Does it Pay to Get a Reverse Mortgage?

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Wednesday, August 6, 2008

Structural Estimation of Life-Cycle Dynamic Problem with Continuous State Variable

- Relevant topic in finance, public economics, IO, marketing
- Computationally challenging

I solved this model with AMPL:

- Easy: reduction in coding errors
- Fast: between 10 and 60 minutes to estimate a dynamic structural model with more than 70,000 variables
- Precise: based on mathematics

Simple Life-Cycle Model: One continuous state variable

- Backward Solution for the True Value Function

The last period value function is known and equal to $V_T(W)$
In periods $t = 1...(T-1)$ the Bellman equation is:

$$V_t(W) = \max_c (u(c)) + \beta E V_{t+1}(W - c))$$

Given V_{t+1}, the Bellman equation implies, for each wealth level W, three equations that determine the optimal consumption, c^*, $V_t(W)$, and $V_t'(W)$:

- Bellman Equation: $V_t(W) = u(c^*) + \beta V_{t+1}(RW - c^*)$
- Euler Equation: $u'(c^*) - \beta R V_{t+1}'(RW - c^*) = 0$
- Envelope Condition: $V'(W) = \beta R V'(RW - c^*)$

Backward Solution for the Approximate Value Function

- Choose a functional form and a finite grid of wealth levels

Time t value function is approximated by

$$V_t(W) = \Phi(W; a_t, \overline{W}_t) = \sum_{k=0}^{7} a_{k+1,t}(W - \overline{W}_t)^k$$

- We would like to find coefficients a_t such that each time t Bellman equation, along with the Euler and Envelope conditions, holds with the Φ approximation

$$\Phi(W; a_t) = \max_c(u(c)) + \beta\Phi_{t+1}(W - c; a_{t+1}))$$

- Define three set of errors, $\lambda_t^b \geq 0, \lambda_{i,t}^e \geq 0, \lambda_t^{env} \geq 0$, that satisfy the following inequalities
  - Bellman Error

$$-\lambda_t^b \leq \Phi(W_{i,t}; a_t) - [u(c_{i,t}^*) + \beta \Phi_{t+1}(RW_{i,t} - c_{i,t}^*; a_{t+1})] \leq \lambda_t^b$$

  - Euler Error

$$-\lambda_{i,t}^e \leq u'(c_{i,t}^*) - \beta R \Phi'(RW_{i,t} - c_{i,t}^*; a_{t+1}) \leq \lambda_{i,t}^e$$

  - Envelope Error

$$-\lambda_t^{env} \leq \Phi'(W_{i,t}; a_t) - \beta \Phi_{t+1}'(RW_{i,t} - c_{i,t}^*; a_{t+1}) \leq \lambda_t^{env}$$

Dynamic Programming with Approximation of the Value Function

- Minimize the sum of the errors:

$$Minimize \sum_t \sum_i \lambda^e_{i,t} + \sum_t \lambda^b_t + \sum_t \lambda^{env}_t$$

subject to:

- Bellman error
- Euler error
- Envelope error

Dynamic Programming with Approximation of the Value Function in AMPL

- Parameters: $W_{i,t}, \beta, R$
- Variables of Interest: $c_{i,t}, a_{k,t}, \lambda_t^b, \lambda_{i,t}^e, \lambda_t^{env}$
- Other Variables: $u_{i,t}, u_{c;i,t}, W_{i,t}^+, \Phi_{i,t}, \Phi_{W;i,t}, \Phi_{i,t}^+, \Phi_{W;i,t}^+$

The DP problem can be written as:

$$Minimize \sum_t \sum_i \lambda_{i,t}^e + \sum_t \lambda_t^b + \sum_t \lambda_t^{env}$$

subject to:

- Bellman Error

$$-\lambda_t^b \leq \Phi_{i,t} - [u_{i,t} + \beta \Phi_{i,t}^+] \leq \lambda_t^b$$

