Dynamic Programming for Portfolio Problems

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Proportional Transaction Cost and CRRA Utility

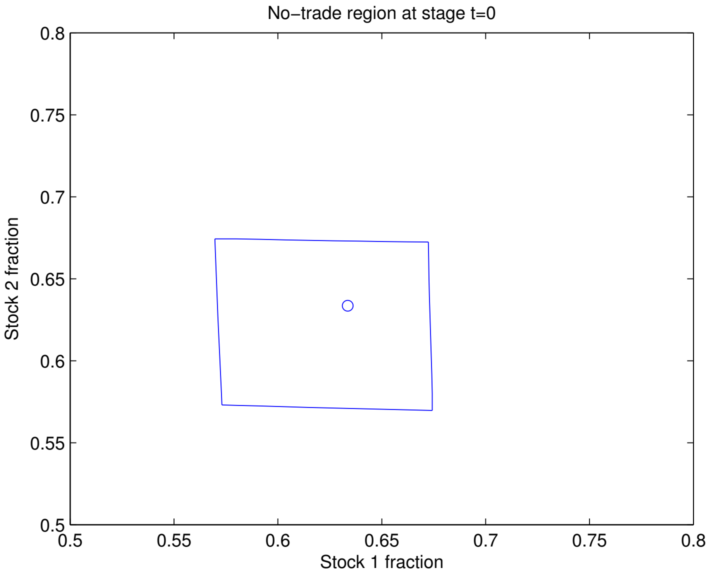
- Separability of wealth W and portfolio fractions x.
- If $u(W) = W^{1-\gamma}/(1-\gamma)$, then $V_t(W_t, x_t) = W_t^{1-\gamma} \cdot g_t(x_t)$.
- If $u(W) = \log(W)$, then $V_t(W_t, x_t) = \log(W_t) + \psi_t(x_t)$.
- "No-trade" region: $\Omega_t = \{x_t : (\delta_t^+)^* = (\delta_t^-)^* = 0\}$, where $(\delta_t^+)^* \geq 0$ are fractions of wealth for buying stocks, and $(\delta_t^-)^* \geq 0$ are fractions of wealth for selling stocks.

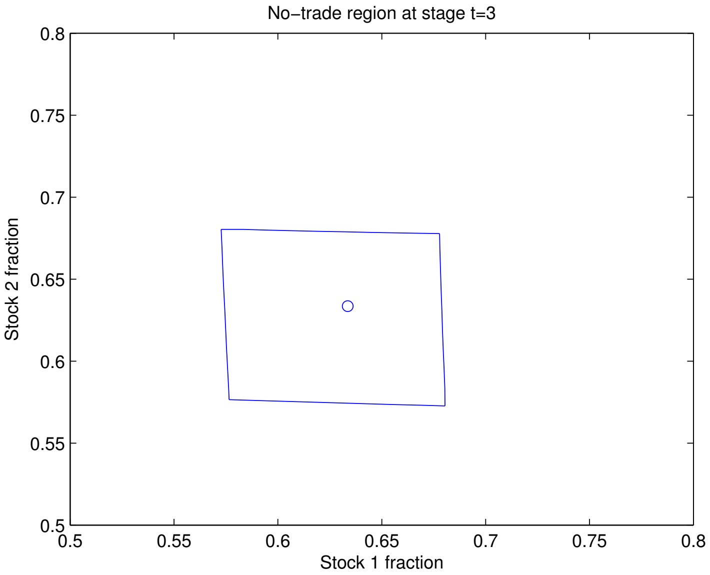
Examples with two stocks

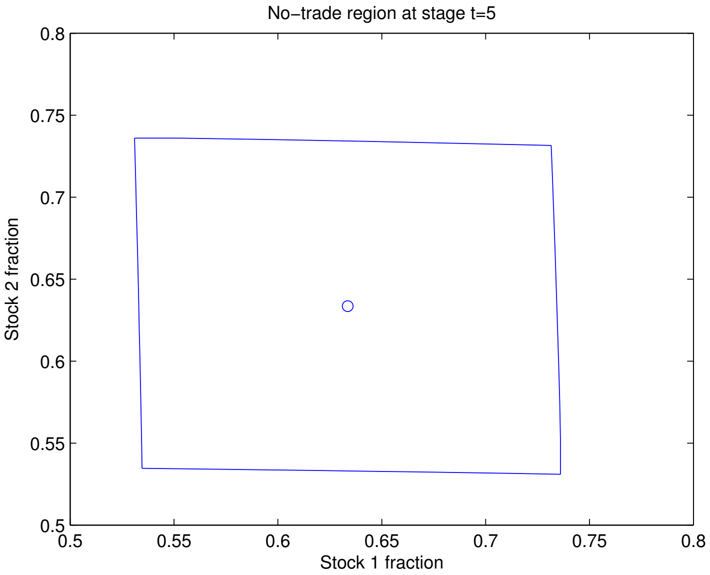
We first look at simple examples:

- Two stocks and one bond
- Investment begins at $t = 0$ and is liquidated at $t = 6$

i.i.d. returns (uniform distribution on [0.87,1.27])

