INVESTIGATING BID PREFERENCES AT LOW-PRICE, SEALED-BID AUCTIONS WITH ENDOGENOUS PARTICIPATION*

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Abstract

At procurement auctions, with bid preferences, qualified firms are treated special. A common policy involves scaling the bids of preferred firms by a discount factor for the purposes of evaluation only. Introducing such an asymmetry has three effects: first, preferred firms may inflate their bids, yet still win the auction; second, nonpreferred firms may bid more aggressively than in the absence of preferences; third, the preference policy can affect participation. For different cost distributions, we solve numerically for the equilibrium bid functions under different discounts and then simulate behaviour. Our approach allows us to quantify the relative importance of the three effects. We find that the participation effect is relatively unimportant and, in most cases, a positive cost-minimizing preference rate exists.

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1. Motivation and Introduction

Government agencies often use procurement auctions to award contracts to firms vying for the right to perform tasks for the government. For example, in the United States, in 2005, the federal government alone spent over $378 billion, nearly $1,300 per person.[1] The most common format for procurement is the low-price, sealed-bid auction. Under this format, firms submit bids in sealed envelopes with the lowest bidder winning the auction—being paid its bid on completion of the task.

However, for over thirty years, under a variety of regulatory regimes, bids have often been treated asymmetrically by favouring the tenders from certain classes of firms. Several reasons surely exist for using bid-preference programmes, some political. For example, a government may want to favour in-state or domestic firms, even though their costs are similar to out-of-state or foreign firms.

In practice, bid preferences have become a common tool of public policy at all levels of government. Preference is often given to small businesses, firms owned by a minority or a woman, veteran-owned firms, and in-state or domestic producers—the list goes on. The most common way to implement bid preferences involves scaling the bids of preferred firms by some discount factor for the purposes of evaluation only; a preferred firm still earns what it bid, if it wins. For example, the United States federal government awards a six percent preference to bids tendered by American firms under the *Buy American Act*; the state of Maryland gives a five percent preference to a firm's bid if it is a qualified small business; the city of Tucson, Arizona awards contracts to firms owned by a minority or a woman, even if their bid exceeds the lowest bid by up to seven percent. By treating qualified firms differently from other bidders, an asymmetry is introduced. One can model these programmes using Harsanyi's (1967/68) theory of noncoöperative games under incomplete information.

It is well known that, in the presence of asymmetries, outcomes at low-price, sealed-bid auctions can be inefficient; see, for example, Vickrey (1961) as well as

[1] Details concerning annual procurement spending can be found from the Federal Data Procurement Center website
`http://www.fpdsng.com/fpr_reports_fy_05.html`

Maskin and Riley (2000a). Giving preference can both introduce an inefficient outcome and increase costs to the contracting agency: even when a nonpreferred firm's bid is the lowest, it may still lose the auction because of the preference adjustment. Thus, firms receiving preferential treatment can inflate their bids and still win the auction; we refer to this as the *preference effect*. In response to the preference policy, nonpreferred firms will behave more competitively than under the equal treatment of bids; we refer to this as the *competitive effect*. When nonpreferred firms bid closer to their costs than under the equal treatment of bids, the government can save money. A third effect also exists; we refer to this as the *participation effect*. Because preferred and nonpreferred firms will have different incentives to participate, even when their costs are the same, bid preferences can induce differential entry into the auction. To quantify the importance of this effect, we make entry endogenous. To wit, the entry decision of each firm depends not only on that firm's cost, but also on the preference awarded it by the government. Our simulation results indicate that the likelihood of entry by a particular class of firm depends on the distribution of costs. Most importantly, a preference policy can either increase or decrease expected procurement costs—it remains a quantitative matter. The relative importance of these opposing effects on expected costs depends on the preference policy and the number of potential bidders in each class as well as the distributions of costs.

Below, we investigate the effects of introducing a commonly-used preference policy into a standard model of a low-price, sealed-bid auction with symmetric potential bidders.[2] Specifically, potential bidders are assumed to draw costs independently from the same distribution, but to belong to one of two classes. We assume that, at the auction, the firms belonging to the first class, referred to as class 1 firms, are given preference for the purposes of evaluation only; the other bidders, referred to as class

[2] In the literature concerned with the structural econometrics of auctions, researchers often refer to symmetric bidders as those for whom cost (valuation) draws are from the same distribution; in that literature, asymmetric bidders are those whose cost (valuation) draws are from different distributions. In the theoretical literature, researchers often refer to symmetric equilibria in which bidders with the same distribution use the same bidding strategy. When we refer to symmetric bidders, we mean bidders whose cost draws are from the same distribution *and* who use the same (monotonic) bidding strategy.

2 firms, receive no preferential treatment.[3] Like Bajari (2001), we then use numerical methods—specifically, a direct optimization strategy often referred to in the literature as the *Mathematical Programming with Equilibrium Constraints* approach or the MPEC approach, for short—to approximate equilibrium (inverse) bid functions. The mathematics of the MPEC approach are summarized in a book by Luo, Pang, and Ralph (1996), while the successful use of the MPEC approach in economics is illustrated by Su and Judd (2008). Our implementation of the MPEC approach allows us to model completely the equilibrium behaviour of firms—specifically,the joint entry and bidding decisions which change endogenously when bid preferences are introduced. Subsequently, we use simulation methods to investigate different policy experiments to determine the effects on expected procurement costs as well as the expected costs of inefficiencies.

In the auction literature, only a few researchers have investigated the effects of bid-preference policies at procurement auctions. McAfee and McMillan (1989) first introduced a model of bidding with preferential treatment. They showed that, in international trade, when foreign (domestic) firms have a cost advantage, a government can stimulate competition and reduce costs by giving preference to domestic (foreign) firms that have a higher cost structure. In addition, when the profits of domestic firms enter the government's objective function along with (minimizing) procurement costs, the government should always offer preferences to domestic firms. McAfee and McMillan did not model entry and they abstracted from any formal analysis of equilibrium bidder behaviour. Recent developments in numerical methods to solve for equilibrium bid functions at low-price, sealed-bid auctions with asymmetric bidders allow us to quantify the effects of the most commonly-observed preference policy. While we assume specific cost distributions, our approach can be extended to any distribution satisfying existence and uniqueness conditions that are now standard

[3] Myerson (1981) has shown that, with symmetric bidders, an optimal auction (one that maximizes the expected utility of the seller) is the Vickrey auction with an optimally-set reserve price. By the revenue equivalence theorem, then, a first-price, sealed-bid auction with the same optimal reserve price is also an optimal auction. Çelik and Yilankaya (2007) have suggested that, when participation is costly, bid preferences may be an optimal policy for the seller; their model of entry is also based on Samuelson (1985), but they abstract from any formal analysis of bidder behaviour.

in the auction literature, including nonparametric distributions.

Flambard and Perrigne (2006) have suggested that a fixed subsidy can reduce the cost of snow removal contracts between 1.34 and 5.82 percent—depending on how the subsidy is funded—for the city of Montréal in the province of Québec, Canada. They also noted that discriminatory reserve prices (posting different price ceilings for bidders of different classes) can reduce costs for the city by 1.18 percent. Flambard and Perrigne suggested that both policies would enhance competition from stronger bidders. Although these authors did not model entry, their argument hinges critically on the entry of potential bidders who would not submit bids in the absence of a subsidy. Our work complements this research in that we consider not only an alternative policy (specifically, a commonly-used preference policy), but also allow for endogenous entry, which depends explicitly on the government's policy.

Recently, some researchers have investigated empirical models and policy issues similar to ours. Specifically, Marion (2007) studied road construction contracts in California. There, the state often grants a five percent discount to bids submitted by prequalified small businesses. The preference policy only applies to contracts that do not use federal funds. Marion compared the government's costs on contracts where the preference policy was implemented to contracts using federal funds where the preference policy was void and found that the five percent bid-preference programme in California increased procurement costs by 3.5 percent. He observed large (nonpreferred) firms bidding less frequently at preference auctions and attributed the increased government cost to reduced participation by large firms. However, in his structural estimation and counterfactual experiments, Marion did not model participation explicitly.

Krasnokutskaya and Seim (2007) have extended Marion's structural framework to admit endogenous entry, modelling the bidding process as a two-stage game in which firms first decide whether to enter the auction, and then determine their optimal bidding strategy. In their model, the firms do not know their individual costs when making the entry decision, but only observe the number of potential bidders. Under these assumptions, Krasnokutskaya and Seim obtained identification and estimated the model under different informational assumptions in the bidding stage of the game.

They found that the share of contracts won by qualified small (preferred) bidders rose as a result of preferential treatment, but that the cost to the government also increased.

Entry decisions are typically incorporated into auction models by considering two-stage games where, in the first stage, bidders decide whether to enter the auction, while in the second stage, entrants submit their bids. The canonical models of entry are those of Samuelson (1985) as well as Levin and Smith (1994). In both models, all firms incur a common entry fee in order to submit a bid, but the two models differ in their assumptions concerning when information is known. Specifically, Samuelson (1985) assumed potential bidders know their private costs *before* deciding whether to incur the fee to enter the auction. On the other hand, Levin and Smith (1994) assumed that potential bidders do not learn their private costs until *after* they have decided to enter the auction.[4] While these models differ only slightly in their timing, they differ drastically in their implications. Li and Zheng (2007) considered a very detailed comparison of the two models from both a theoretical and an empirical perspective. They developed a unified estimation framework within which a model-selection procedure was used to distinguish between the two competing models. Li and Zheng found "very strong" evidence against the Levin and Smith (1994) model, leading us to adopt the framework proposed by Samuelson (1985).

Based on our research, we can decompose the total effect of a bid-preference programme on the buyer's cost into three parts and then investigate the relative importance of each part. We are, thus, able to illustrate the importance of those endogenous changes that most significantly affect the cost and the efficiency of the procurement auction.

The remainder of our paper is organized as follows: in the next section, we develop a model of equilibrium bidding at low-price, sealed-bid auctions when bid preferences exist and entry is endogenous. Purposeful behaviour in equilibrium is characterized by a system of ordinary differential equations (ODEs) that cannot

[4] Athey, Levin, and Seira (2004) have extended the model of Levin and Smith (1994) so that potential bidders draw entry fees from a common distribution, allowing entry fees to vary across bidders. Krasnokutskaya and Seim (2007) have adopted this model of entry in their research.

be solved analytically. In section 3, we describe the numerical methods we used to approximate the optimal bid functions; these methods extend a computational algorithm first suggested by Bajari (2001). In section 4, we summarize the properties of the approximated bid functions, while in section 5, we use the approximate bid functions, in conjunction with simulation methods, to quantify the effects that a bid-preference programme will have on the equilibrium behaviour of different classes of bidders. Subsequently, in section 6, we summarize our results and conclude. In an appendix to the paper, we present a proof too detailed for inclusion in the text of the paper.

