Dynamic Programming with Shape Preservation

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Shape Preservation

Goal: Suppose

- theory tells us that a value function is concave and increasing, and
- the data from value function iteration is consistent with the shape information

then

- we want the approximation to satisfy the shape information
- which presumably will improve stability, and
- presumably improve accuracy

Challenge #1: Neither interpolation nor regression preserves shape of the data

- These methods create approximations that minimize a norm of the errors, choosing among a vector space of functions
- Shapes define cones of functions: monotone, concave, etc.
- Cones do not have countable bases, not even if we limit ourselves to convex combinations of basis functions

Challenge #2: Shape constraints are infinitistic

- f monotonically increasing means $f'(x) > 0$ for $a \leq x \leq b$
- f concave means $f''(x) < 0$ for $a \leq x \leq b$
- Finding a function of a particular shape is a problem with an infinite number of constraints!

Shape-preserving Chebyshev Interpolation: Attempt 1

LP model for shape-preserving Chebyshev Interpolation: Suppose

- Data from max step is (z_i, v_i), and consistent with concavity and monotonicity, and
- we want an interpolating Chebyshev polynomial, $\sum_{j=0}^{m-1} c_j T_j(z)$ with same shape

Idea: Add shape constraints to interpolation problem at shape points $y_i, i = 1, ..., m'$ where $m' > m$:

$$\min_{c_j} \quad \sum_{j=0}^{m-1} c_j$$

$$\text{s.t.} \quad \sum_{j=0}^{n} c_j T_j'(y_i) > 0 > \sum_{j=0}^{n} c_j T_j''(y_i), \qquad i = 1, \ldots, m',$$

$$\sum_{j=0}^{n} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m,$$

Problem: Overdetermined. Existence not guaranteed.

Shape-preserving Chebyshev Interpolation: Attempt 2

Add more basis functions, for a total of $n > m$, enough so that we are sure there will be a shape-preserving interpolating polynomial:

$$\min_{c_j} \quad \sum_{j=0}^{n} c_j$$

$$\text{s.t.} \quad \sum_{j=0}^{n} c_j T_j'(y_i) > 0 > \sum_{j=0}^{n} c_j T_j''(y_i), \qquad i = 1, \ldots, m',$$

$$\sum_{j=0}^{n} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m,$$

Problem: Undetermined. Too many solutions, particularly since we don't know minimal number required extra basis functions.

Shape-preserving Chebyshev Interpolation: Attempt 3

Select a nice solution by penalizing the high-order basis elements:

$$\begin{aligned}
\min_{c_j} \quad & \sum_{j=0}^{m-1} c_j + \sum_{j=m}^{n} (j+1-m)^2 c_j \\
\text{s.t.} \quad & \sum_{j=0}^{n} c_j T_j'(y_i) > 0 > \sum_{j=0}^{n} c_j T_j''(y_i), \qquad i = 1, \ldots, m', \\
& \sum_{j=0}^{n} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m,
\end{aligned}$$

Problem: No theoretical problem - for fixed n, generally get a unique solution. (Note: different n could produce different function)

Shape-preserving Chebyshev Interpolation: Success

LP model for shape-preserving Chebyshev Interpolation: Improve computational performance by

- using the pure interpolant, $\hat{c}_j$, as the initial guess, and
- decompose coefficients into positive and negative parts

$$
\begin{aligned}
\min_{c_j, c_j^+, c_j^-} \quad & \sum_{j=0}^{m-1} (c_j^+ + c_j^-) + \sum_{j=m}^{n} (j+1-m)^2 (c_j^+ + c_j^-) \\
\text{s.t.} \quad & \sum_{j=0}^{n} c_j T_j'(y_i) > 0 > \sum_{j=0}^{n} c_j T_j''(y_i), \qquad i = 1, \ldots, m', \\
& \sum_{j=0}^{n} c_j T_j(z_i) = v_i, \quad i = 1, \ldots, m, \\
& c_j - \hat{c}_j = c_j^+ - c_j^-, \quad j = 0, \ldots, m-1, \\
& c_j = c_j^+ - c_j^-, \quad j = m, \ldots, n, \\
& c_j^+ \geq 0, \quad c_j^- \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