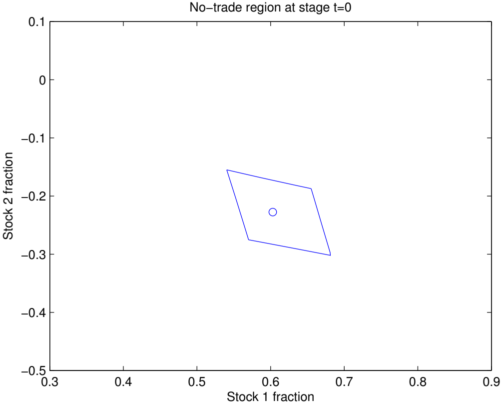


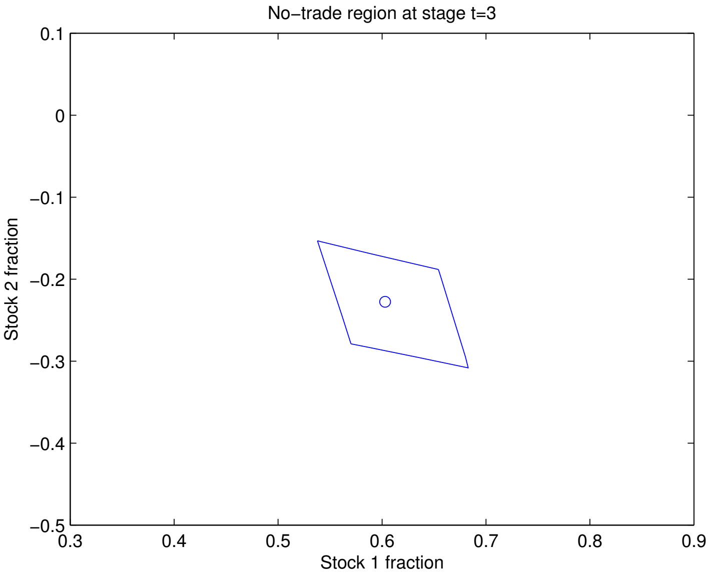

No−trade region at stage t=3
Stock 2 fraction
Stock 1 fraction
0.5
0.55
0.6
0.65
0.7
0.75
0.8

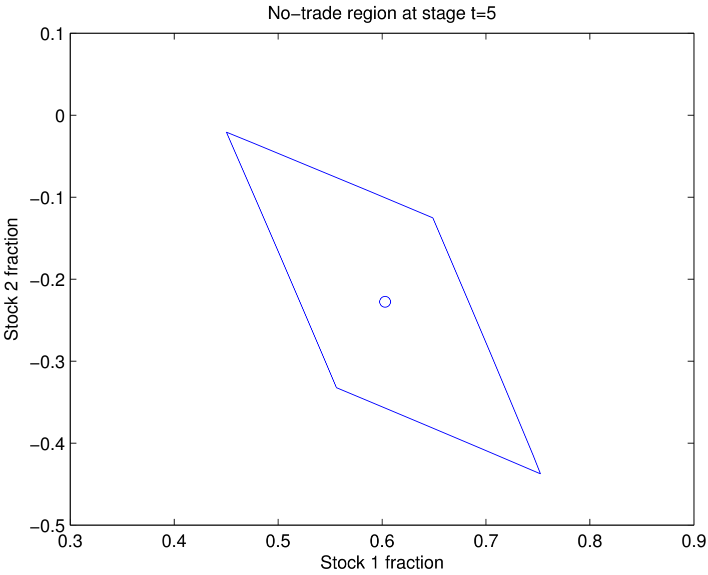

No−trade region at stage t=5
Stock 2 fraction
Stock 1 fraction
0.5
0.55
0.6
0.65
0.7
0.75
0.8

Correlated returns (discrete)

- **Return:** $R_1 = 0.85, 1.08, 1.25$; $R_2 = 0.88, 1.06, 1.2$;
- **Joint Probability Matrix:** $\begin{bmatrix} 0.08 & 0.1 & \\ 0.12 & 0.4 & 0.12 \\ & 0.1 & 0.08 \end{bmatrix}$

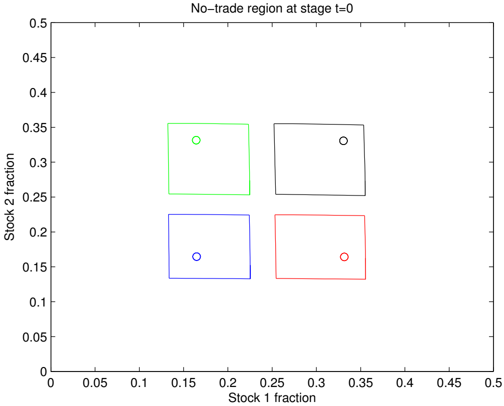


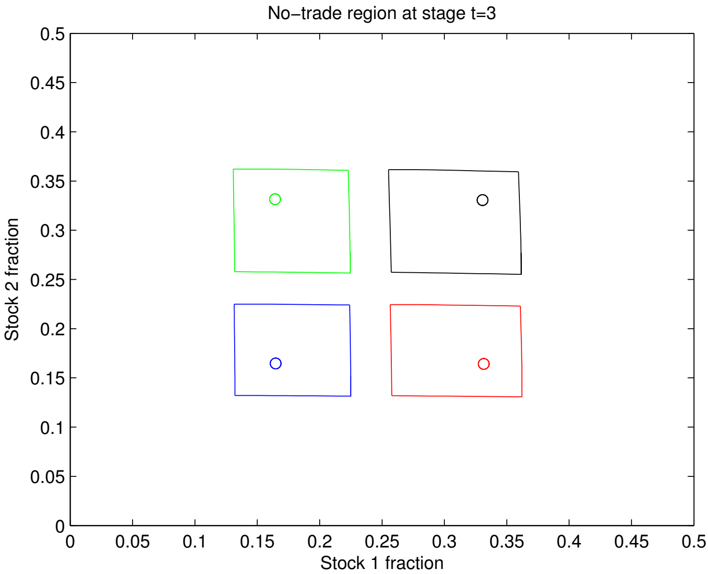

No−trade region at stage t=3
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Stock 2 fraction
Stock 1 fraction