2. Theoretical Model

Consider a government agency that seeks to complete an indivisible task at the lowest cost. The agency invites sealed-bid tenders from n potential suppliers—firms. The bids are opened more or less simultaneously and the contract is awarded to the lowest bidder who wins the right to perform the task. The agency then pays the winning firm its bid on completion of the task.

Suppose each firm has a private cost. Assume that each firm knows its private cost, but not those of its competitors. Assume that firm i's cost C_i is an independent draw from the cumulative distribution function $F(c)$, which is continuous, having an associated positive probability density function $f(c)$ that has compact support $[\underline{c}, \bar{c}]$ where $\underline{c}$ is strictly positive. Assume that the number of potential bidders n as well as the cumulative distribution function of costs $F(c)$ and the common support $[\underline{c}, \bar{c}]$ are common knowledge. This environment is often referred to as the symmetric *independent private-cost paradigm* (IPCP).

Suppose potential bidders are risk-neutral. Thus, when firm i submits bid b_i, it receives the following payoff:

$$U_i(b_1, \ldots, b_n, C_i) = \begin{cases} b_i - C_i, & \text{if } b_i < b_j \text{ for all } j \neq i \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Assume that firm i chooses b_i to maximize its expected profit

$$\mathcal{E}(U_i|b_i) = (b_i - C_i)\,\Pr(\mathsf{win}|b_i). \tag{2.2}$$

Within the IPCP the Bayes–Nash equilibrium bid function of the i^{th} bidder has been characterized by Holt (1980) as well as Riley and Samuelson (1981); it is the solution to a commonly-encountered ODE and has the following closed-form:

$$\beta(c) = c + \frac{\int_c^{p_0} [1 - F(u)]^{n-1} du}{[1 - F(c)]^{n-1}} \tag{2.3}$$

where p_0 is a price ceiling—a maximum acceptable bid that has been imposed by the buyer. In general, we often implicitly assume no price ceiling exists, in which case p_0 equals the highest cost $\bar{c}$. However, in our application, we admit the possibility of a binding price ceiling because it is required in an example involving the uniform distribution of costs.

2.1. Incorporating Bid Preferences

Now relax the assumption that all bidders are treated the same and incorporate bid preferences, thus introducing an asymmetry into the model. Consider the most commonly-used preference programme under which the bids of preferred firms are treated differently for the purposes of evaluation only. In particular, the bids of preferred firms are typically scaled by some discount factor which is one plus a preference rate denoted ρ. When the preference rate ρ is a choice variable of the buyer, he can control the importance of an asymmetry in the auction. This is different from what occurs in a standard auction model with asymmetric bidders—bidders whose cost (valuation) draws are from different distributions. In those models, the asymmetry is exogenously fixed.

What happens to bidder behaviour when the buyer gives preference to a subset of bidders, class 1 bidders, by scaling their bids by $(1 + \rho)$? Suppose there are n_1 preferred bidders and n_2 typical (nonpreferred) bidders, where $(n_1 + n_2)$ equals n. The preference policy reduces the bids of class 1 firms for the purposes of evaluation only; a winning firm is still paid its bid, on completion of an awarded contract.

To be concrete, consider the following example: suppose two firms participate at an auction, one a preferred firm who has tendered a bid of \$104.99, and the other, a nonpreferred firm who has tendered a bid of \$100. At a standard, low-price, sealed-bid auction, the nonpreferred bidder would win the auction because it has submitted the lowest bid. At an auction with a preference rate ρ of 0.05, the government would consider the preferred firm's tender a bid of (\$104.99/1.05), or \$99.99. The preference rate ρ is used for the purposes of evaluation only. In this example, the preferred firm would win the auction and be paid \$104.99 on completion of the task. Note that the *ex post* outcome is inefficient and more costly for the government than under the equal treatment of bids.

Consider the decision problem faced by a representative bidder of each class at a low-price, sealed-bid auction with preferences. Each bidder draws a firm-specific cost independently from $F(c)$. Each firm then chooses its bid b to maximize (2.2), but an asymmetry is introduced into the auction game through the term $\Pr(\mathsf{win}|b)$. Suppose that all bidders of class j use a (class-symmetric) monotonically increasing strategy $\beta_j(\cdot)$, where $j \in \{1, 2\}$. This assumption imposes structure on the probability of winning an auction, conditional on a particular strategy $\beta_j(\cdot)$, which then determines the bid b_j given a class j firm's cost draw. In particular, for a class 1 bidder,

$$\Pr(\mathsf{win}|b_1) = \left(1 - F[\beta_1^{-1}(b_1)]\right)^{n_1-1} \left(1 - F\left[\beta_2^{-1}\left(\frac{b_1}{1+\rho}\right)\right]\right)^{n_2}, \tag{2.4}$$

while for a class 2 bidder

$$\Pr(\mathsf{win}|b_2) = \left[1 - F\left(\beta_1^{-1}\left[(1+\rho)b_2\right]\right)\right]^{n_1} \left(1 - F[\beta_2^{-1}(b_2)]\right)^{n_2-1}. \tag{2.5}$$

When these probabilities are substituted into equation (2.2), expected-utility maximization yields the following pair of first-order conditions (FOCs):

$$\frac{\partial \mathcal{E}(U_1)}{\partial b_1} = 0 = 1 - \left[b_1 - \beta_1^{-1}(b_1)\right] \left[\frac{(n_1-1) f\left[\beta_1^{-1}(b_1)\right] \beta_1^{-1'}(b_1)}{\left(1 - F\left[\beta_1^{-1}(b_1)\right]\right)} + \frac{n_2 f\left[\beta_2^{-1}\left(\frac{b_1}{1+\rho}\right)\right] \frac{1}{1+\rho} \beta_2^{-1'}\left(\frac{b_1}{1+\rho}\right)}{\left(1 - F\left[\beta_2^{-1}\left(\frac{b_1}{1+\rho}\right)\right]\right)}\right] \tag{2.6}$$

and

$$\frac{\partial \mathcal{E}(U_2)}{\partial b_2} = 0 =$$

$$1 - \left[b_2 - \beta_2^{-1}(b_2)\right] \left[\frac{n_1 f\left(\beta_1^{-1}\left[(1+\rho)b_2\right]\right)(1+\rho)\beta_1^{-1'}\left[(1+\rho)b_2\right]}{\left[1 - F\left(\beta_1^{-1}\left[(1+\rho)b_2\right]\right)\right]} + \frac{(n_2-1) f\left[\beta_2^{-1}(b_2)\right]\beta_2^{-1'}(b_2)}{\left(1 - F\left[\beta_2^{-1}(b_2)\right]\right)} \right]. \tag{2.7}$$

In equilibrium, this pair of FOCs yields a system of ODEs. When the appropriate boundary conditions are imposed, this system will determine the optimal inverse-bid functions. However, as Bajari (2001) has pointed out, the Lipschitz conditions are not satisfied for these equations in the neighborhood of $\bar{c}$, the upper bound of the cost support. Thus, we must resort to numerical methods to solve this system. However, a technical problem arises: one of the boundary conditions, $\underline{b}$, the lower bound of support of bids, is unknown and endogenous—a function of the economic environment, most importantly the preference rate ρ.

Lebrun (1999), Maskin and Riley (2000b, 2003), as well as Bajari (2001) (who added a differentiability assumption on the inverse-bid functions) have shown that, in an asymmetric model without bid preferences, if the class-specific cumulative distribution functions $F_j(\cdot)$ have the same support $[\underline{c}, \bar{c}]$ and the class-specific probability density functions $f_j(\cdot)$ are continuously differentiable and bounded away from zero on the common support, then the equilibrium inverse-bid functions will satisfy the following conditions.

1. *Right-Boundary Condition:*

   For $j = \{1, 2\}$, $\beta_j^{-1}(\bar{c}) = \bar{c}$.

2. *Left-Boundary Condition:*

   There exists an unknown constant $\underline{b}$ such that for $j = \{1, 2\}$, $\beta_j^{-1}(\underline{b}) = \underline{c}$.

3. *Intermediate-Cost Condition:*

   For $j = \{1, 2\}$ and for all $b \in (\underline{b}, \bar{c})$, $\beta_j^{-1}(b)$ will solve the FOCs (2.6) and (2.7) when ρ is zero.

Condition 1 simply requires that a bidder who draws the highest possible cost $\bar{c}$ also bids $\bar{c}$. Condition 2 requires that any firm who draws the lowest possible cost $\underline{c}$ tenders the same bid $\underline{b}$, regardless of its cost distribution. Any bid below $\underline{b}$ would be suboptimal because the firm could strictly increase the bid by ε and still win the auction with certainty, while at the same time increasing its profits. When a bidder has a cost draw of $c \in [\underline{c}, \bar{c})$, the firm can win the auction with positive probability. Condition 3 characterizes an important trade-off: higher bids result in higher profits if the firm wins the contract, but higher bids also reduce the probability of winning the contract. The firm chooses its optimal bid to satisfy one of the FOCs (2.6) and (2.7) where ρ is zero when no preference is shown. Furthermore, the equilibrium inverse-bid functions are unique.

Most observed preference policies use a constant preference rate to adjust the bids of qualified firms for the purposes of evaluation only. To incorporate bid preferences in the model, using this common preference rule, the above conditions must be adjusted to depend on the class of the firm. Reny and Zamir (2004) have extended the results concerning equilibrium bid functions in a general asymmetric environment; these results apply to the bid-preference case. Specifically, under the common preference policy, the equilibrium inverse-bid functions will satisfy the following revised conditions.

4. *Right-Boundary Conditions:*

   a) For all nonpreferred bidders of class 2, $\beta_2^{-1}(\bar{c}) = \bar{c}$;

   b) for all preferred bidders of class 1, $\beta_1^{-1}(\bar{b}) = \bar{c}$, where $\bar{b} = \bar{c}$ if $n_1 > 1$, but when $n_1 = 1$, then $\bar{b}$ is determined by

$$\bar{b} = \underset{b}{\operatorname{argmax}} \left[(b - \bar{c}) \left(1 - F_2 \left[\beta_2^{-1} \left(\frac{b}{1+\rho} \right) \right] \right)^{n_2} \right].$$

5. *Left-Boundary Conditions:*

   There exists an unknown constant $\underline{b}$ such that

   a) for all nonpreferred bidders of class 2, $\beta_2^{-1}(\underline{b}) = \underline{c}$;

   b) for all preferred bidders of class 1, $\beta_1^{-1}\left[(1+\rho)\underline{b}\right] = \underline{c}$.

