Dynamic Programming with Shape Preservation and Hermite Information

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Shape-preserving Chebyshev Interpolation

LP model for shape-preserving Chebyshev Interpolation:

$$
\begin{aligned}
\min_{c_j} \quad & \sum_{j=0}^{m-1} (c_j^+ + c_j^-) + \sum_{j=m}^{n} (j+1-m)^2 (c_j^+ + c_j^-) \\
\text{s.t.} \quad & \sum_{j=0}^{n} c_j T_j'(y_i) > 0 > \sum_{j=0}^{n} c_j T_j''(y_i), \qquad i = 1, \ldots, m', \\
& \sum_{j=0}^{n} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m, \\
& c_j - \hat{c}_j = c_j^+ - c_j^-, \quad j = 0, \ldots, m-1, \\
& c_j = c_j^+ - c_j^-, \quad j = m, \ldots, n, \\
& c_j^+ \geq 0, \quad c_j^- \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

Optimal Growth Models

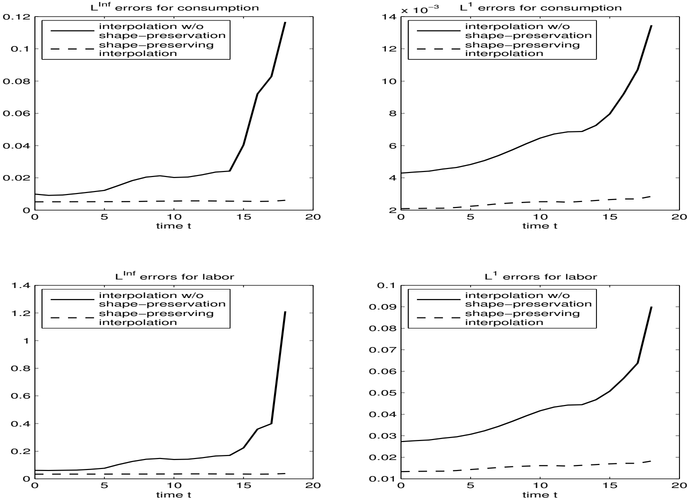
- Optimal Growth Problem:

$$V_0(k_0) = \max_{c,l} \sum_{t=0}^{T-1} \beta^t u(c_t, l_t) + \beta^T V_T(k_T),$$
$$\text{s.t.} \quad k_{t+1} = F(k_t, l_t) - c_t, \quad 0 \le t < T$$

- DP model of optimal growth problem:

$$V_t(k) = \max_{c,l} \; u(c, l) + \beta V_{t+1}(F(k, l) - c)$$

Errors of NDP with Chebyshev interpolation (shape-preserving or not)



Multi-Stage Portfolio Optimization

- W_t: wealth at stage t; stocks' random return: $R = (R_1, \ldots, R_n)$; bond's riskfree return: R_f;
- $S_t = (S_{t1}, \ldots, S_{tn})^\top$: money in the stocks; $B_t = W_t - e^\top S_t$: money in the bond,
- $W_{t+1} = R_f(W_t - e^\top S_t) + R^\top S_t$
- Multi-Stage Portfolio Optimization Problem:

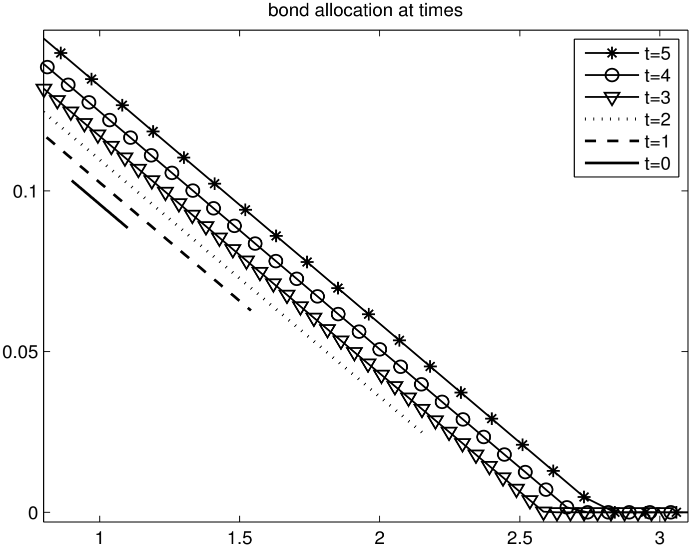
$$V_0(W_0) = \max_{X_{\mathbf{t}}, 0 \leq t < T} E\{u(W_T)\}$$