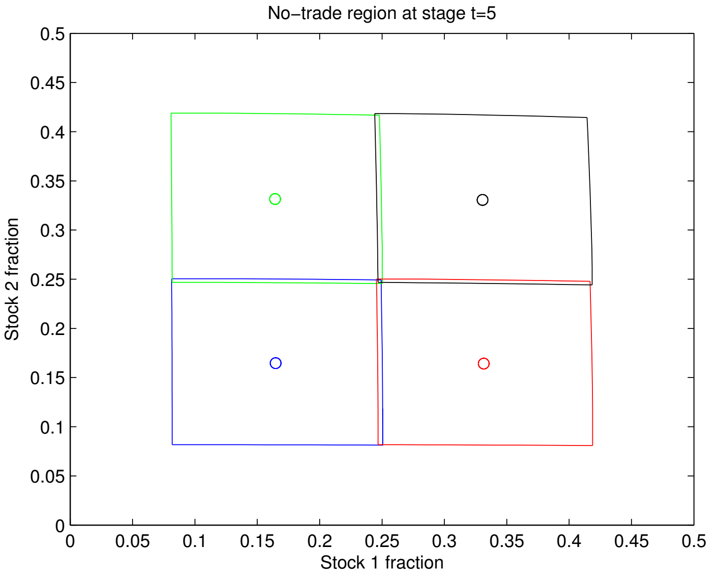

No−trade region at stage t=5
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
Stock 2 fraction
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Stock 1 fraction

Stochastic mean return

- Log-normal return: $\mathcal{N}\left(\mu_i - \sigma_i^2/2, \sigma_i^2\right)$ with $\mu_i = 0.06$ or 0.08 and $\sigma_i = 0.2$, for $i = 1, 2$
- Transition probability matrix of μ: $\begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$

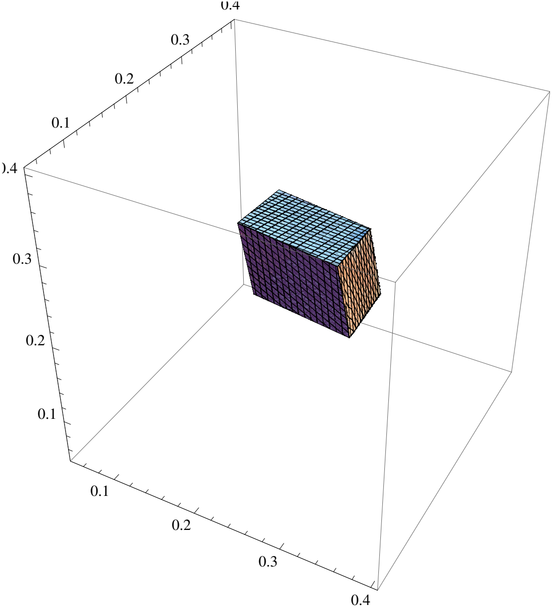


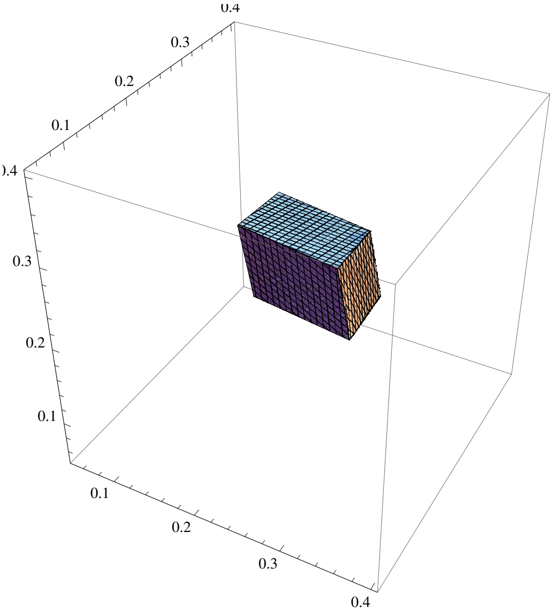

No−trade region at stage t=3
Stock 2 fraction
Stock 1 fraction
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

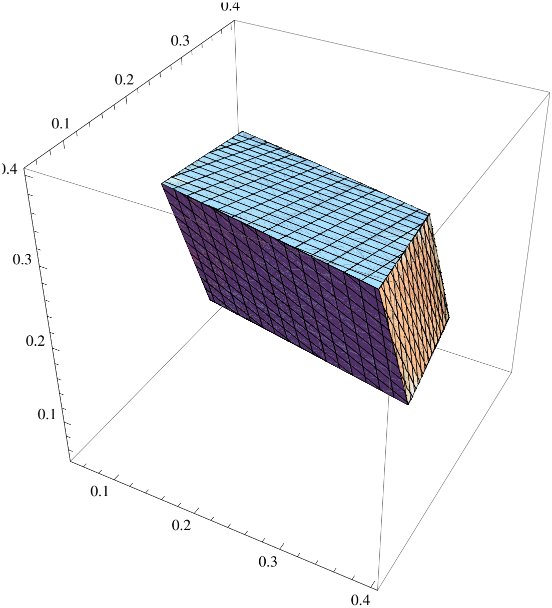

No−trade region at stage t=5
Stock 2 fraction
Stock 1 fraction
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Three stocks

