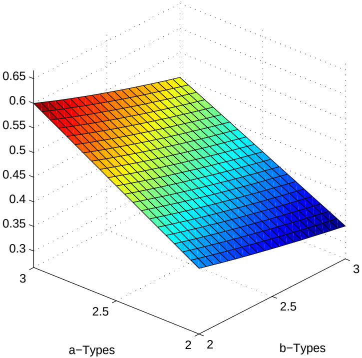


FIGURE: Optimal Supply Schedule

Computation - Issues

- Larger Dimension (d'oh!)
  - Number of possible policies explodes
  - Cannot rely on local incentive compatibility
  - Number of IC constraints explodes
- Small feasible region relative to action space
  - Hard to find a feasible starting point
  - Algorithm often gets stuck

Computation - AMPL to the Rescue!

- 49 grid points
- KNITRO with Active Set algorithm
- 98 variables, 2401 constraints
- Multistart option - 50 runs
- Best solution chosen

AMPL OUTPUT

```
Final objective value = 1.88400489913399e+01
Final feasibility error (abs / rel) = 5.50e-13 / 7.00e-14
Final optimality error (abs / rel) = 1.01e-08 / 5.78e-09
```

When things go well:

```
Iter Objective FeasError OptError ||Step|| CGits
-------- -------------- ---------- ---------- ---------- -------
446 1.883564e+01 3.600e-08 6.101e-01 3.056e-02 2
447 1.883601e+01 3.324e-10 6.422e-01 1.910e-03 6
448 1.883670e+01 1.220e-09 5.466e+00 5.759e-03 1
449 1.883723e+01 5.542e-09 1.527e+00 5.040e-03 4
450 1.883764e+01 9.788e-11 6.268e-01 5.039e-03 1
451 1.883828e+01 4.610e-09 3.196e-02 8.818e-03 3
452 1.883923e+01 1.554e-08 3.980e-02 1.764e-02 1
453 1.883979e+01 1.669e-08 4.290e-02 1.764e-02 1
454 1.883992e+01 2.584e-08 2.794e-02 1.180e-02 5
455 1.884004e+01 2.525e-08 2.867e-02 1.179e-02 2
456 1.884005e+01 8.475e-10 1.118e-02 4.480e-03 9
457 1.884005e+01 4.582e-12 3.906e-05 1.363e-05 7
458 1.884005e+01 2.220e-15 9.310e-08 2.196e-08 6
```

When things go bad..

```
Iter Objective FeasError OptError ||Step|| CGits
-------- -------------- ---------- ---------- ---------- -------
0 -4.821871e+01 1.046e+01
1 -6.597676e+00 5.046e+00 9.596e+03 6.716e+00 2
2 6.938318e+00 4.733e+00 1.434e+05 9.334e+00 1
3 -5.364320e+01 3.293e+00 7.311e+04 1.598e+01 2
4 -7.743043e+01 1.711e+00 1.033e+05 6.115e+00 1
5 -5.054798e+01 1.481e-01 1.386e+02 2.865e+00 0
6 -4.105956e+01 7.316e-02 4.901e+01 8.087e-01 0
7 -3.547193e+01 1.743e-02 1.398e+01 4.493e-01 1
8 -3.303212e+01 4.191e-03 1.466e+01 2.151e-01 2
9 -3.110397e+01 5.802e-03 2.018e+01 2.646e-01 1
10 -2.808023e+01 2.913e-04 9.807e+00 2.665e-01 2
11 -2.663157e+01 5.511e-05 1.125e+01 1.436e-01 2
12 -2.593095e+01 1.710e-05 4.918e+01 7.349e-02 2
13 -2.591876e+01 1.687e-05 4.895e+00 1.148e-03 3
14 -2.591589e+01 9.421e-06 8.968e+00 2.871e-04 8
15 -2.590045e+01 3.179e-08 9.182e+00 1.426e-03 1
16 -2.590009e+01 3.119e-08 1.756e+00 4.975e-05 1
17 -2.589618e+01 7.424e-09 4.210e+00 3.809e-04 1
18 -2.589618e+01 2.366e-09 8.687e+00 1.387e-04 3
19 -2.589313e+01 2.970e-09 3.071e+00 2.774e-04 1
20 -2.588970e+01 4.187e-09 3.685e+00 3.334e-04 1
21 -2.588957e+01 4.226e-09 1.428e+00 1.248e-05 1
22 -2.588956e+01 4.412e-09 2.508e-02 5.990e-07 1
```

More Graphs!

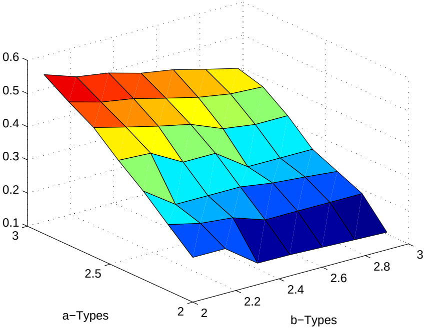


Figure: Optimal Supply Schedule with Types

Distortions

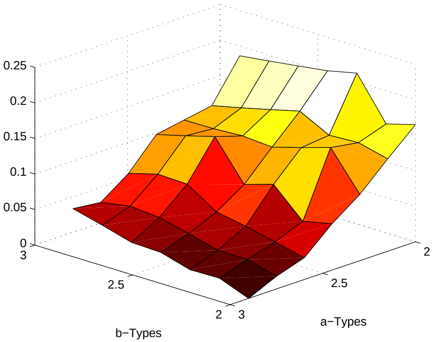


Figure: Supply w. Types vs Supply with full info

Sense and Sensitivity

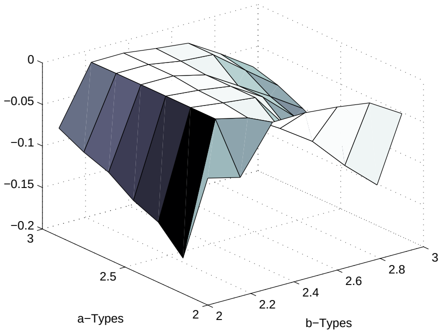


Figure: Type (2.6,2.6) - sample IC constraint

SENSE AND SENSITIVITY II

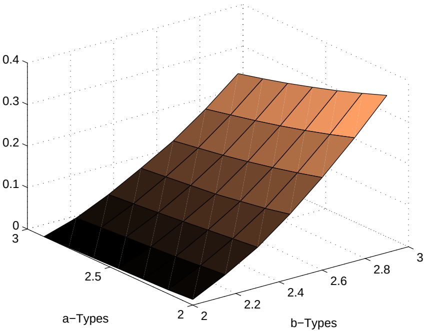


FIGURE: Slack in IR constraint by type (mind the supply distortion!)