- Euler Error

$$-\lambda_{i,t}^e \leq u_{c;i,t} - \beta R \Phi_{W;i,t}^+ \leq \lambda_{i,t}^e$$

- Envelope Error

$$-\lambda_t^{env} \leq \Phi_{W;i,t}^+ - \beta \Phi_{W;i,t}^+ \leq \lambda_t^{env}$$

AMPL code for a Simple Life-Cycle Model

```
param nT:=3; #number of time periods
param r:=1.04; #interest rate
param β:=0.96; #discount factor
# Chebychev nodes
param xmin {1..nT};
param xmax{it in 1..nT};
param pi:=3.14159265;
param nx:=28; # number of nodes
param rcheb {i in 1..nx}:=(-cos((2*i-1)*pi/(2*nx)))/cos(((2*nx-1)/2*nx)*pi); #roots
param x {i in 1..nx,it in 1..nT }:=0.5*(rcheb[i]+1)*(xmax[it]-xmin[it])+xmin[it]; #nodes
param xbar{it in 1..nT}:=(xmin[it]+xmax[it])/2;
param nK:=8; #number of points in the polynomial
var ax{ik in 1..nK,it in 1..nT-1}; #polynomial coefficients
var cons{ix in 1..nx,it in 1..nT-1}>=0.001; #consumption
var VV1{ix in 1..nx,it in 2..nT}; #next-period value function
var VVprime1 {ix in 1..nx,it in 1..nT}; #derivative of next-period value function
var u{ix in 1..nx,it in 1..nT-1}= log(1+cons[ix,it]); #utility function
var uprime{ix in 1..nx,it in 1..nT-1}=1/(1+cons[ix,it]); #derivative of utility function
```

AMPL code for a Simple Life-Cycle Model

```
# polynomials
# time t
param phi{ik in 1..nK, ix in 1..nx,it in 1..nT-1}=(x[ix,it]-xbar[it])^(ik-1); #basis
var poly{ix in 1..nx,it in 1..nT-1}=sum{ik in 1..nK} (ax[ik,it]*phi[ik,ix,it]); #seventh-order polynomial
var fapp{ix in 1..nx,it in 1..nT-1}=poly[ix,it]; #function approximation of value function
param phiprime {ik in 1..nK, ix in 1..nx,it in 1..nT-1}=(ik-1)*(x[ix,it]-xbar[it])^(ik-2);
var polyprime{ix in 1..nx,it in 1..nT-1}=sum{ik in 1..nK} (ax[ik, it]*phiprime[ik,ix,it]);
var fappprime{ix in 1..nx,it in 1..nT-1}=polyprime[ix,it]; # derivative
#time t+1
var phi1{ik in 1..nK,ix in 1..nx,it in 2..nT-1}=(r*x[ix,it-1]-cons[ix,it-1]-xbar[it])^(ik-1);
var poly1{ix in 1..nx,it in 2..nT-1}=sum{ik in 1..nK} (ax[ik,it]*phi1[ik,ix,it]);
var fapp1{ix in 1..nx,it in 2..nT-1}=poly1[ix,it]; #function approximation of next period value function
var phi1prime {ik in 1..nK, ix in 1..nx,it in 2..nT-1}=(ik-1)*(r*x[ix,it-1]-cons[ix,it-1]-xbar[it])^(ik-2);
#bellman error
var bell1{ix in 1..nx,it in 1..nT-1}= u[ix,it]+beta*VV1[ix,it+1];
var bellerror{ix in 1..nx,it in 1..nT-1}=fapp[ix,it]-bell1[ix,it];
var lambdab{it in 1..nT-1}>=0;
#euler error
var eulererror1{ix in 1..nx,it in 1..nT-1}=uprime[ix,it]-beta*VVprime1[ix,it+1];
var lambdae1{ix in 1..nx,it in 1..nT-1}>=0;
#envelope error
var envelopeerror{ix in 1..nx,it in 1..nT-1}=polyprime[ix,it]-beta*VVprime1[ix,it+1];
var lambdaenv{it in 1..nT-1}>=0;
```