- Problem: 3 stocks + 1 bond.
- Investment begins at $t = 0$ and is liquidated at $t = 6$
- Log-normal return: $\mathcal{N}\left(\mu - \sigma^2/2,\, \Lambda\Sigma\Lambda\right)$ with $\sigma = (0.16, 0.18, 0.2)^\top$, $\mu = (0.07, 0.08, 0.09)^\top$, Λ is the diagonal matrix of σ, and
$$\Sigma = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.314 \\ 0.1 & 0.314 & 1 \end{bmatrix}.$$
- degree-7 complete Chebyshev approximation
- product Gauss-Hermite quadrature rule with 9 nodes in each dimension


0.4
0.3
0.2
0.1
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4


0.4
0.3
0.2
0.1
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4


0.3
0.2
0.1
0.3
0.2
0.1
0.1
0.2
0.3
0.4

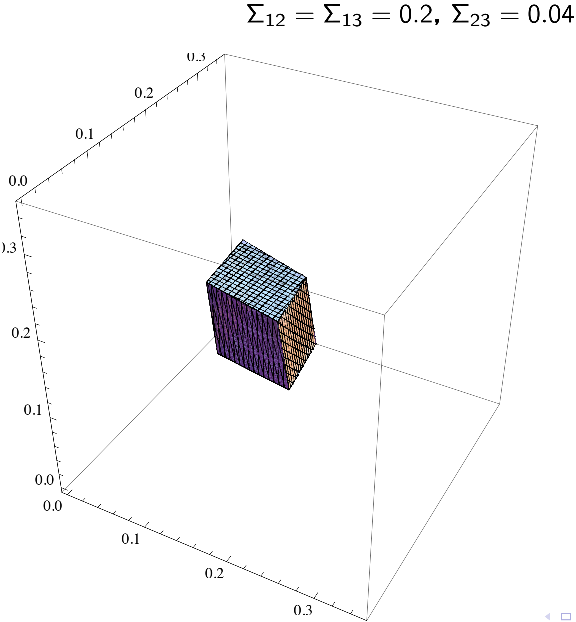
Portfolio with Transaction Costs and Consumption

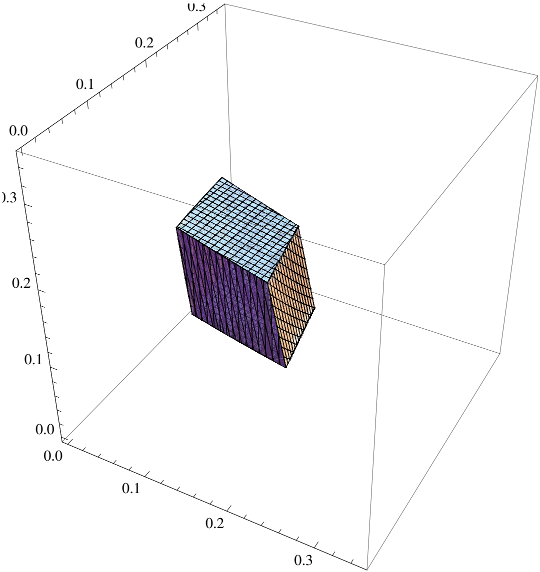
- Problem: 3 stocks + 1 bond, 50 periods
- No-trade regions:

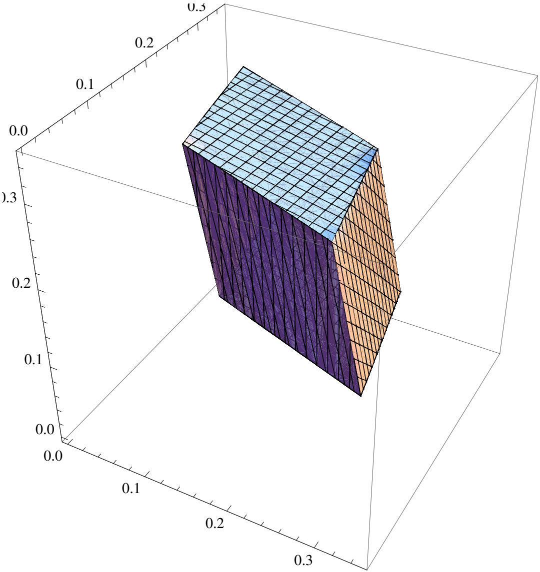
$$\Omega_{49} \approx [0.16, 0.33]^3, \Omega_{48} \approx [0.14, 0.24]^3, \ldots,$$
$$\Omega_t \approx [0.19, 0.27]^3, \text{ for } t = 0, \ldots, 15,$$

- Merton's ratio: $x^* = (\Lambda\Sigma\Lambda)^{-1}(\mu - r)/\gamma = (0.25, 0.25, 0.25)^\top$