6. *Intermediate-Cost Conditions:*

   a) For all nonpreferred bidders of class 2 and for all $b \in (\underline{b}, \bar{c}/(1+\rho))$, $\beta_2^{-1}(b)$ will solve the FOC (2.7), but for $b \in [\bar{c}/(1+\rho), \bar{c})$ nonpreferred bidders will bid their cost—*i.e.*, $\beta_2^{-1}(c) = c$;

   b) for all preferred bidders of class 1 and for all $b \in ((1+\rho)\underline{b}, \bar{b})$, $\beta_1^{-1}(b)$ will solve the FOC (2.6).

Condition 4 illustrates that, with a preference policy, nonpreferred bidders will bid their cost when they have the highest cost. When just one preferred firm competes with nonpreferred firms, that firm finds it optimal to submit a bid that is greater than the highest cost because the preference rate will reduce the bid and allow the preferred firm to win the auction with some probability. However, when more than one firm receives preference, it is optimal for preferred firms to bid their costs at the right boundary. This argument is demonstrated in Appendix A.1, the intuition being that, with more than two preferred firms, the optimal bid can be interpreted as a game of Bertrand competition where the firms continue to undercut one another until each firm bids its cost in equilibrium. Condition 5 requires that, when a nonpreferred firm draws the lowest cost, it tenders the lowest possible bid $\underline{b}$, whereas a preferred firm submits $(1+\rho)\underline{b}$. This condition can be explained by a similar argument to the standard left-boundary condition, taking into account that preferred bids get adjusted using ρ. The last condition requires that the FOCs must hold at all intermediate bids. To ensure consistency across solutions in our application, like Krasnokutskaya and Seim (2007), we assume in condition 6.a) that nonpreferred players bid their costs if those costs are in the range $(\bar{c}/(1+\rho), \bar{c})$. Because of the preferential treatment and condition 4.b), assuming more than one bidder receives preferential treatment, nonpreferred players cannot win the auction when they bid higher than $[\bar{c}/(1+\rho)]$.

2.2. Endogenous Entry

In the theoretical literature concerned with entry into low-price, sealed-bid (first-price, sealed-bid) auctions, researchers have considered variations on two basic models. To motivate entry, however, it is assumed that bidders must pay some fee in order to submit a tender. All bidders are assumed to know this fee as well as the cost distribution and the number of potential bidders.[5] Samuelson (1985) assumed that potential bidders know their private costs *before* deciding whether to enter the auction. In contrast, Levin and Smith (1994) assumed that potential bidders first decide whether to enter the auction; only on having entered the auction do entrants learn their private costs. The entry games in these two models, which differ only in the timing, have drastically different implications for the strategies of the firms and, hence, optimal policy choices of buyers. Under the Samuelson (1985) model, equilibrium behaviour in the entry game is characterized by a pure strategy: firms enter only if their costs are below a certain threshold, otherwise they do not pay the entry fee. In the model of Levin and Smith (1994), equilibrium behaviour in the entry game is characterized by a mixed strategy: firms enter each auction with a certain probability.

Li and Zheng (2007) developed a structural framework within which the two models can be estimated. A Bayesian model-selection procedure was then used to choose the appropriate model. Li and Zheng found "very strong" evidence against the model of Levin and Smith (1994). In light of this evidence, we have chosen to employ the Samuelson (1985) model of entry.[6]

Consider the symmetric IPCP (without preferences) outlined above. As before, firm i learns its own cost c_i, but now each firm must incur an entry fee, perhaps a bid-preparation cost, κ. Having paid the entry fee, however, firm i does *not* learn the number of firms ($\leq n$) that also chose to pay the fee and, thus, participate.

In equilibrium, there exists a unique threshold cost c^*: firms with costs below c^*

[5] Here, we only describe models of entry within the symmetric IPCP because they relate directly to our research. Economic theorists have also considered models of entry involving asymmetric bidders; *e.g.*, Tan and Yilankaya (2006) have shown that, at a second-price auction, when the underlying cost distributions are concave, a unique "intuitive" equilibrium exists in which strong bidders are more likely to enter than weak bidders. Their proof can be adapted to show there exists a unique intuitive equilibrium in low-price, sealed-bid auction games, provided the underlying distributions are convex. Very few distributions satisfy this convexity assumption and the intuitive equilibrium can break-down when bid preferences are given to weak bidders.

[6] We should note that Krasnokutskaya and Seim (2007) have estimated a model with bid preferences in which entry follows the model of Athey, Levin, and Seira (2004), which in turn generalizes the model of Levin and Smith (1994).

pay the entry fee and bid at the auction, while firms with costs above c^* choose not to enter the auction. A firm with cost c^* is indifferent between submitting a bid, or not, so its expected profit is zero. Hence,

$$[\beta(c^*) - c^*][1 - F(c^*)]^{n-1} - \kappa = 0$$

where $\beta(\cdot)$ denotes the (second-stage) equilibrium bid function. Because no bidder with a cost higher than c^* enters the auction and because all bidders employ a common equilibrium bidding strategy, which is strictly increasing in cost c, the bidder with threshold cost c^* submits the highest bid, equal to the buyer's price ceiling p_0, which equals the highest cost $\bar{c}$ when no price ceiling exists. Thus, the threshold c^* is determined by the following equation:

$$(p_0 - c^*)[1 - F(c^*)]^{n-1} - \kappa = 0. \tag{2.8}$$

2.3. Endogenous Entry with Bid Preferences

Given the set of boundary conditions stated above, the system of equations defined by the FOCs characterizes the equilibrium bidding strategies for preferred and non-preferred bidders in equations (2.6) and (2.7), respectively. The asymmetry induced by a preference policy results in different equilibrium bidding strategies, depending on the class of the firm. Under equal bid treatment, the firms have the same threshold cost determining the entry condition; however, as we shall see, the preference policy creates a disparity between the entry threshold costs. Denote the threshold costs of preferred and nonpreferred firms by c_1^* and c_2^*, respectively. Incorporating entry decisions requires further adjustment of the right-boundary conditions presented above: admitting entry will induce bidders with sufficiently high cost draws not to enter the auction. Thus, the highest bids will be submitted by the threshold bidders. Adapting the right-boundary conditions presented above to account for endogenous entry yields the following relevant boundary conditions in a model having a preference policy as well as endogenous entry.

7. *Right-Boundary Conditions, with Entry:*

   a) For all nonpreferred bidders of class 2, $\beta_2^{-1}(p_0) = c_2^*$;

   b) for all preferred bidders of class 1, $\beta_1^{-1}(p_0) = c_1^*$ when $n_1 > 1$

   where p_0 denotes the buyer's price ceiling which, in general, we assume to be nonbinding and equal $\bar{c}$, except where this is impossible.

8. *Left-Boundary Conditions:*

   There exists an unknown constant $\underline{b}$ such that

   a) for all nonpreferred bidders of class 2, $\beta_2^{-1}(\underline{b}) = \underline{c}$;

   b) for all preferred bidders of class 1, $\beta_1^{-1}\left[(1+\rho)\underline{b}\right] = \underline{c}$.

9. *Intermediate-Cost Conditions:*

   a) For all nonpreferred bidders of class 2 and for all $b \in (\underline{b}, p_0)$, $\beta_2^{-1}(b)$ will solve the first-order condition (2.7);

   b) for all preferred bidders of class 1 and for all $b \in ((1+\rho)\underline{b}, p_0)$, $\beta_1^{-1}(b)$ will solve the first-order condition (2.6).

Note that the right-boundary conditions now require the threshold bidders of each class to submit the highest bid. Also, the entry decision already ensures nonpreferred players with high costs (those above c_2^*) do not enter the auction. Previously, we assumed they bid their costs because entry was costless. By accounting for the first-stage entry decision, firms reconsider their bids in the second-stage. Because a threshold bidder will only win when no other bidder chooses to enter the auction, that firm will submit the highest bid possible.

Given the new boundary conditions and the discussion in the preceding subsection, the zero-profit condition defining the threshold cost for a preferred firm can be expressed as

$$[p_0 - \beta_1^{-1}(p_0)] \left(1 - F\left[\beta_1^{-1}(p_0)\right]\right)^{n_1-1} \left(1 - F\left[\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right)\right]\right)^{n_2} - \kappa = 0.$$

Note that, using the upper boundary conditions,

$$\beta_1^{-1}(p_0) = c_1^*,$$

and, by monotonicity of the nonpreferred bid function,

$$\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right) < \beta_2^{-1}(p_0) = c_2^*.$$

Thus, the threshold cost for preferred firms is determined by

$$(p_0 - c_1^*)\left[1 - F(c_1^*)\right]^{n_1 - 1}\left(1 - F\left[\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right)\right]\right)^{n_2} - \kappa = 0. \tag{2.9}$$

For a nonpreferred player, the zero-profit condition can be written

$$\left[p_0 - \beta_2^{-1}(p_0)\right]\left[1 - F\left(\beta_1^{-1}[(1+\rho)p_0]\right)\right]^{n_1}\left(1 - F\left[\beta_2^{-1}(p_0)\right]\right)^{n_2 - 1} - \kappa = 0.$$

Note, too, that a preferred player will never enter if his cost is higher than c_1^*, so, from a nonpreferred player's perspective, the probability of a preferred player's ever bidding greater than p_0 is zero. Thus, the threshold cost for a nonpreferred bidder is determined by

$$(p_0 - c_2^*)\left[1 - F(c_1^*)\right]^{n_1}\left[1 - F(c_2^*)\right]^{n_2 - 1} - \kappa = 0. \tag{2.10}$$

Together, these two conditions determine the thresholds at which preferred and nonpreferred bidders choose to enter the auction. While it is difficult to deduce further details concerning bidder behaviour without explicit distributional assumptions, we can state the following lemma.

> **Lemma 1**: The threshold of a preferred bidder c_1^* will not equal the threshold of a nonpreferred bidder c_2^* when the bid-preference rate ρ is positive.