AMPL code for a Simple Life-Cycle Model

```
minimize obj:sum {ix in 1..nx,it in 1..nT-1} (lambdae1[ix,it]+lambdab[it]+lambdaenv[it]);
subject to consterror3{ix in 1..nx,it in 1..nT-1}: eulererror1[ix,it]>=-lambdae1[ix,it];
subject to consterror4{ix in 1..nx,it in 1..nT-1}: eulererror1[ix,it]<=lambdae1[ix,it];
subject to consterror5{ix in 1..nx,it in 1..nT-1}: bellerror[ix,it]>=-lambdab[it];
subject to consterror6{ix in 1..nx,it in 1..nT-1}: bellerror[ix,it]<=lambdab[it];
subject to consterror9{ix in 1..nx,it in 1..nT-1}: envelopeerror[ix,it]>=-lambdaenv[it];
subject to consterror10{ix in 1..nx,it in 1..nT-1}: envelopeerror[ix,it]<=lambdaenv[it];
subject to consbound1{ix in 1..nx, it in 1..nT-1}: r*x[ix,it]-cons[ix,it]>=xmin[it+1];
subject to consbound2{ix in 1..nx, it in 1..nT-1}: r*x[ix,it]-cons[ix,it]<=xmax[it+1];
#last period value function and derivatives
subject to VVTconst1{ix in 1..nx}: VV1[ix,nT]= log(1+x[ix, nT-1]-cons[ix, nT-1]);
subject to VTprime1 {ix in 1..nx}: VVprime1[ix,nT]=1/(1+x[ix, nT-1]-cons[ix, nT-1]);
# period 2..nT-1 value function and derivatives
subject to VVTminus1{ix in 1..nx,it in 2..nT-1}: VV1[ix,it]= fapp1[ix,it];
subject to VTminus1prime{ix in 1..nx, it in 2..nT-1}:
VVprime1[ix, it]= sum{ik in 1..nK} (ax[ik, it]*phi1prime[ik,ix,it]);
data;
param xmin:=
1 1
2 0.5
3 0.01;
param xmax:=
1 10
2 10
3 10;
```

Continuous and Discrete State Variables

Let W be a continuous state variable and J be a discrete state variable.
Time t value function is approximated by

$$V_t\,(W,J) = \Phi(W,J;a_t,\overline{W}_t) = \sum_{k=0}^{7} a_{k+1,t}(W-\overline{W}_t)^k$$

The constrained optimization approach to a life-cycle model with continuous and discrete state variables is:

$$\textit{Minimize} \sum_i \sum_j \sum_t \lambda^e_{i,j,t} + \sum_j \sum_t \lambda^b_{j,t} + \sum_j \sum_t \lambda^{env}_{j,t}$$

subject to

- Bellman Error:

$$-\lambda^b_{j,t} \leq \Phi(W,J;a_t) - [u(c^*,J) + \beta\Phi(RW-c^*,J;a_{t+1})] \leq \lambda^b_{j,t}$$

- Euler Error

$$-\lambda^e_{i,j,t} \leq \; u'(c^*,J) - \beta R\Phi'(RW-c^*,J;a_{t+1}) \leq \lambda^{env}_{i,j,t}$$

- Envelope Error:

$$-\lambda^{env}_{j,t} \leq \Phi'(W,J;a_t) - \beta\Phi'(RW-c^*,J;a_{t+1}) \leq \lambda^{env}_{j,t}$$