$\Sigma_{12} = \Sigma_{13} = 0.2,\ \Sigma_{23} = 0.04$

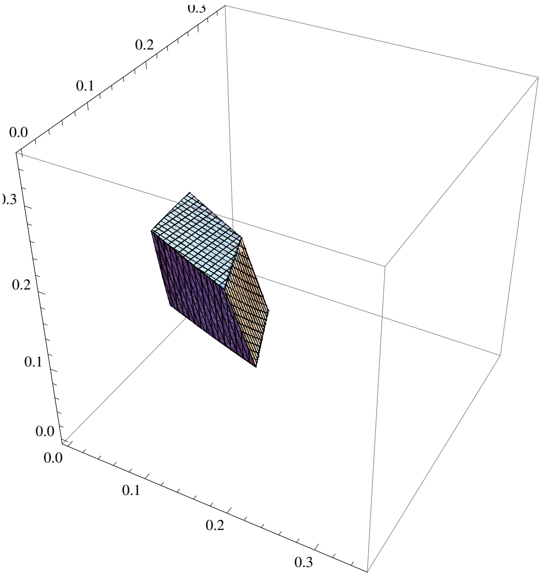


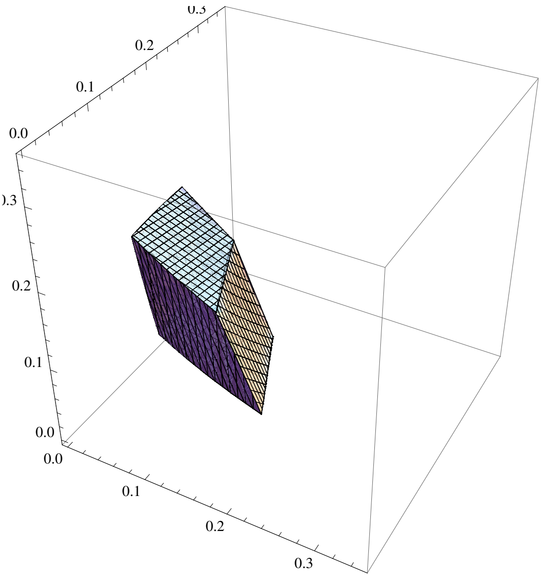

0.0
0.1
0.2
0.3
0.0
0.1
0.2
).3
0.0
0.1
0.2
0.3



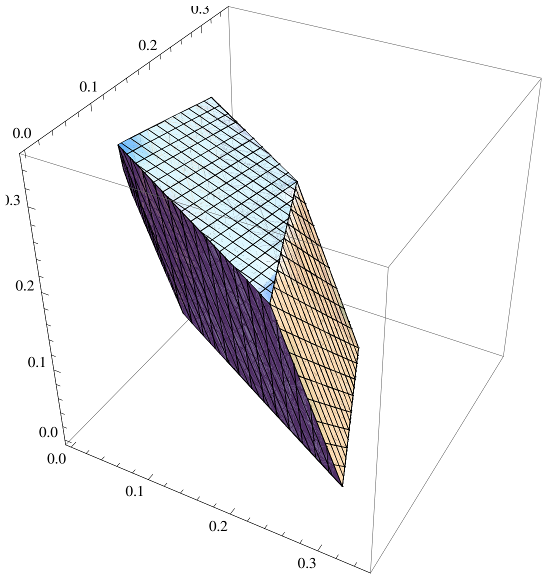
0.3
0.2
0.1
0.0
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3

More correlation: $\Sigma_{12} = \Sigma_{13} = 0.4$, $\Sigma_{23} = 0.16$





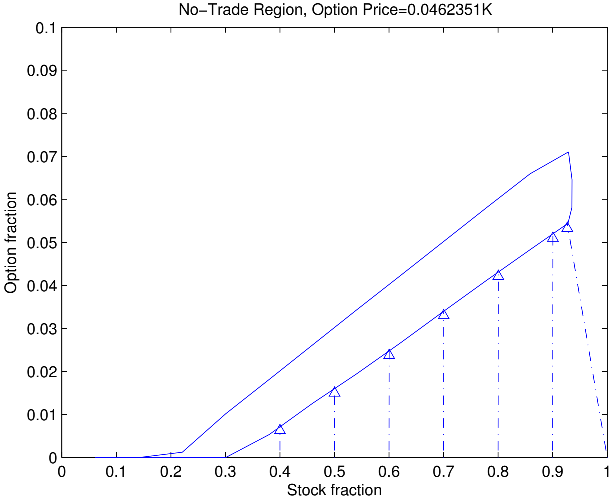
0.3
0.2
0.1
0.0
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3


0.0
0.1
0.2
0.3
0.0
0.1
0.2
0.0
0.1
0.2
0.3

Portfolios with options

- Problem: 1 stock, 1 put option on the stock, and 1 bond
- Investment begins at $t = 0$ and is liquidated at $t = 6$ months
- Proportional transaction costs in stock and option trades
- Return of option is based on pricing of the option.
  - We use the binomial lattice method to price the option
  - Stock price will be one exogenous state variable
- Separability of wealth W and portfolio fractions x and stock price S
  - If $u(W) = W^{1-\gamma}/(1-\gamma)$, then $V_t(W_t, S_t, x_t) = W_t^{1-\gamma} \cdot g_t(S_t, x_t)$.
  - If $u(W) = \log(W)$, then $V_t(W_t, S_t, x_t) = \log(W_t) + \psi_t(S_t, x_t)$.