Proof: Equating the zero-profit condition of nonpreferred bidders from equation (2.10) to the zero-profit condition of preferred bidders from equation (2.9) yields

$$(p_0 - c_2^*)\left[1 - F(c_1^*)\right]^{n_1}\left[1 - F(c_2^*)\right]^{n_2 - 1} - \kappa =$$
$$(p_0 - c_1^*)\left[1 - F(c_1^*)\right]^{n_1 - 1}\left(1 - F\left[\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right)\right]\right)^{n_2} - \kappa.$$

Cancelling terms, yields

$$\begin{aligned}(p_0 - c_2^*)\,[1 - F(c_1^*)]\,[1 - F(c_2^*)]^{n_2-1} &= (p_0 - c_1^*)\left(1 - F\left[\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right)\right]\right)^{n_2} \\ &> (p_0 - c_1^*)\left(1 - F\left[\beta_2^{-1}(p_0)\right]\right)^{n_2} \\ &= (p_0 - c_1^*)\,[1 - F(c_2^*)]^{n_2}\end{aligned}$$

where the inequality obtains by the monotonicity of the (inverse) bid function and the restriction that ρ be positive. Rearranging terms yields

$$\frac{(p_0 - c_2^*)}{(p_0 - c_1^*)} > \frac{[1 - F(c_2^*)]}{[1 - F(c_1^*)]}, \tag{2.11}$$

so c_1^* cannot equal c_2^*.

Note that Lemma 1 *cannot* be made stronger than it is. We show this by example. Suppose the underlying distribution were uniform on the interval $[\underline{c}, \bar{c}]$, then the condition in equation (2.11) becomes

$$\frac{(p_0 - c_2^*)}{(p_0 - c_1^*)} > \frac{(\bar{c} - c_2^*)}{(\bar{c} - c_1^*)},$$

which can be rearranged as

$$\bar{c}(c_1^* - c_2^*) > p_0(c_1^* - c_2^*).$$

This inequality requires c_1^* to be greater than c_2^* and also p_0 to be less than the high cost $\bar{c}$. This condition is also satisfied when p_0 is greater than $\bar{c}$ and c_2^* is greater than c_1^*. However, were p_0 to exceed $\bar{c}$, then the price ceiling would be nonbinding and equilibrium behaviour would require players to bid at most $\bar{c}$, in which case the original inequality would not hold.

In our experience, conditions on the price ceiling are rare. The distribution of costs will determine which class of bidders has the higher threshold. For example, were an exponential (with hazard rate λ) cost distribution assumed instead, then the condition in equation (2.11) would become

$$\frac{(p_0 - c_2^*)}{(p_0 - c_1^*)} > \exp[-\lambda(c_2^* - c_1^*)],$$

which reduces to

$$\log(p_0 - c_2^*) + \lambda c_2^* > \log(p_0 - c_1^*) + \lambda c_1^*.$$

From this alone, it is unclear which threshold cost is higher: it will depend on the values of the buyer's price ceiling (or the upper bound of the cost support $\bar{c}$) and the hazard-rate parameter λ.

3. Numerical Methods

We adapted Bajari's (2001) third computational algorithm to approximate bid functions for each class of bidder using the cumulative distribution and probability density functions of costs in conjunction with the FOCs (2.6) and (2.7) to solve a free boundary-value problem. Under Bajari's approach, it is assumed that the inverse-bid functions can be represented by a (standard basis) polynomial whose coefficients are determined numerically by minimizing a sum-of-squared-residuals function evaluated over a (uniform) grid of points. We improve on this method by employing Chebyshev polynomials and by casting the problem within the MPEC approach advocated by Su and Judd (2008).[7]

Su and Judd have suggested using the MPEC approach to estimate the unknown parameters of theoretical economic models. As in previous research, such as Rust (1987), this involves choosing the structural parameters and, thus, endogenous economic variables to maximize the likelihood of having observed the data, subject to the constraints that the endogenous economic variables are consistent with an equilibrium for the structural parameters. In our application, we use the MPEC approach to discipline the set of Chebyshev coefficients so that the FOCs defining the inverse-bid functions are approximately satisfied, subject to constraints that the entry and boundary conditions defining the equilibrium strategies are satisfied. Of course,

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[7] In chapter 11 of his book, Judd (1998) has described how to approximate functional equations using projection methods; *inter alia* are included descriptions of collocation methods as well as Galerkin and least-squares methods. We thank an anonymous referee for pointing out that our approach can be interpreted as the Chebyshev collocation method, given an appropriate choice for the number of points, relative to the degree of the polynomial(s) and the number of boundary conditions. When the number of points is larger than the number of coefficients, overidentification obtains in the least-squares method, and this can be reliably exploited.

these conditions are functions of the endogenously-determined low bid $\underline{b}$ as well as the thresholds for preferred and nonpreferred bidders, c_1^* and c_2^*. We solve for these endogenous variables in conjunction with the Chebyshev coefficients by formulating the problem as a constrained optimization problem. The approach exploits the benefits of standard numerical optimization methods and provides us with approximate solutions to the intractable system of ODEs that define the inverse-bid functions. Thus, we can account for how behaviour changes endogenously when a preference policy is introduced into a standard model within the symmetric IPCP having entry.

In our application, two different classes of bidders exist, where n_1 is the number of potential preferred bidders and n_2 is the number of potential nonpreferred bidders, with n equalling $(n_1 + n_2)$. Thus, two FOCs exist, resulting in a system of linear ODEs for the inverse-bid functions. These equations can be written in matrix form as

$$\mathbf{c} = [\mathbf{A}(b)]^{-1}\boldsymbol{\iota}_2 \tag{3.1}$$

where $\mathbf{c}$ is the (2×1) vector of derivatives of $\beta_j^{-1}(b)$, $\boldsymbol{\iota}_2$ is a (2×1) vector of ones, and $\mathbf{A}(b)$ is a (2×2) matrix with a typical (k, ℓ)-element given by

$$a_{k\ell}(b) = \left[b - \beta_k^{-1}(b)\right] \frac{[(n_\ell - 1)\mathbb{1}(k = \ell) + n_\ell \mathbb{1}(k \neq \ell)] f\left(\beta_\ell^{-1}\left[b h_{k\ell}(\rho)\right]\right)}{1 - F\left(\beta_\ell^{-1}\left[b h_{k\ell}(\rho)\right]\right)} \; h_{k\ell}(\rho)$$

where $\mathbb{1}(\mathsf{A})$ denotes the indicator function of the event A. Here, $h_{k\ell}(\rho)$ is a function of ρ, which depends on whether class k and ℓ are favoured firms. When k and ℓ are the same (*i.e.*, both preferred or both nonpreferred under the government policy), then $h_{k\ell}(\rho)$ is one. However, when one is preferred and the other is nonpreferred, then $h_{k\ell}(\rho)$ is $[1/(1 + \rho)]$, and when one is nonpreferred and the other is preferred $h_{k\ell}(\rho)$ is $(1 + \rho)$. One component of the system of equations (3.1) could be expressed explicitly after computing the inverse of $\mathbf{A}(b)$ and expanding the system as

$$\begin{aligned} \beta_k^{-1}(b) =& \frac{1 - F\left[\beta_k^{-1}(b)\right]}{(n-1) f\left[\beta_k^{-1}(b)\right]} \\ & \left(\frac{-[n - (n_k + 1)]}{b - \beta_k^{-1}(b)} + \sum_{\ell \neq k} \frac{n_\ell}{b - \frac{1}{h_{k\ell}(\rho)} \beta_\ell^{-1}\left[b h_{k\ell}(\rho)\right]} \right). \end{aligned} \tag{3.2}$$

For his third algorithm, Bajari (2001) assumed that the inverse-bid functions can be represented by polynomials. Under this assumption, one can evaluate the polynomials at a grid of points and then use a nonlinear least-squares solver to find the coefficients that make a quadratic objective function smallest. In our method, we assumed that the inverse-bid function of each player can be represented by a Chebyshev polynomial. We then minimized the sum-of-squared-residuals function subject to the entry and boundary conditions of the preferred and nonpreferred firms. Specifically, we assumed that the inverse-bid function for class j can be expressed as

$$\beta_j^{-1}(b_t;\boldsymbol{\alpha}_j,\underline{b},c_1^*,c_2^*) = \underline{b} + \sum_{m=0}^{M} \alpha_{j,m}\mathbb{T}_m[x(b_t)] \qquad \text{for } j=1,2; t=1,\ldots,T \tag{3.3}$$

where $x(\cdot)$ lies in the interval $[-1,1]$ and where, for clarity, we have explicitly defined it as a transformation of the bid under consideration. Here, $\mathbb{T}_m(\cdot)$ denotes the m^{th} Chebyshev polynomial and the vector $\boldsymbol{\alpha}_j$ collects the polynomial coefficients for a firm of class j, with T representing the number of bids considered in the algorithm. Collect the vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ in $\boldsymbol{\alpha}$. The points $\{b_t\}_{t=1}^T$ were chosen to be the Chebyshev points on the interval $[\underline{b},p_0]$ when the nonpreferred firms' FOC was considered and the Chebyshev points on the interval $[(1+\rho)\underline{b},p_0]$ when the preferred firms' FOC was considered. The transformation $x(\cdot)$ simply maps the Chebyshev points from the interval of interest to the interval $[-1,1]$, following Judd (1998), which is the domain on which Chebyshev polynomials are defined. In particular, for nonpreferred firms

$$x_t \equiv x(b_t) = \frac{2b_t - \underline{b} - p_0}{p_0 - \underline{b}},$$

while for preferred firms

$$x_t \equiv x(b_t) = \frac{2b_t - (1+\rho)\underline{b} - p_0}{p_0 - (1+\rho)\underline{b}}.$$

Note that the true value of $\underline{b}$ is unknown *a priori* and is endogenously determined. The other endogenous variables—the thresholds c_1^* and c_2^*—have not been discussed yet because they are determined via the entry conditions, which are imposed as constraints on the minimization problem.

Thus, in our application with two classes, we solve for $2(M+1)$ polynomial coefficients and three endogenous variables—the common lower bound $\underline{b}$ as well as the thresholds c_1^* and c_2^*. One can rewrite the system of ODEs characterized by (3.1) as

$$\mathbf{g}(b) \equiv \boldsymbol{\iota}_2 - \mathbf{A}(b)\mathbf{c} = \mathbf{0}_2, \tag{3.4}$$

where $\mathbf{g}(b)$ is a (2×1) vector. At an exact solution, the elements of the vector $\mathbf{g}(b)$ are zero for each player at every bid b.