Empirical Part

- We have continuous data on wealth and consumption
- We assume that the measurement error in consumption is normally distributed with mean 0 and unknown variance σ^2
- We can use the Euler Equation to recover the predicted value of consumption
- The probability that household n chooses consumption $c_{n,tp}$ in period tp is:

$$\Pr(c_{n,tp}|W_{n,tp}^{data}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(c_{n,tp}^{data} - c_{n,tp}^{pred})^2}{2\sigma^2}}$$

Therefore the Log-Likelihood is given by:

$$\mathcal{L}(\theta) = \sum_{n=1}^{N} \sum_{tp=1}^{TP} \log \Pr(c_{n,tp}|W_{n,tp}^{data}, \theta)$$

Structural Estimation with DP

- Conventional Approach
  1. take a guess of structural parameters
  2. solve DP
  3. calculate loglikelihood
  4. repeat 1,2,3 until loglikelihood is maximized
- The constrained optimization approach to structural estimation with dynamic programming is:

$$Max\mathcal{L} - Penalty \cdot \Lambda$$

subject to:

Euler error

Bellman error

Envelope error

where $\Lambda = \sum_t \sum_i \lambda^e_{i,t} + \sum_t \lambda^b_t + \sum_t \lambda^{env}_t$

Does it pay to get a Reverse Mortgage?

- For most retirees the house is their major asset.
- A reverse mortgage(RM) allows one to convert equity in a house property into an income stream, without making the periodic loan payments or moving.
- Build a structural, dynamic model of retirees' consumption, housing and moving decisions and solve using MPEC (Mathematical Programming with Equilibrium Constraints)
- Welfare analysis shows that RM provides liquidity and longevity insurance. However, there are very high start up costs and moving becomes a risky proposition.

Reverse Mortgage

- RM are home loans that do not have to be repaid as long as the borrower lives in the house
- But the minimum between the house value and the principal plus the cumulated interest has to be paid back if the retiree definitively moves out or dies (non-recourse loans)
- Potential Market: 30.8 million households with at least one member age 62 and older in 2006
- Actual Market: 265,234 federally insured RM in 2007, about 1% of the 30.8 million households

Is the Reverse Mortgage a Fair Contract?

Option Payment: Lump Sum at the Beginning

- Initial Cost for the Lender: $\overline{B}$
- Closing Cost: $F = \lambda H_{it} + f$
- Outstanding Debt: $G_{it}^{RM} = \overline{B} \sum_{j=1..t} (1 + i_D)^{t-j}$
- Repayment in Present Value: $RM_{it} = \frac{\min(H_{it}, G_{it}^{RM})}{R^{t-j}}$
- Expected Gain for the Lender:
  $EGain_{j,i} = F + \sum_{t=j+1..T} \eta_{i,t-1}[(1-\eta_{i,t})(1-m_{i,t}) + \eta_{i,t} m_{i,t}] RM_{i,t}$
- Closure contract at age 62, H=$100,000

Under this setting:

- Annually Adj HECM: $\overline{B}$=$31,000 EG=$64,000 EG(F=0)=$54,000
- Monthly Adj HECM: $\overline{B}$=$47,000 EG=$74,000 EG(F=0)=$63,000
- FM HomeKeeper: $\overline{B}$=$10,000 EG=$30,000 EG(F=0)=$22,000

Model

- Preferences

$$U_{it}(C_{it}, H_{it}) = \frac{(C_{it}^{1-\omega} H_{it}^{\omega})^{1-\gamma}}{1-\gamma} + \varepsilon_{it}(d_{it})$$

where $\varepsilon_{it}(d_{it})$ is a vector of unobserved utility components associated to the discrete housing choice and it is Extreme Value Type I distributed

- Budject Constraint

$$W_{it+1} = (1+r)W_{it} + ss - C_{it} - \psi_{it} - M_{it}$$

- Choice set:
  - Continuous C_{it}
  - Discrete d_{it}: Move-Stay D_{it}^{M}, Own-Rent D_{it}^{O}, House H_{it}