- Bellman Equation:

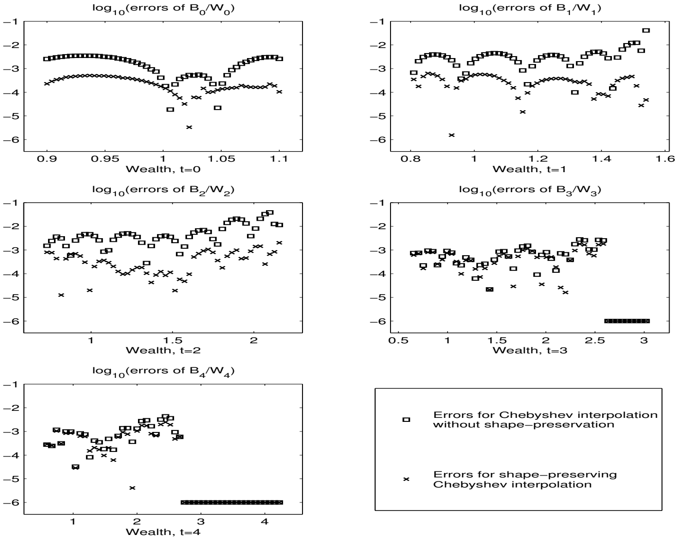
$$V_t(W) = \max_{S} \; E\{V_{t+1}(R_f(W - e^\top S) + R^\top S)\}$$

W: state variable; S: control variables.

Exact optimal bond allocation



Errors of Optimal Stock Allocations (shape-preserving or not)



Envelope Theorem

- Envelope theorem: Let

$$\begin{aligned} V(x) &= \max_y \; f(x,y) \\ & \quad \text{s.t. } g(x,y) = 0. \end{aligned}$$

Let $y^*(x)$ be the optimizer and $\lambda^*(x)$ be the shadow price.

$$\frac{\partial V}{\partial x} = \frac{\partial f}{\partial x}(x, y^*(x)) + \lambda^*(x)^\top \frac{\partial g}{\partial x}(x, y^*(x)).$$

Derivative of Value Functions in General Models

- For an optimization problem,

$$V(x) = \max_{y} f(x,y) \quad \text{s.t. } g(x,y)=0, h(x,y)\geq 0,$$

add a trivial control variable z and a trivial constraint $x - z = 0$:

$$V(x) = \max_{y,z} f(z,y) \quad \text{s.t. } g(z,y)=0, h(z,y)\geq 0, x-z=0.$$

- Then by the envelope theorem, we get

$$V'(x) = \lambda,$$

where λ is the shadow price for the trivial constraint $x - z = 0$.

Numerical DP Algorithm with Hermite Interpolation

Initialization. Choose the approximation nodes, $X_t = \{x_{it} : 1 \le i \le m_t\}$ for every $t < T$, and choose a functional form for $\hat{V}(x; \mathbf{b})$. Let $\hat{V}(x; \mathbf{b}^T) \equiv V_T(x)$.

Step 1. Maximization step. For each $x_i \in X_t$, $1 \le i \le m_t$, compute

$$\begin{aligned} v_i \quad = \quad & \max_{a_i \in \mathcal{D}(y_i, t), y_i} \quad u_t(y_i, a_i) + \beta E\{\hat{V}(x_i^+; \mathbf{b}^{t+1}) \mid y_i, a_i\}, \\ & \quad \text{s.t.} \qquad x_i - y_i = 0, \end{aligned}$$

and $\quad s_i = \lambda_i$, where λ_i is the shadow price of the constraint $x_i - y_i = 0$.

Step 2. Hermite fitting step. Compute the $\mathbf{b}^t$ such that $\hat{V}(x; \mathbf{b}^t)$ approximates (x_i, v_i, s_i) data.

Derivative of Value Functions in Optimal Growth Models

- For the optimal growth problem,

$$\begin{aligned} V_t(k) = \max_{k^+,c,l,y} \quad & u(c,l) + \beta V_{t+1}(k^+), \\ \text{s.t.} \quad & F(y,l) - c - k^+ = 0, \\ & k - y = 0, \end{aligned}$$

  with k^+, c and l as control variables, and y is the dummy variable.
- Formula for computing $V_t'(k)$:

$$V_t'(k) = \lambda,$$

  where λ is the shadow price for the dummy constraint $k - y = 0$, and given directly by optimization packages.