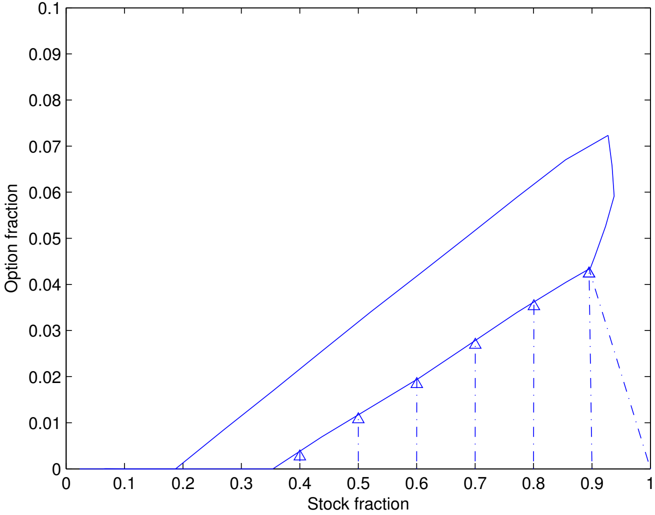
1 stock and 1 at-the-money put option at $t = 0$ (liquidate at $t = 6$ months)



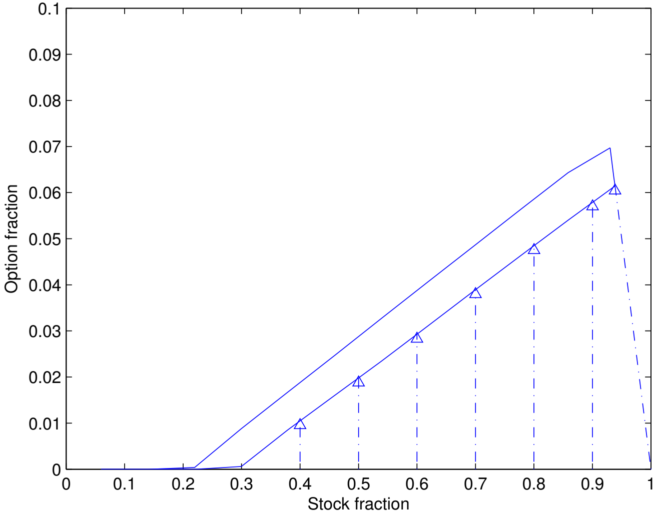
- **Put option: strike** K**, expiration time** T**, payoff** $\max(K - S_T, 0)$
- **stock price** S**, utility** $u(W) = -W^{-2}/2$

Dependence on transaction costs

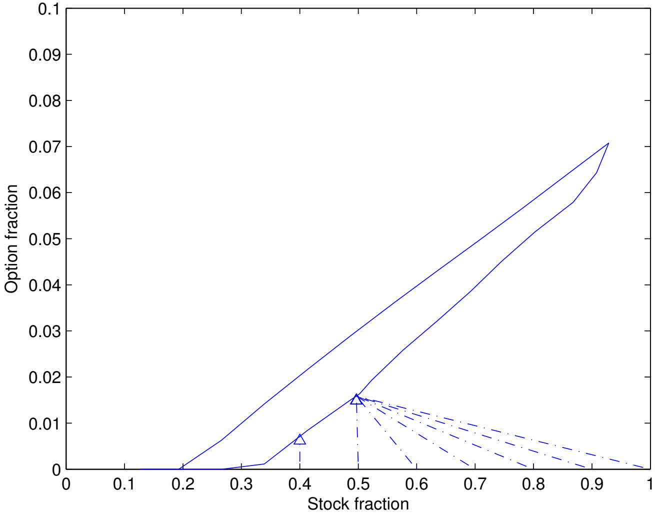
1 at-the-money put option at $t = 0$
transaction cost ratios: $\tau_1 = 0.01$, $\tau_2 = 0.02$



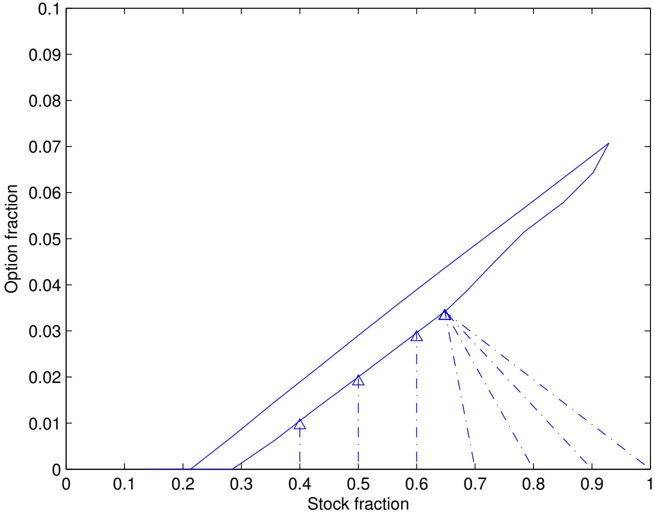
1 at-the-money put option at $t = 0$
transaction cost ratios: $\tau_1 = 0.01$, $\tau_2 = 0.005$



1 at-the-money put option at $t = 0$ (liquidate at $t = 6$) transaction cost ratios: $\tau_1 = 0.005$, $\tau_2 = 0.01$