Housing Expenses

- Per-Period Cost

$$\psi_{it} = [D_i^O \psi^{own} + (1 - D_i^O)\psi^{rent}][D_{it}^M H_{it+1} + (1 - D_{it}^M)H_{it}]$$

- Moving Cost

$$M_{it} = D_{it}^M D_{it}^O [D_{it+1}^O H_{it+1} - H_{it} + H_{it+1}\phi(D_{it+1}^O)] + D_{it}^M (1 - D_{it}^O) H_{it+1}\phi^{rent}$$

where the transaction costs are:

$$\phi(D_{it+1}^O) = [D_{it+1}^O \phi^{own} + (1 - D_{it+1}^O)\phi^{rent}]$$

Value Function

$$V_{it}(X_{it}, \varepsilon_{it}) = \max_{d_{it}, C_{it}} \frac{(C_{it}^{1-\omega}(D_{it}^M H_{it+1}^{\omega} + (1-D_{it}^M) H_{it}^{\omega}))^{1-\gamma}}{1-\gamma} + \varepsilon_{it}(d_{it}) + \beta(n_{t+1} E[V_{it}(W_{it+1}, (D_{it}^M H_{it+1} + (1 - D_{it}^M) H_{it}), D_{it+1}^O, \varepsilon_{it+1})] + b(TW_{it+1}))$$

subject to

$$W_{it+1} = (1+r)W_{it} + ss - C_{it} - \psi_{it} - M_{it}$$

$$C_{it} \geq C_{MIN}$$

State Space

$$X_{it} = \{W_{it}, H_{it}, D_{it}^O, Age_{it}\}$$

Preference parameters to estimate

$$\theta = \{\gamma, \omega, \sigma\}$$

- Assumption I: Additivity

$$U(d_{it}, C_{it}, X_{it}, \theta) = \frac{(C_{it}^{1-\omega} H_{it}^{\omega})^{1-\gamma}}{1-\gamma} + \varepsilon_{it}(d_{it})$$

- Assumption II: Conditional Independence

$$f(X_{it+1}, \varepsilon_{it+1} | d_{it}, C_{it}, X_{it}, \varepsilon_{it}, \theta) = q(\varepsilon_{it+1} | X_{it+1}) g(X_{it+1} | d_{it}, C_{it}, X_{it}, \theta)$$

- Bellman Equation:

$$V_t(X_{it}, \varepsilon_{it}) = \max_{d_{it}, C_{it}} [U(d_{it}, C_{it}, X_{it}, \theta) + \varepsilon_{it}(d_{it}) + \beta \eta_{it+1} EV(X_{it+1})]$$

$$= \max_{d_{it}} \left\{ \left[\max_{C_{it}} \{ [U(d_{it}, C_{it}, X_{it}, \theta) + \beta \eta_{it+1} EV(X_{it+1})] | d_{it} \} \right] + \varepsilon_{it}(d_{it}) \right\}$$

- Inner Maximization (consumption conditional on housing)

$$r(X_{it}, d_{it}) = \max_{C_{it}} [U(d_{it}, C_{it}, X_{it}, \theta) + \beta \eta_{it+1} EV_{t+1}(X_{it+1})] | d_{it}]$$

- Outer Maximization (housing)
  Conditional Choice Probabilities

$$P(j | X_{it}, \theta) = \frac{\exp\{r(X_{it}, j, \theta)\}}{\sum_{k \in d_{it}(X_{it})} \exp\{r(X_{it}, k, \theta)\}}$$

  where

$$EV_{t+1}(X_{it+1}) = \ln \left[\sum_{k \in d_t(X_t)} \exp\{r(X_{it}, k, \theta)\} \right]$$

