Dynamic Programming with Piecewise Linear Interpolation

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May 24, 2012

Piecewise Linear Interpolation
If Lagrange data $\{(x_i, v_i) : i = 1, \ldots, m\}$ is given, then its piecewise linear interpolation is

$$\hat{V}(x) = b_{j,0} + b_{j,1}x, \quad \text{if} \;\; x \in [x_j, x_{j+1}],$$

where

$$\begin{aligned} b_{j,1} &= \frac{v_{j+1} - v_j}{x_{j+1} - x_j}, \\ b_{j,0} &= v_i - b_{j,1}x_i, \end{aligned}$$

for $j = 1, \ldots, m-1$.

In the maximization step of numerical DP algorithms, one could directly solve the maximization problem

$$v_i = \max_a \; u(x_i, a) + \beta \; \hat{V}\left(y; \mathbf{b}^{+}\right)$$

where

$$y = g(x_i, a)$$

Problem: $\hat{V}(x; \mathbf{b}^{t+1})$ is not differentiable, making it difficult to solve the optimization problem for a.

Min-Function Approach

- The differentiability problem is solved as follows:

$$
\begin{aligned}
v_i \quad = \quad \max_{a,w,y} \; & u(x_i, a) + \beta \; w \\
\text{s.t.} \qquad & y = g(x_i, a) \\
& w \leq b_{j,0}^{+} + b_{j,1}^{+} y, \qquad 1 \leq j < m
\end{aligned}
$$

- optimization solvers can still solve the new model quickly
  - The objective function is smooth
  - inequality constraints are linear and sparse
  - we can apply fast Newton-type optimization algorithms to solve this problem if g is also smooth.
  - although this new model adds $(m-1)$ linear inequality constraints, few of them will be active at any iteration
- this way does not need to find the interval where y locates, while the spline approximation of value function must

Convex-Set Approach

- Both previous methods need to calculate coefficients; this is very complicated for multi-dimensional piecewise linear interpolation.
- Convex set approach never computes coefficients of approximation:

$$
\begin{aligned}
v_i = \max_{\mu_j \geq 0, a, w, y} \quad & u(x_i, a) + \beta\ w, \\
\text{s.t.} \quad & y = g(x_i, a), \\
& y = \sum_{j=1}^{m} \mu_j x_j^+ \\
& w \leq \sum_{j=1}^{m} \mu_j v_j^+ \\
& \sum_{j=1}^{m} \mu_j = 1
\end{aligned}
$$