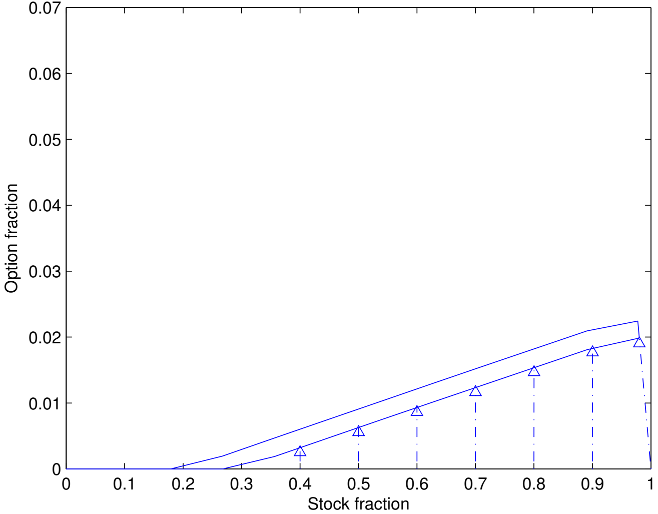


1 at-the-money put option at $t = 0$
transaction cost ratios: $\tau_1 = 0.005$, $\tau_2 = 0.005$

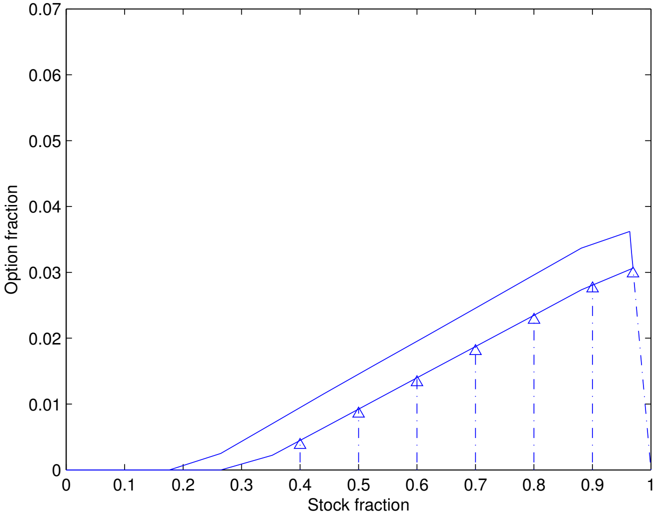


Dependence on horizon

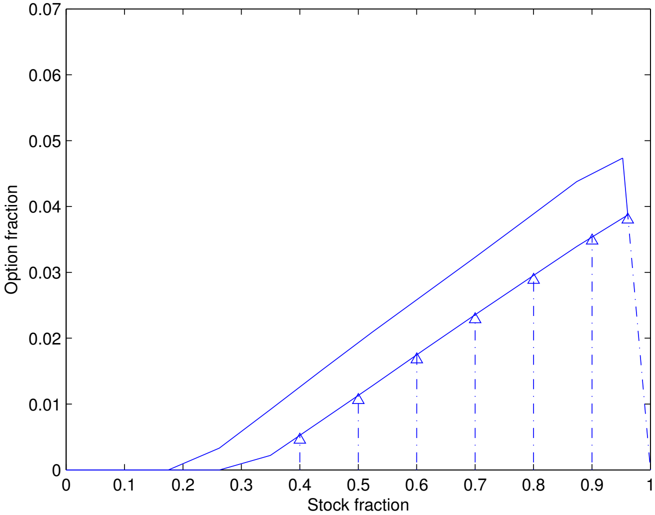
1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$)
expiration time: $T = 1$ month



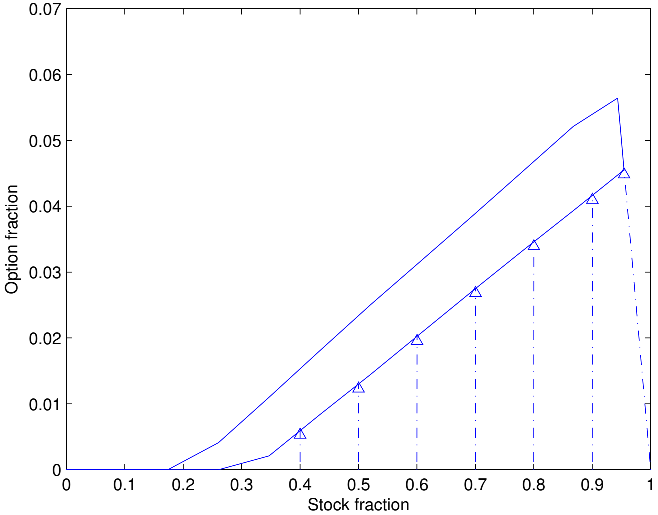
1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$)
expiration time: $T = 2$ months