Dynamic Programming

- Minimize the Sum of Errors

$$\Lambda = \sum_t \sum_i \sum_j \sum_Q \lambda^{e}_{i,j,Q,t} + \sum_t \sum_j \sum_Q \lambda^{b}_{j,Q,t} + \sum_t \sum_j \sum_Q \lambda^{env}_{j,Q,t} + \sum_t \sum_i \sum_j \sum_Q \sum_d \lambda^{cons}_{i,j,d,Q,t}$$

subject to:

$$-\lambda^{e}_{i,j,Q,t} \leq EulerEquation_{i,j,Q,t} \leq \lambda^{e}_{i,j,Q,t}$$

$$-\lambda^{b}_{j,Q,t} \leq BellmanEquation_{j,Q,t} \leq \lambda^{b}_{j,Q,t}$$

$$-\lambda^{env}_{j,Q,t} \leq EnvelopeCondition_{j,Q,t} \leq \lambda^{env}_{j,Q,t}$$

$$-\lambda^{cons}_{i,j,d,Q,t} \leq PolicyFunction_{i,j,d,Q,t} \leq \lambda^{cons}_{i,j,d,Q,t}$$

Go to the Appendix

Solving DP and Estimation with the MPEC

$$\underset{\theta,a,c}{Max} \mathcal{L}(\theta) - Penalty \cdot \Lambda$$

subject to:
Euler errors
Bellman error
Envelope error
Policy function error

where

$$\mathcal{L}(\theta) = \sum_{n=1}^{N} \sum_{tp=1}^{TP} \log \Pr(d_{n,tp}^{H}, c_{n,tp} | W_{n,tp}^{data}, H_{n,tp}^{data}, \theta)$$

We assume that there is a measurement error in consumption $\sim N(0, \sigma^2)$

Go to the Appendix II

Data

The Health and Retirement Study (HRS) and The Consumption and Activities Mail Survey (CAMS). US data (2000-2005)

- We select a group of 175 households that could be the potential target segment for Reverse Mortgage
- Characteristics: 62 years old or older, single, retiree,homeowner, complete information about consumption and financial situation

	Percentiles			Min	Max	Mean
	25%	50%	75%			
H	$40,000	$70,000	$92,000	$ 2,500	$170,000	$71,000
W	$5,000	$17,500	$63,000	$0	$276,548	$45,950
H/W	0.86	2.5	7.5	0.11	1500	23.4
C	$6,270	$9,774	$15,090	$110	$84,380	$13,873
ss	$6,972	$9,468	$11,340	$0	$ 24.701	$9,087
Age	69	74	79	64	86	74

Results

Parameter	Estimate	Asymptotic s.e.	Bootstrap s.e
γ	3.87	$(1.07e-009)$	(0.04)
ω	0.85	$(1.82e-009)$	(0.0002)
σ	0.87	$(6.82e-004)$	(0.05)

- The asymptotic standard errors, computed using a finite difference approach, show a big small-sample bias
- We use a bootstrap procedure to reduce the small-sample bias

Simulation of Welfare Gain from Reverse Mortgage

The welfare gain from a reverse mortgage as been calculated as a percentage increase in the initial non-housing financial wealth that makes the household without reverse mortgage as well off in expected utility terms as with the reverse mortgage.

- Welfare Gain as a Function of the House Value

House Value	Financial Wealth	Welfare Gain
$40,000	$10,300	-37.95%
$80,000	$26,000	3.16%
$120,000	$48,800	25.56%

Simulation of Welfare Gain from Reverse Mortgage

- Welfare Gain as a Function of the Initial Financial Wealth

Financial Wealth	Median FW	Welfare Gain	House Value
LW ($<$ \$10,000)	\$1,540	-63.7%	\$40,000
MW (\$10,000 - \$80,000)	\$26,500	13.44%	\$80,000
HW ($>$ \$80,000)	\$138,000	34.61%	\$80,000

- PROS of RM: liquidity and longevity insurance
- CONS of RM: high closing cost and moving risk
- In the moving case, households with low financial wealth are significanlty worse off. Closing a RM contract dramatically affects any future consumption and housing decision

Simulation of Welfare Gain from Renting

- The simulation shows that some households would be better off from a reverse mortgage
- After about 20 years from its first appearance, the reverse mortgage market is still at 1% of its potentiality
- What would be the welfare gain if the household chooses to en-cash the savings locked in the house by moving out and renting the same size house?