The algorithm works as follows: first, construct a vector of T bids from the region of feasible bids; transform this vector to the interval $[-1, 1]$. In doing so, we choose the Chebyshev nodes. For each bid, determine $\mathbf{g}(b_t)$. Define the objective function Q to be

$$Q\left(\boldsymbol{\alpha}, \underline{b}, c_1^*, c_2^*\right) \equiv \sum_{t=1}^{T} \sum_{j=1}^{2} \left[g_j\left(b_t\right)\right]^2.$$

The objective is then to choose the $2(M+1)$ polynomial coefficients as well as $\underline{b}$, c_1^*, and c_2^* to minimize Q subject to the entry conditions

$$[p_0 - \beta_1^{-1}(p_0)]\left(1 - F[\beta_1^{-1}(p_0)]\right)^{n_1-1}\left(1 - F\left[\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right)\right]\right)^{n_2} = \kappa$$

and

$$[p_0 - \beta_2^{-1}(p_0)]\left(1 - F[\beta_1^{-1}(p_0)]\right)^{n_1}\left(1 - F[\beta_2^{-1}(p_0)]\right)^{n_2-1} = \kappa,$$

as well as the previously-discussed boundary conditions

$$\beta_2^{-1}(\underline{b}) = \underline{c},\ \beta_1^{-1}\left[(1+\rho)\underline{b}\right] = \underline{c},\ \beta_2^{-1}(p_0) = c_2^*,\ \text{and}\ \beta_1^{-1}(p_0) = c_1^*$$

where the inverse bid functions are now represented by the Chebyshev polynomials specified by equation (3.3). The quadratic form Q is simply the sum of squared residuals, so this approach converts the problem of solving a system of ODEs to solving a constrained nonlinear minimization problem. A direct optimization routine can then be used to solve for the $[2(M+1)+(2+1)]$ unknowns given an initial guess.

4. Bid-Function Approximations

In our application, we considered auctions involving five potential bidders, each of whom draws a cost independently from the same distribution. Krasnokutskya and Seim (2007) have reported that, from January 2002 to April 2005, about forty percent of the firms who bid on California Department of Transportation (Caltrans) projects were given preference. Thus, we assumed that two of the five bidders receive preferential treatment, while the remaining three bidders are considered nonpreferred. We considered four different cost distributions, all of which had compact support on the interval $[1, 4]$. These distributions were truncated over this cost support and the probability density functions are bounded away from zero. Specifically, we present results for the following truncated distributions: (shifted) exponential, normal, uniform, and Weibull. The probability density functions and cumulative distribution functions for these distributions are depicted in figures 1 and 2, respectively.

The uniform distribution has a mean of 2.5, as does the truncated normal, whose standard-deviation parameter is 0.5. The truncated exponential distribution has been shifted (often referred to as a two-parameter exponential distribution) by $\underline{c}$ equal to one, and has a mean of two. The truncated Weibull distribution has a mean of three. We set p_0 equal to $\bar{c}$, four, in all cases except those involving the uniform distribution, where it was set equal to 3.99.

We used the structural estimates of Li and Zheng (2007) to inform our choice of the entry fee. Specifically, Li and Zheng found that the average ratio of the entry fee to the private value of bidders in their sample was about 7.48 percent. We, therefore, set the entry fee κ equal to 0.2, which is eight percent of the average cost draw of the uniform and normal distributions, ten percent of the average exponential cost draw, and 6.67 percent of the average cost when costs are distributed according to the truncated Weibull law.

We implemented our methods using the programming language AMPL, choosing the polynomial coefficients and endogenous variables as

$$\left(\hat{\boldsymbol{\alpha}}, \hat{\underline{b}}, \hat{c}_1^*, \hat{c}_2^*\right) = \underset{\boldsymbol{\alpha}, \underline{b}, c_1^*, c_2^*}{\operatorname{argmin}} \sum_{t=1}^{T} \sum_{j=1}^{2} \left[g_j\left(b_t\right)\right]^2$$

Figure 1

Truncated Probability Density Functions

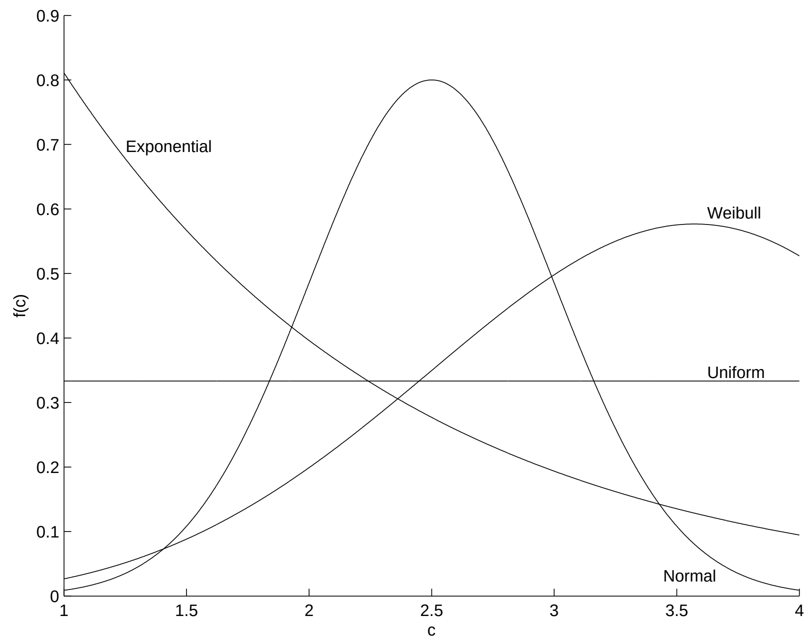
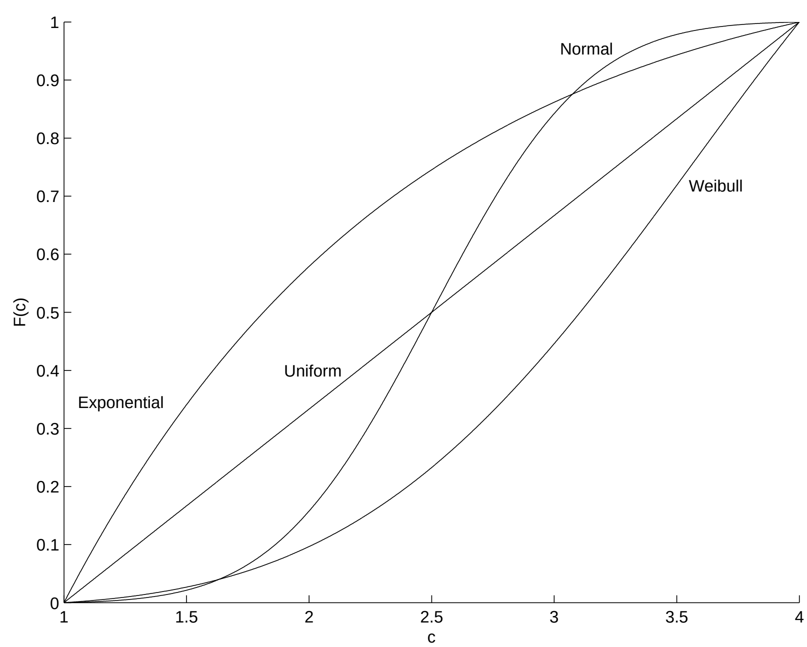


Figure 2

Truncated Cumulative Distribution Functions



subject to

$$
\begin{aligned}
[p_0 - \beta_1^{-1}(p_0)] \left(1 - F[\beta_1^{-1}(p_0)]\right)^{n_1 - 1} \left(1 - F\left[\beta_2^{-1}\left(\frac{p_0}{1+\rho}\right)\right]\right)^{n_2} &= \kappa \\
[p_0 - \beta_2^{-1}(p_0)] \left(1 - F[\beta_1^{-1}(p_0)]\right)^{n_1} \left(1 - F[\beta_2^{-1}(p_0)]\right)^{n_2 - 1} &= \kappa \\
\beta_2^{-1}(\underline{b}) &= \underline{c} \\
\beta_1^{-1}\left[(1+\rho)\underline{b}\right] &= \underline{c} \\
\beta_2^{-1}(p_0) &= c_2^* \\
\beta_1^{-1}(p_0) &= c_1^*.
\end{aligned}
$$

In addition, we imposed monotonicity and rationality (players bid higher than their costs) on the inverse bid functions at all T points. Using AMPL has a number of advantages: first, its user interface admits choice among a variety of nonlinear optimization solvers, including SNOPT and MINOS, without having to modify code significantly. Second, AMPL can also perform automatic differentiation on nonlinear programming problems. Third, the language is free. In fact, users can run the code for free using the NEOS Server online. The code for this problem ran in about one second.[8]

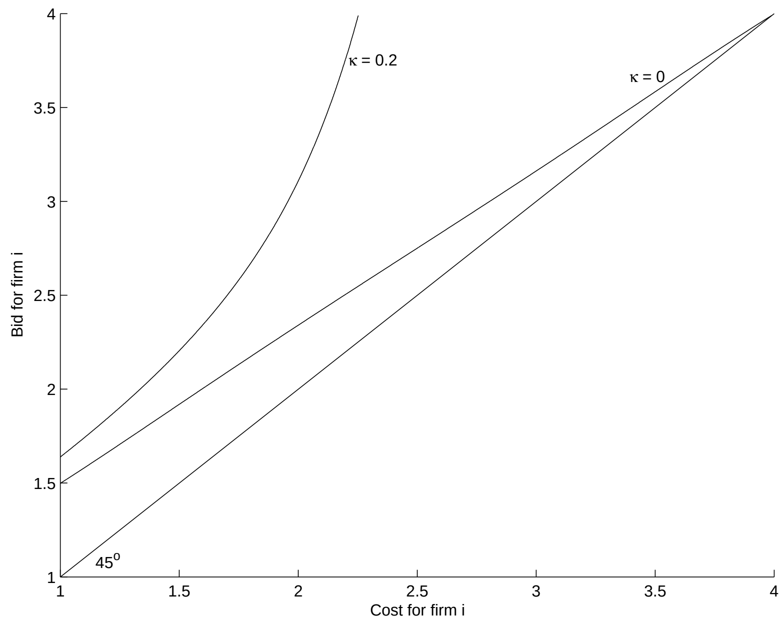
The equilibrium bid functions when the preference rate is zero and the costs of bidders are drawn from the uniform distribution are depicted in figure 3. Because the preference rate is zero, the bid functions of preferred and nonpreferred players are identical. In this figure, we highlight the effect of a positive entry fee on behaviour at a low-price, sealed-bid auction with symmetric players. Without preferences, this problem simplifies to a symmetric auction with entry; the equilibrium bid function can then be represented as

$$\beta(c) = c + \frac{\int_c^{c^*} [1 - F(u)]^{n-1} \, du}{[1 - F(c)]^{n-1}} + \frac{[1 - F(c^*)]^{n-1}}{[1 - F(c)]^{n-1}} (p_0 - c^*).$$

[8] To promote future research, all programmes and a guide to running the programmes can be downloaded from

`http://myweb.uiowa.edu/tphubbar`

Figure 3
Example of Computed Bid Functions for
Uniform Case with No Preference Policy



Note that, without entry, this simplifies to equation (2.3) because c^* equals p_0. This closed-form representation illustrates that, in this model, without specific restrictions on the underlying distribution, bidding could become more or less aggressive than in the absence of an entry fee. The second term on the right-hand side of the equation becomes smaller as the integral in the numerator is over a smaller range (because c^* is less than p_0), while the third term on the right-hand side of the equation is positive and does not exist in the model when entry fees are zero. The bid functions in figure 3 depict clearly that, when the underlying cost distribution is uniform, bidders are less aggressive when entry is costly than when it is cheap. In fact, except for bidders with the lowest costs, all firms inflate their bids by more than the entry fee κ of 0.2.