Optimal Growth Models

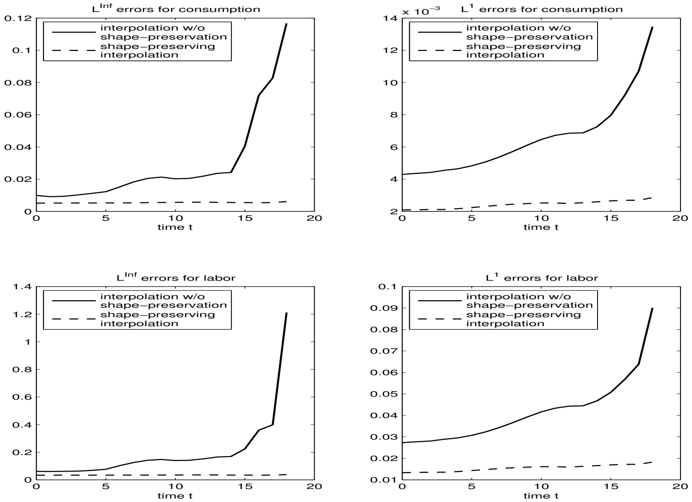
- Optimal Growth Problem:

$$V_0(k_0) = \max_{c,l} \sum_{t=0}^{T-1} \beta^t u(c_t, l_t) + \beta^T V_T(k_T),$$
$$\text{s.t.} \quad k_{t+1} = F(k_t, l_t) - c_t, \quad 0 \le t < T$$

- DP model of optimal growth problem:

$$V_t(k) = \max_{c,l} \; u(c, l) + \beta V_{t+1}(F(k, l) - c)$$

Errors of NDP with Chebyshev interpolation (shape-preserving or not)



Multi-Stage Portfolio Optimization

- W_t: wealth at stage t; stocks' random return: $R = (R_1, \ldots, R_n)$; bond's riskfree return: R_f;
- $S_t = (S_{t1}, \ldots, S_{tn})^\top$: money in the stocks; $B_t = W_t - e^\top S_t$: money in the bond,
- $W_{t+1} = R_f(W_t - e^\top S_t) + R^\top S_t$
- Multi-Stage Portfolio Optimization Problem:

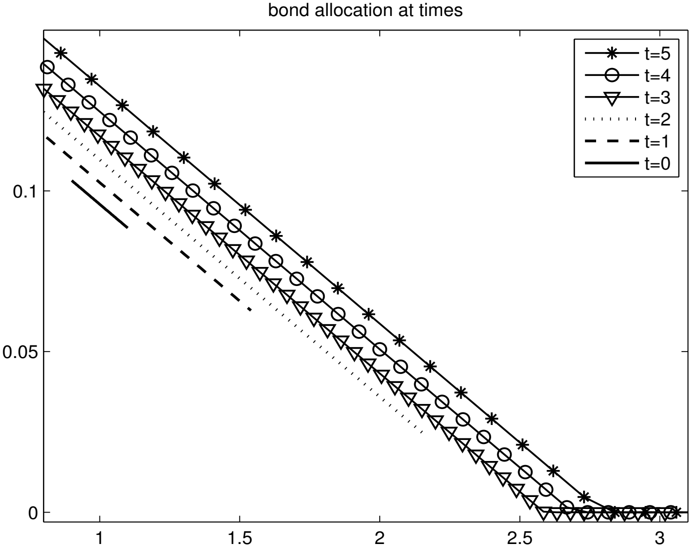
$$V_0(W_0) = \max_{X_{\mathbf{t}}, 0 \le t < T} E\{u(W_T)\}$$

- Bellman Equation:

$$V_t(W) = \max_{S} \quad E\{V_{t+1}(R_f(W - e^\top S) + R^\top S)\}$$

W: state variable; S: control variables.

Exact optimal bond allocation



Errors of Optimal Stock Allocations (shape-preserving or not)