Chebyshev-Hermite Interpolation

- If we have Hermite data $\{(x_i, v_i, s_i) : i = 1, \ldots, m\}$ on $[a, b]$, then the following system of $2m$ linear equations produces coefficients for degree $2m - 1$ Chebyshev polynomial interpolation on the Hermite data:

$$\sum_{j=0}^{2m-1} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m,$$

$$\frac{2}{b-a} \sum_{j=0}^{2m-1} c_j T_j'(z_i) = s_i, \quad i = 1, \ldots, m,$$

where $z_i = \frac{2x_i - a - b}{b-a}$ $(i = 1, \ldots, m)$ are the Chebyshev nodes in $[-1, 1]$, and $T_j(z)$ are Chebyshev basis polynomials.

Relative errors of optimal solutions of numerical DP with Chebyshev interpolation on m Chebyshev nodes using Lagrange vs. Hermite data

			error of c_0^*		error of l_0^*	
γ	η	m	Lagrange	Hermite	Lagrange	Hermite
0.5	0.1	5	1.1(−1)	1.2(−2)	1.9(−1)	1.8(−2)
		10	6.8(−3)	3.1(−5)	9.9(−3)	4.4(−5)
		20	2.3(−5)	1.5(−6)	3.2(−5)	2.3(−6)
0.5	1	5	1.4(−1)	1.4(−2)	6.1(−2)	5.6(−3)
		10	7.7(−3)	3.7(−5)	3.1(−3)	1.6(−5)
		20	2.6(−5)	6.5(−6)	1.1(−5)	3.0(−6)
2	0.1	5	5.5(−2)	6.1(−3)	2.7(−1)	3.6(−2)
		10	3.5(−3)	2.1(−5)	2.0(−2)	1.2(−4)
		20	1.6(−5)	1.4(−6)	9.1(−5)	7.6(−6)
2	1	5	9.4(−2)	1.1(−2)	1.3(−1)	1.7(−2)
		10	5.7(−3)	3.9(−5)	9.2(−3)	6.1(−5)
		20	2.8(−5)	4.7(−6)	4.3(−5)	8.0(−6)
8	0.1	5	2.0(−2)	2.2(−3)	3.6(−1)	4.9(−2)
		10	1.2(−3)	8.5(−6)	2.7(−2)	1.9(−4)
		20	6.1(−6)	1.0(−6)	1.4(−4)	4.4(−6)
8	1	5	6.6(−2)	7.2(−3)	3.4(−1)	4.5(−2)
		10	3.0(−3)	2.6(−5)	2.0(−2)	1.7(−4)
		20	2.0(−5)	0.0(−7)	1.3(−4)	2.1(−7)

Note: $a(k)$ means $a \times 10^k$.

Derivative of Value Functions in Portfolio Optimization

- For the multi-stage portfolio optimization problem,

$$\begin{aligned} V_t(W) &= \max_{B,S} \; E\{V_{t+1}(R_f B + R^\top S)\}, \\ & \quad \text{s.t. } \; W - B - e^\top S = 0, \end{aligned}$$

  with the bond allocation B and the stock allocation S.
- Formula for computing $V_t'(W)$:

$$V_t'(W) = \lambda,$$

  where λ is the shadow price for the constraint $W - B - e^\top S = 0$.

Shape-preserving Hermite Spline Interpolation

- Using Hermite data $\{(x_i, v_i, s_i) : i = 1, \ldots, m\}$,

$$\hat{V}(x; \mathbf{c}) = c_{i1} + c_{i2}(x - x_i) + \frac{c_{i3} c_{i4}(x - x_i)(x - x_{i+1})}{c_{i3}(x - x_i) + c_{i4}(x - x_{i+1})},$$

  when $x \in [x_i, x_{i+1}]$, where

$$\begin{aligned} c_{i1} &= v_i, \\ c_{i2} &= \frac{v_{i+1} - v_i}{x_{i+1} - x_i}, \\ c_{i3} &= s_i - c_{i2}, \\ c_{i4} &= s_{i+1} - c_{i2}, \end{aligned}$$

  for $i = 1, \ldots, m - 1$.

- $\hat{V}(x; \mathbf{c})$ is shape-preserving: When $x \in (x_i, x_{i+1})$, if the value function is increasing and concave, then the rational function spline interpolation is also increasing and concave in each (x_i, x_{i+1}).
- $\hat{V}(x; \mathbf{c})$ is a rational function on each interval (x_i, x_{i+1}), and $\mathcal{C}^1$ globally.

Errors of Optimal Bond Allocations (Lagrange vs Hermite vs Shape-preserving+Hermite)