House Value	Financial Wealth	Welfare Gain
$40,000	$10,300	498%
$80,000	$26,000	366%
$120,000	$48,800	205%

Simulation of Welfare Gain from Renting

Financial Wealth	Median FW	Welfare Gain	House Value
LW ($<$ \$10,000)	\$1,540	5200%	\$40,000
MW (\$10,000 - \$80,000)	\$26,500	215%	\$80,000
HW ($>$ \$80,000)	\$138,000	23%	\$80,000

- Households with very low financial wealth have a significant welfare loss from closing a reverse mortgage contract, due to the moving risk, the high transaction costs and the fact that they could borrow only a percentage of the house value.
- On the other side, if the equity in their house is released by moving out and renting, they would be able to en-cash the full amount of the saving locked in their house without increasing their level of indebtedness and without incurring the moving risk.

Conclusions

- Innovative structural, dynamic model of retirees' consumption, housing and moving decisions.
- First MPEC approach applied to an empirical strucural model with dynamic programming problem and continuous state variables.
- Reverse mortgage provides liquidity and longevity insurance. However moving becomes a much riskier proposition, especially for households with low financial wealth.
- Common belief is that Reverse mortgage is for "house rich but cash poor" households. This paper shows otherwise.

  Thank you

Appendix: DP with Approximation of the Value Function

- Euler Equations:

$$u'(c^*_{dN}, H) - \beta\eta_{t+1}RV'_{t+1}(RW - c^*_{dN} - \psi + ss; H, Q) = 0$$

$$u'(c^*_{dMhq}, h) - \beta\eta_{t+1}RV'_{t+1}(RW - c^*_{dMhq} - \psi - M + ss; h, q) = 0$$

- Bellman Equation:

$$V_t(W, H, Q) = \ln\left\{\exp(\widehat{V}_{dN,t}) + \sum_q \sum_h \exp(\widehat{V}_{dMhq,t})\right\}$$

- Envelope Condition:

$$V'_t(W, H, Q) = \Pr(NM|W, H, Q) \cdot \widehat{V}'_{dN,t} + \sum_q \sum_h \Pr(Mhq|W, H, Q) \cdot \widehat{V}'_{dMhq,t}$$

Go back

- Value Function Approximation

$$V_t\,(W,H,Q) = \Phi(W,H,Q;a_t,\overline{W}_t) = \sum_{k=0}^{7} a_{k+1,H,Q,t}(W-\overline{W}_t)^k$$

- Policy Function Approximation

$$c^*_{d,t}\,(W,H,Q) = \Phi(W,H,Q;b_{d,t},\overline{W}_t) = \sum_{k=0}^{7} b_{k+1,H,Q,d,t}(W-\overline{W}_t)^k$$

We would like to find coefficients a_t and $b_{d,t}$ such that each time t Bellman equation, along with the Euler and Envelope conditions, holds with the Φ approximation

Go back

- Euler Errors

$$-\lambda^e_{i,j,Q,t} \leq u'(c^*_{i,j,d^N,t}, H_{j,t}) - \beta R \Phi'(RW_{i,t} - c^*_{i,j,d^N,t} - \psi + ss; H_{j,t}, Q_t; a_{t+1}) \leq \lambda^e_{i,j,Q,t}$$

$$-\lambda^e_{i,j,Q,t} \leq u'(c^*_{i,j,d^{Mhq},t}, H_{t+1}) - \beta R \Phi'(RW_{i,t} - c^*_{i,j,d^{Mhq},t} - \psi - M + ss; H_{t+1}, Q_{t+1}; a_{t+1}) \leq \lambda^e_{i,j,Q,t}$$