When bid preferences are introduced and there is no entry fee at the auction, class 1 firms then begin to inflate their bids because they know the government will then discount them for the purposes of evaluation: the preference effect is important. In figure 4, we depict the equilibrium bid functions of class 1 firms as the preference rate changes, when the cost distribution is uniform. As the preference rate increases, the preferred firms inflate their bids. This preference effect is particularly distinct for bidders with low costs. In contrast, in figure 5, we depict the equilibrium bid functions for class 1 firms using the same preference rates, but where entry is costly. Note that, when the cost distribution is uniform, the likelihood of preferred firms' entering the auction increases as the preference rate increases. Firms with costs near the low end of the distribution use the increase in the preference rate to inflate their bids by more than when no entry fee exists. However, firms with costs near the respective threshold cost for each class bid more aggressively, knowing that firms with higher cost draws are willing to enter the auction given the higher preference rate. The behaviour of firms with cost draws near the average cost draw, conditional on entry, is less distinct when isolating the behaviour of preferred firms—their bidding strategies at interior points are determined simultaneously with the bidding strategies of nonpreferred firms.

Note that introducing a bid-preference policy results in a competitive effect: because the class 2 firms know they are at a disadvantage, they bid more aggressively than under equal treatment of bids. In figure 6, we depict the equilibrium bid functions for the class 2 firms as the preference rate changes, using the same cost distribution and preference policies of earlier figures, when no entry fee existed. In contrast to the preferred bidders, it is the nonpreferred firms with exceptionally low costs that change their bidding strategies the least. The theoretical restriction that $\beta_1(\bar{c})$ equals $\bar{c}$ (when n_1 is greater than one) implies that a nonpreferred firm will never win the auction if it draws a cost in the range $\left(\frac{\bar{c}}{1+\rho}, \bar{c}\right]$. In line with this restriction, the nonpreferred firms bid closer to their costs as the point $\frac{\bar{c}}{1+\rho}$ is approached. Note, too, because no nonpreferred firm can win with a cost in the range $\left(\frac{\bar{c}}{1+\rho}, \bar{c}\right]$, any bidding strategy for these costs corresponds to a Nash equilibrium bid function.

In figure 7, we depict the equilibrium bid functions for the nonpreferred firms using the same preference rates, but where entry is costly. When the cost distribution is uniform, preferred firms are less and less likely to enter the auction, which can be seen by the decrease in the threshold cost. The competitive effect is noticeable for low-cost firms, in this uniform case, when a positive entry fee exists. The firms with very low cost-draws have a high probability of winning the auction, so they

Figure 4
Endogenous Changes in Preferred Players' Behaviour
No Entry Cost ($\boldsymbol{\kappa = 0}$)

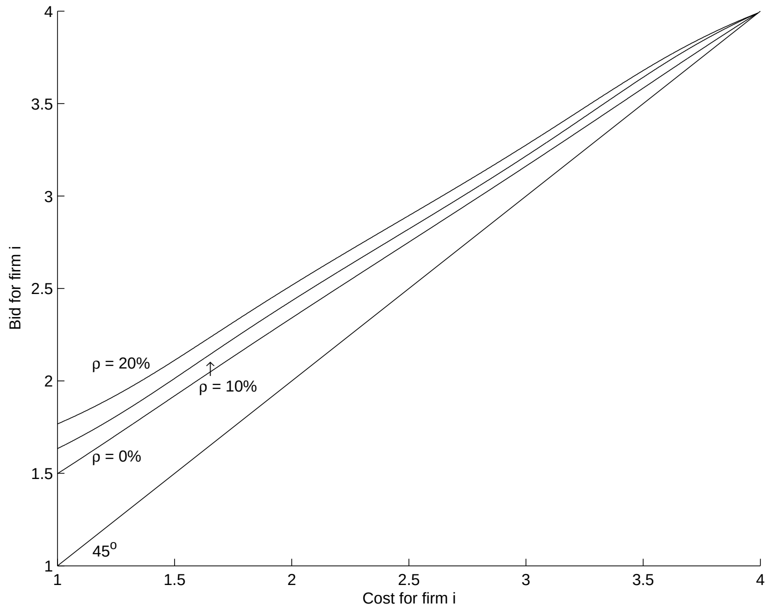
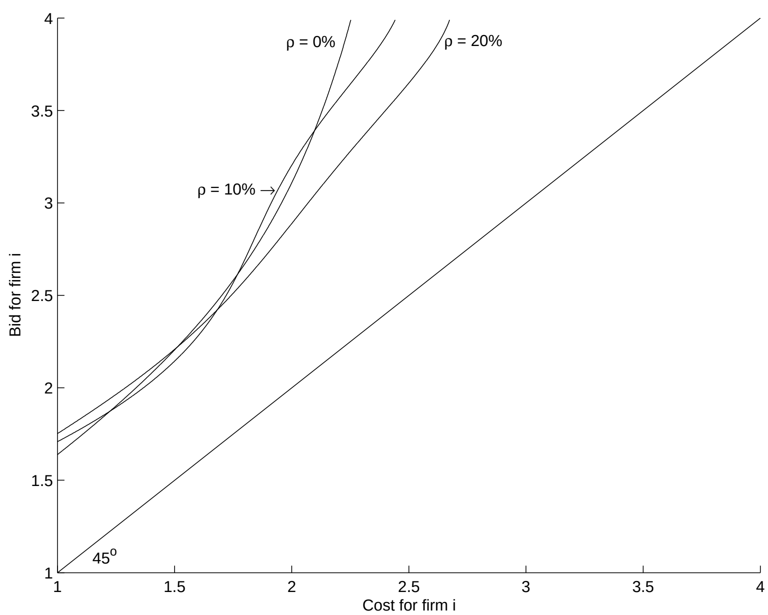


Figure 5
Endogenous Changes in Preferred Players' Behaviour
Positive Entry Cost ($\boldsymbol{\kappa = 0.2}$)



respond more aggressively than would otherwise be the case in order to ensure that the preference policy does not prevent them from winning the auction. Nonpreferred firms with costs near the respective threshold increase their bids, hoping to win the auction when no other firms enter.

Space limitations prevent us from presenting more than a few of the approximated bid functions. Changing the underlying distribution, or the composition of the potential bidders at the auction, or the preference rate, or the entry fee, or the number of points used in the algorithm, or the degree of the Chebyshev polynomials can be undertaken relatively effortlessly. While we solved for the optimal bid functions when the number of preferred players n_1 was two and the number of nonpreferred players n_2 was three, we varied the preference rate ρ from zero to fifty percent. We also considered entry fees of either zero or 0.2. In the next section, we present the results of our simulation experiments using the exponential, normal, uniform, and Weibull laws.

5. Simulation Experiments

After approximating the equilibrium bid functions without and with entry fees and for different preference rates, we then simulated the outcomes at auctions to quantify the trilogy of effects—preference, competitive, and participation. We assumed costs were drawn from either the exponential, the normal, the uniform, or the Weibull distributions presented in the previous section. We used the same simulated draws for models with no entry fee and those with endogenous entry. In particular, we were interested in estimating the expected procurement cost to the buyer and the inefficiency caused under each scenario.

Specifically, we simulated data from $10,000$ auctions for each potential bidder using the inverse cumulative distribution function method. This involved generating a sample of $10,000$ uniform random numbers from the interval $[0, 1]$ for each player. The random draws represented values of the cumulative distribution function. Recall that the cumulative distribution function of any continuous random variable is distributed uniformly on the interval $[0, 1]$. For each uniform draw, we calculated an associated random cost by finding the cost such that the cumulative distribution function $F(\cdot)$

Figure 6

Endogenous Changes in Nonpreferred Players' Behaviour

No Entry Cost ($\kappa = 0$)

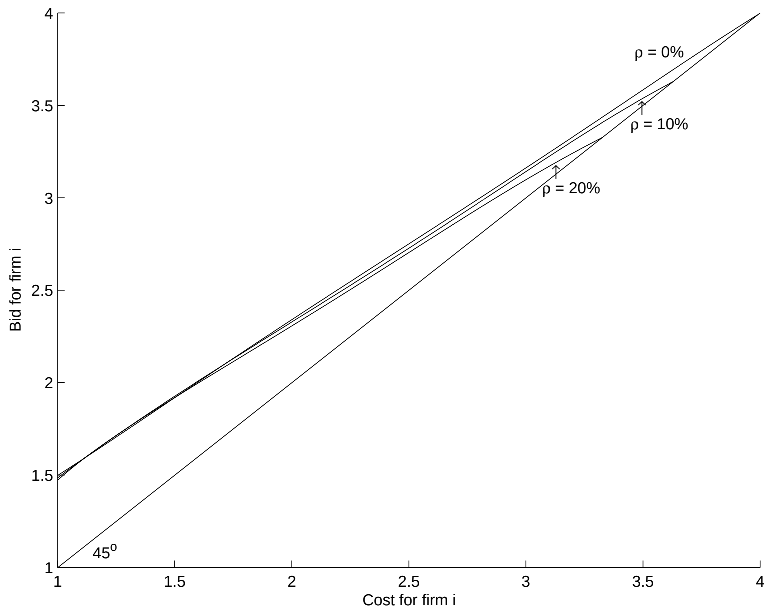
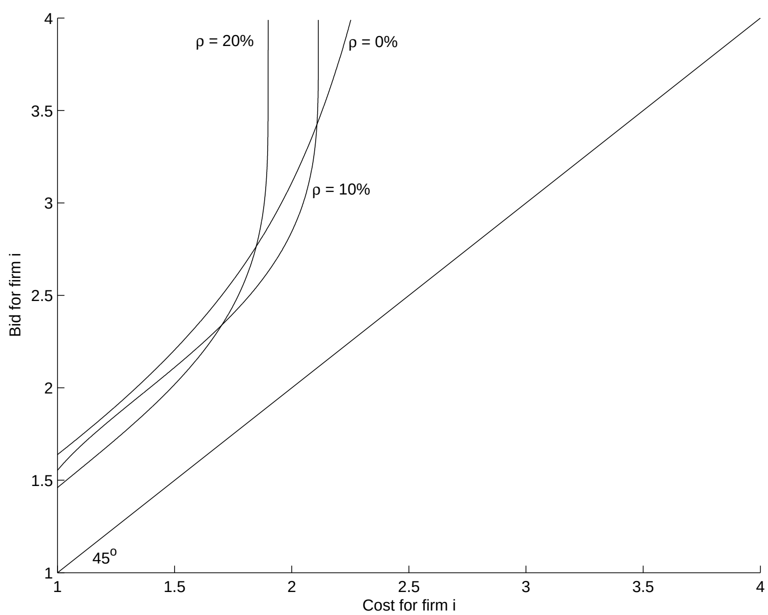