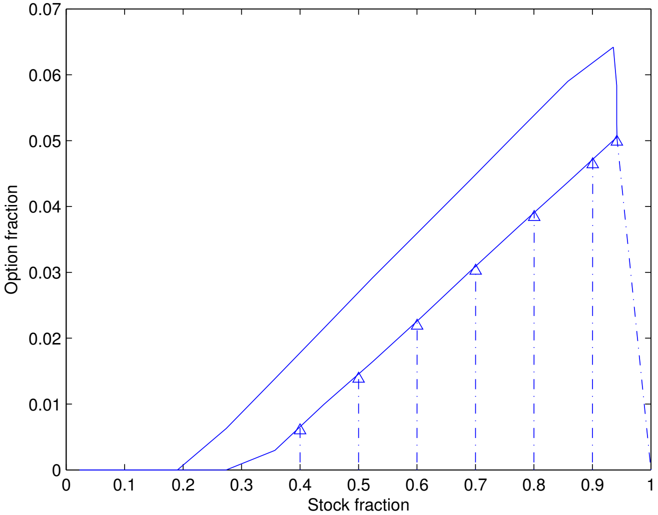
1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$) expiration time: $T = 3$ months



1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$)
expiration time: $T = 4$ months

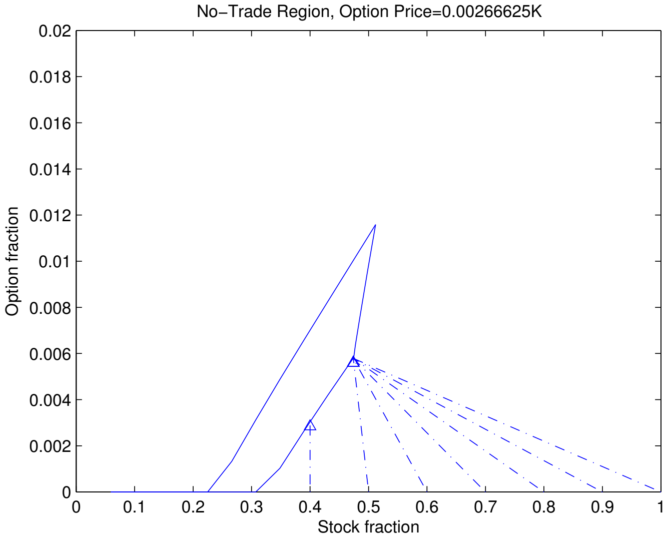


1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$) expiration time: $T = 5$ months



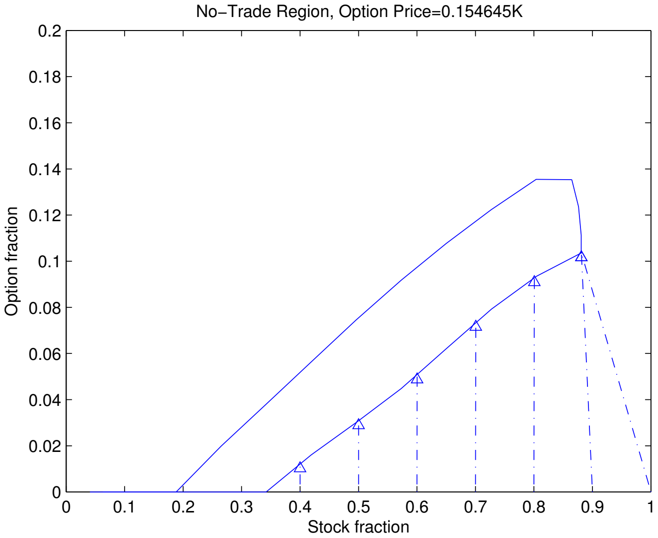
Dependence on strike price ($K = 0.8S_0$)

1 put option at $t = 0$ (liquidate at $t = 6$, $\tau_1 = \tau_2 = 0.01$)



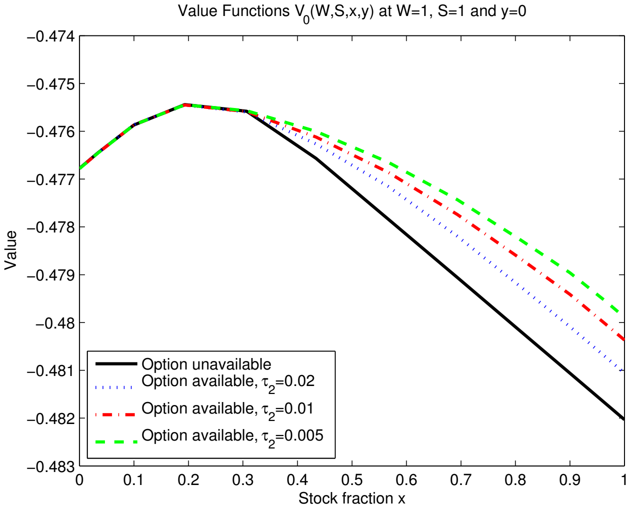
$K = 1.2S_0$

1 put option at $t = 0$ (liquidate at $t = 6$, $\tau_1 = \tau_2 = 0.01$)



Social value of options

Value functions with/without options at $t = 0$ (liquidate at $t = 6$)



- (x, y): **fractions of money in stock and option**
- $\tau_1 = 0.01$ **and** τ_2: **transaction cost ratios of stock and option**

Summary

- Developed a NDP method with shape-preserving approximation, stabler
- Developed a NDP method with Hermite interpolation, more accurate and more time-saving
- Developed a parallel NDP algorithm, running over hundreds of computers, almost linear speed-up
- Solved arbitrary-number-of-period and many-asset dynamic portfolio problems with transaction costs
- Solved arbitrary-number-of-period dynamic portfolio problems with options and transaction costs