- Bellman Error

$$-\lambda^b_{j,Q,t} \leq \Phi(W_{i,t}, H_{j,t}, Q_t; a_t) - \ln\left\{\exp(\widehat{V}_{i,j,d^N,t}) + \sum_q \sum_h \exp(\widehat{V}_{i,j,d^{Mhq},t})\right\} \leq \lambda^b_{j,Q,t}$$

where

$$\widehat{V}_{i,j,d^N,t} = u(c^*_{i,j,d^N,t}, H_{j,t}) + \beta \eta_{t+1} \Phi(RW_{i,t} - c^*_{i,j,d^N,t} - \psi + ss; H_{j,t}, Q_t; a_{t+1})$$

$$\widehat{V}_{i,j,d^{Mhq},t} = u(c^*_{i,j,d^{Mhq},t}, H_{t+1}) + \beta \eta_{t+1} \Phi(RW - c^*_{i,j,d^{Mhq},t} - \psi - M + ss; H_{t+1}, Q_{t+1}; a_{t+1})$$

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- Envelope Error

$$-\lambda^{env}_{j,Q,t} \leq \Phi'(W_{i,t}, H_{j,t}, Q_t; a_t) - \{f_{i,j,d^N,t} \cdot \Phi'(RW_{i,t} - c^*_{i,j,d^N,t} - \psi + ss; H_{j,t}, Q_t; a_{t+1})$$

$$+ \sum_q \sum_h [f_{i,j,d^{Mhq},t} \cdot \Phi'(RW_{i,t} - c_{i,j,d^{Mhq},t} - \psi - M; H_{t+1}, Q_{t+1}; a_{t+1})]\} \leq \lambda^{env}_{j,Q,t}$$

where

$$f_{i,j,d,t} = \Pr(d|W_{i,t}, H_{j,t}, Q_t) = \frac{\exp(\widehat{V}_{i,j,d,t})}{\exp(\widehat{V}_{i,j,d^N,t}) + \sum_q \sum_h \exp(\widehat{V}_{i,j,d^{Mhq},t})}$$

- Policy Function Error

$$-\lambda^{cons}_{i,j,Q,d,t} \leq \Phi(W_{i,t}, H_{j,t}, Q_t; b_t) - c^*_{i,j,d,t}(W_{i,t}, H_{j,t}, Q_t) \leq \lambda^{cons}_{i,j,Q,d,t}$$

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Loglikelihood

- Measurement Error in Consumption

$$Pr(c_{n,t}|d^H_{n,tp}, W^{data}_{n,tp}, H^{data}_{n,tp}, Q^{data}_{n,tp}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(c^{data}_{n,tp} - c^{pred}_{n,tp})^2}{2\sigma^2}}$$

- Discrete Choice Probability

$$Pr(d^H_{n,tp}|W^{data}_{n,tp}, H^{data}_{n,tp}, Q^{data}_{n,tp}) = \frac{e^{V_{d,n,tp}}}{\sum_m e^{Vm,n,tp}}$$

- Joint Probability of Housing and Consumption Choice

$$Pr(d^H_{n,tp}, c_{n,tp}|W^{data}_{n,tp}, H^{data}_{n,tp}, Q^{data}_{n,tp}) = Pr(d^H_{n,tp}|W^{data}_{n,tp}, H_{n,tp}, Q^{data}_{n,tp}) \cdot Pr(c_{n,t}|d^H_{n,tp}, W^{data}_{n,tp}, H^{data}_{n,tp}, Q^{data}_{n,tp})$$

- Log-Likelihood

$$\mathcal{L}(\theta) = \sum_{n=1}^{N} \sum_{tp=1}^{TP} \log Pr(d^H_{n,tp}, c_{n,tp}|W^{data}_{n,tp}, H^{data}_{n,tp}, Q^{data}_{n,tp}, \theta)$$

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