Figure 7

Endogenous Changes in Nonpreferred Players' Behaviour

Positive Entry Cost ($\kappa = 0.2$)



evaluated at the cost equals the random draw. We then used the approximated bid functions for the player and the example under consideration to calculate each bid $\hat{b}_{i\ell}$, for player i at auction ℓ, given $\hat{c}_{i\ell}$. The bids from preferred players were then adjusted given the government's preference policy in each specific case. The government cost was determined for each simulated auction after comparing the adjusted preferred bids and the nonpreferred bids. We then averaged over auctions to get the average procurement cost for the government in each example considered. In the event of misallocation, where a preferred (nonpreferred) firm won the auction even though a nonpreferred (preferred) firm had a lower cost, we also approximated two measures of inefficiency for each preference scenario k using the following two formulæ:

$$\begin{aligned} R_k = \sum_{\ell=1}^{I_k} \Bigg[& \mathbb{1}\left(\frac{b_{\min_{1\ell}}}{1+\rho_k} \leq b_{\min_{2\ell}} \cap c_{\min_{1\ell}} \geq c_{\min_{2\ell}}\right)(c_{1\ell} - c_{2\ell}) \\ & + \mathbb{1}\left(\frac{b_{\min_{1\ell}}}{1+\rho_k} \geq b_{\min_{2\ell}} \cap c_{\min_{1\ell}} \leq c_{\min_{2\ell}}\right)(c_{2\ell} - c_{1\ell})\Bigg] \Big/ \sum_{\ell=1}^{L_k} w_\ell \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} S_k = \sum_{\ell=1}^{I_k} \Bigg[& \mathbb{1}\left(\frac{b_{\min_{1\ell}}}{1+\rho_k} \leq b_{\min_{2\ell}} \cap c_{\min_{1\ell}} \geq c_{\min_{2\ell}}\right)(c_{1\ell} - c_{2\ell}) \\ & + \mathbb{1}\left(\frac{b_{\min_{1\ell}}}{1+\rho_k} \geq b_{\min_{2\ell}} \cap c_{\min_{1\ell}} \leq c_{\min_{2\ell}}\right)(c_{2\ell} - c_{1\ell})\Bigg] \Big/ \sum_{\ell=1}^{I_k} w_\ell \end{aligned} \tag{5.2}$$

where L_k is the number of auctions at which at least one bidder submits a bid for scenario k, I_k is the number of inefficient auctions for scenario k, and $b_{\min_{1\ell}}$ and $b_{\min_{2\ell}}$ are the lowest bids tendered by a preferred and nonpreferred player, respectively, at auction ℓ. Here, w_ℓ denotes the winning bid at auction ℓ. Thus, $b_{\min_{j\ell}}$, where j is $\{1, 2\}$, is in the set

$$\Gamma_j \equiv \{b_{\min_{j\ell}} |\; b_{\min_{j\ell}} \leq b_{ji\ell} \;\forall \text{ bidders } i \text{ of preference class } j \text{ at auction } \ell\}.$$

The inefficiency level R_k (S_k) provides a measure of the average difference in costs when inefficient allocations obtain in scenario k relative to the winning bids (the winning bids at inefficient auctions). The indicator component ensures that when

Table 1

Average Government Cost

	Model	$\rho = 0\%$	$\rho = 5\%$	$\rho = 10\%$	$\rho = 15\%$	$\rho = 20\%$
No Entry Cost	Exponential	1.4493	1.4594	1.4789	1.5024	1.5305
		(0.2243)	(0.2281)	(0.2324)	(0.2386)	(0.2473)
	Normal	2.2562	2.2701	2.2955	2.3278	2.3650
		(0.1614)	(0.1683)	(0.1928)	(0.2247)	(0.2603)
	Uniform	1.9173	1.9401	1.9666	1.9970	2.0288
		(0.3545)	(0.3540)	(0.3576)	(0.3682)	(0.3817)
	Weibull	2.7496	2.7905	2.8386	2.8930	2.9502
		(0.2641)	(0.2659)	(0.2845)	(0.3176)	(0.3567)
Entry Cost	Exponential	1.7080	1.6878	1.6687	1.6602	1.6511
		(0.4814)	(0.4663)	(0.4414)	(0.4167)	(0.3891)
	Normal	2.6957	2.6339	2.5738	2.4951	2.4526
		(0.3698)	(0.3621)	(0.3473)	(0.3386)	(0.3124)
	Uniform	2.1825	2.1506	2.1256	2.1195	2.1168
		(0.5071)	(0.4891)	(0.4816)	(0.4965)	(0.5337)
	Weibull	3.2328	3.1585	3.1046	3.0887	3.0833
		(0.2971)	(0.3046)	(0.3420)	(0.3710)	(0.3952)

the auction outcome is efficient, there is no contribution to the sum; however, with inefficient allocations the difference in costs between the winning bidder (with higher cost) and the lowest-cost bidder is considered. These measures can then be interpreted as the average *economic loss* in scenario k, under different normalizations.

The means and standard deviations (in parentheses) of expected government procurement costs for each case are reported in table 1 for commonly-observed preference rates.[9] Results from the models with no entry fee and those with an entry fee are included. Note that, when entry is costless, the expected procurement cost to the government increases monotonically (for all distributions) as the preference rate increases; *i.e.*, the preference effect dominates the competitive effect. In fact, this finding was originally noted by McAfee and McMillan (1989) in Corollary 4.[10] When

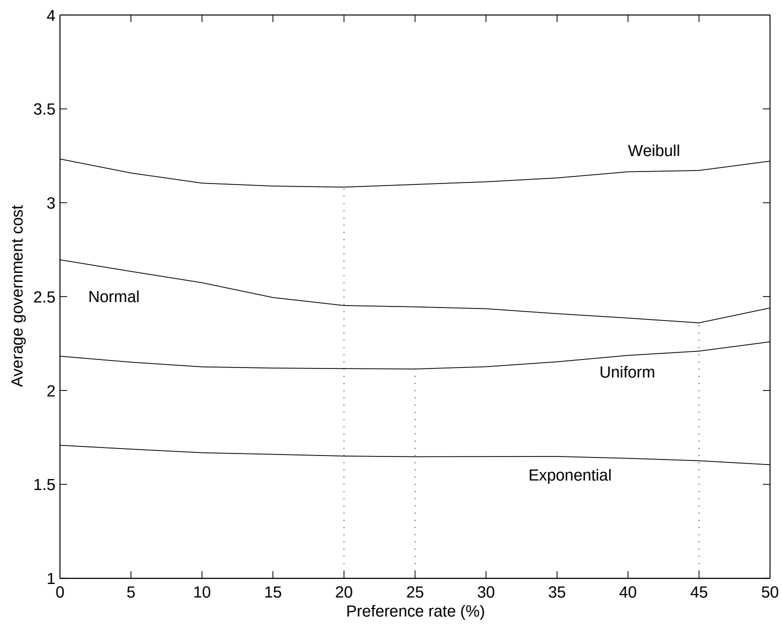
[9] From now on, unless stated explicitly, in table 1, we present the average procurement cost considering only auctions where at least one bidder participated. If, instead, we assumed that, when no bidder participated at the auction the government bought at price p_0, then the trends in these results remain the same, but the levels increase.

[10] McAfee and McMillan showed that no preferential treatment should be awarded to bidders (assuming the government wants to minimize its expected procurement cost) if the cost distribution of one class of bidders is a multiplicative transformation of the cost distribution of the other class; *i.e.*,

$$F_i(c) = \theta F_j(c) \quad \theta > 0.$$

In the symmetric IPCP, θ equals one, so this condition is satisfied.

Figure 8
Average Government Cost



the entry fee is positive, it is clear that the expected costs to the government increase, but that a preference policy can lower expected costs. Thus, the result of McAfee and McMillan breaks-down when entry is endogenous. In fact, expected procurement costs decrease monotonically for all preference rates considered in the table. In figure 8, we depict the expected costs to the government as the preference rate changes for each distribution using an expanded set of preference rates; we considered ρs between zero and fifty percent.[11] Note that there exist "optimal" preference rates at which the expected costs to the government are minimized (although we do not mean to suggest this is the reason these policies are used in practice) when the cost distribution is normal (45 percent), uniform (25 percent), or Weibull (20 percent). If the cost distribution is exponential, then the average government cost continues to decrease when ρ equals 0.5. Clearly, the optimal preference rate depends on the distribution of costs for firms.

In table 2, we present some statistics of interest for the standard preference rates

[11] While the reader may find a preference rate of fifty percent absurdly high, Flambard and Perrigne (2006) have reported such rates in defense contracts.

presented above when there is a positive entry fee. For all of the distributions we considered, the competitive effect exists: the preference policy induces nonpreferred firms to behave more competitively than under equal treatment of bids. This can be seen by the decrease in the expected bid of a nonpreferred player (conditional on entry) when the preference rate increases. When the cost distribution is uniform or Weibull, preferred bidders use the preference policy to inflate their bids. In the exponential and normal cases, however, the average bid by a preferred bidder is less than under equal treatment of bids. In these cases, preferred firms are also less likely to enter the auction than they were before. These observations are not unrelated: the average bid of preferred players is decreasing because only the preferred firms with the lowest costs enter the auction—the preferred entry threshold decreases in the preference rate. This occurs because the competitive behaviour of nonpreferred firm makes preferred firms less and less likely to win the auction.

Preference policies are often enacted by governments to encourage participation from under-represented groups of bidders. Our simulation results illustrate that these policies are effective if the cost distributions are uniform or Weibull: the preference policy increases the probability of a preferred firm's entering the auction. The opposite effect occurs when the cost distributions are exponential or normal: a preference policy reduces entry by preferred firms, in contrast to its original purpose. Note that, in all of the cases we considered, the expected number of bidders at the auctions remained fairly stable, but the composition of the bidders changed as Lemma 1 would predict. When preferential treatment induces entry by preferred (nonpreferred) firms, nonpreferred (preferred) firms are less likely to enter.

In table 3, we report the proportion of inefficient allocations that obtained in each of the experiments. In all cases, the relative frequency of inefficiency increases in the preference rate. For entry fees, we present two numbers in each case, the first is the fraction of auctions, where at least one bid is submitted, that is inefficient, while the second is the fraction of all simulated auctions that is inefficient, because either the auction was won by a bidder with a higher cost than another bidder or no firms chose to enter the auction, which we assumed would be inefficient as the government would then have to pay p_0 to an outside party with cost p_0 for the task to be performed.

Table 2

Other Statistics of Interest with Positive Entry Cost

	Model	$\rho = 0\%$	$\rho = 5\%$	$\rho = 10\%$	$\rho = 15\%$	$\rho = 20\%$
Exponential	Pr(preferred enters)	0.4572	0.4440	0.4382	0.4411	0.4387
	Pr(nonpreferred enters)	0.4582	0.4683	0.4725	0.4706	0.4724
	$\mathcal{E}$(number of bidders)	2.2891	2.2930	2.2939	2.2942	2.2946
	$\mathcal{E}$(preferred bid)	2.0700	2.0420	1.9997	1.9531	1.9222
	$\mathcal{E}$(nonpreferred bid)	2.0814	2.0453	2.0067	1.9724	1.9298
Normal	Pr(preferred enters)	0.4077	0.3851	0.3797	0.3619	0.3517
	Pr(nonpreferred enters)	0.4074	0.4248	0.4282	0.4425	0.4513
	$\mathcal{E}$(number of bidders)	2.0374	2.0446	2.0440	2.0512	2.0572
	$\mathcal{E}$(preferred bid)	2.9264	2.9102	2.8755	2.8542	2.7640
	$\mathcal{E}$(nonpreferred bid)	2.9321	2.8343	2.7270	2.6051	2.5309
Uniform	Pr(preferred enters)	0.4188	0.4674	0.4817	0.5162	0.5607
	Pr(nonpreferred enters)	0.4189	0.3820	0.3689	0.3391	0.2975
	$\mathcal{E}$(number of bidders)	2.0942	2.0807	2.0700	2.0498	2.0138
	$\mathcal{E}$(preferred bid)	2.5117	2.6224	2.6690	2.6916	2.7179
	$\mathcal{E}$(nonpreferred bid)	2.5203	2.3476	2.2311	2.1443	2.0235
Weibull	Pr(preferred enters)	0.3591	0.4055	0.4495	0.4713	0.4869
	Pr(nonpreferred enters)	0.3559	0.3253	0.2947	0.2774	0.2652
	$\mathcal{E}$(number of bidders)	1.7860	1.7868	1.7829	1.7746	1.7692
	$\mathcal{E}$(preferred bid)	3.3856	3.3993	3.4385	3.4696	3.4861
	$\mathcal{E}$(nonpreferred bid)	3.3846	3.2310	3.0693	2.9616	2.8778

Table 3

Proportion of Inefficient Auctions

	Model	$\rho = 0\%$	$\rho = 5\%$	$\rho = 10\%$	$\rho = 15\%$	$\rho = 20\%$
No Entry Cost	Exponential	0.0000	0.0469	0.0928	0.1317	0.1709
	Normal	0.0000	0.0636	0.1229	0.1747	0.2205
	Uniform	0.0000	0.0294	0.0582	0.0876	0.1111
	Weibull	0.0000	0.0384	0.0798	0.1189	0.1526
Entry Cost	Exponential	0.0000	0.0337	0.0655	0.1001	0.1292
		0.0454	0.0770	0.1077	0.1405	0.1686
	Normal	0.0000	0.0394	0.0523	0.0898	0.1148
		0.0718	0.1075	0.1191	0.1532	0.1761
	Uniform	0.0000	0.0284	0.0525	0.0786	0.1053
		0.0652	0.0937	0.1166	0.1427	0.1669
	Weibull	0.0000	0.0452	0.0734	0.1007	0.1318
		0.1088	0.1491	0.1713	0.1944	0.2224

Note that, for this latter proportion, inefficiencies arise even when the preference rate is zero because no bidder chooses to enter the auction given its cost draw. Without question, the incidence of inefficiencies induced by the preference policy is relatively large, but the economic importance of the misallocations is relatively small.

In table 4, we report two measures of inefficiency in each scenario, computed using the formulæ of equation (5.1) and (5.2). The first set of values represent the total inefficiency across auctions where at least one bid is submitted, normalized by the sum of all winning bids, while the second set of values are normalized by the sum of winning bids from only the inefficient auctions. The inefficiency values are typically less in the models with entry using these measures. Thus, incorporating a preference policy in an auction without entry will usually overstate the economic cost given this comparison. Note, too, that while the preference policy certainly increases the frequency and value of inefficiency, from an economic perspective the cost of the inefficiency is quite small, representing less than one percent of the average winning bid over all auctions, and less than seven percent of the winning bid when an inefficiency does obtain. For the cases with a positive entry fee, we report one additional measure which is computed as the total inefficiency, now including the cases where the government awards the contract to an outside party at p_0 should no bidder enter the auction, normalized by the sum of winning bids, including the auctions where the buyer paid p_0. This measure is consistent with the second set of values from table 3. Note, first, that these numbers are significantly higher than all other entries (either with or without an entry cost). However, these numbers are much more stable as the preference rate changes; *i.e.*, much of the inefficiency arises because of the positive entry cost and not because of the preference policy.

6. Conclusion

Within the IPCP, we investigated the effects of a commonly-observed bid-preference policy in a standard model of a low-price, sealed-bid procurement auction when bidder participation is endogenous. We found an important preference effect—preferred bidders inflate their bids—as well as an important competitive effect—nonpreferred bidders behave more aggressively than under equal treatment of bids in an effort counteract the preferential policy. The importance of the participation effect was small and depended on the distribution of costs. Under four assumed distributions, ones often employed in structural-econometric research, we found preference policies are not expensive in terms of economic cost (inefficiency) and can lead to cost savings for the government. However, in practice, these policies are not motivated as a way to reduce government expenditures, but rather as a means to increase participation

Table 4

Average Value of Inefficiency

	Model	$\rho = 0\%$	$\rho = 5\%$	$\rho = 10\%$	$\rho = 15\%$	$\rho = 20\%$
	Exponential	0.0000	0.0010	0.0038	0.0075	0.0125
		0.0000	0.0195	0.0369	0.0509	0.0645
No Entry	Normal	0.0000	0.0013	0.0048	0.0094	0.0150
		0.0000	0.0193	0.0353	0.0487	0.0610
Cost	Uniform	0.0000	0.0006	0.0022	0.0049	0.0077
		0.0000	0.0174	0.0319	0.0464	0.0572
	Weibull	0.0000	0.0007	0.0030	0.0063	0.0102
		0.0000	0.0167	0.0324	0.0463	0.0579
	Exponential	0.0000	0.0004	0.0016	0.0040	0.0067
		0.0000	0.0119	0.0233	0.0376	0.0484
		0.0515	0.0518	0.0539	0.0559	0.0589
Entry	Normal	0.0000	0.0007	0.0013	0.0033	0.0056
		0.0000	0.0162	0.0233	0.0347	0.0454
		0.0384	0.0393	0.0405	0.0430	0.0455
Cost	Uniform	0.0000	0.0006	0.0017	0.0038	0.0088
		0.0000	0.0176	0.0299	0.0398	0.0623
		0.0409	0.0439	0.0461	0.0507	0.0568
	Weibull	0.0000	0.0010	0.0025	0.0045	0.0073
		0.0000	0.0199	0.0295	0.0393	0.0482
		0.0319	0.0338	0.0353	0.0372	0.0403

from under-represented classes of bidders. Our research has shown that a preference policy need not be the best way to achieve this goal. In fact, in the exponential and normal cases, the policy reduced entry by preferred bidders.

As a first step in considering the effects of a bid-preference policy, we chose to focus on behaviour within the symmetric IPCP. Within this paradigm, the bid-preference policy is the only cause of inefficiencies. Within the asymmetric IPCP, differences in the cost distributions of bidders could also introduce other inefficiencies. Extending our model to the asymmetric IPCP is a natural next step for future research.

A. Appendix

In this appendix, we prove that, with a positive preference rate ρ, a preferred bidder who draws the highest cost will bid more than the high cost if and only if he is the only preferred player at the auction.

When n_1 exceeds one, more than one player receives preference at the auction and each preferred player solves the following problem:

$$\max_b \; (b-c)\left(1-F\left[\beta_1^{-1}(b)\right]\right)^{n_1-1}\left(1-F\left[\beta_2^{-1}\left(\frac{b}{1+\rho}\right)\right]\right)^{n_2}. \tag{A.1}$$

Rearranging the FOC for a representative player implies:

$$b = c + \frac{1}{\left(\frac{(n_1-1)f\left[\beta_1^{-1}(b)\right]\beta_1^{-1'}(b)}{\left(1-F\left[\beta_1^{-1}(b)\right]\right)} + \frac{n_2 f\left[\beta_2^{-1}\left(\frac{b}{1+\rho}\right)\right]\frac{1}{1+\rho}\beta_2^{-1'}\left(\frac{b}{1+\rho}\right)}{\left(1-F\left[\beta_2^{-1}\left(\frac{b}{1+\rho}\right)\right]\right)}\right)}. \tag{A.2}$$

As $b \to \bar{b}$, $\beta_1^{-1}(b) \to \beta_1^{-1}(\bar{b})$ since $\beta_1^{-1}(\cdot)$ is a continuous, monotonically increasing function. By definition, $\beta_1^{-1}(\bar{b})$ is $\bar{c}$, so

$$\lim_{b \to \bar{b}} F\left[\beta_1^{-1}(b)\right] = F(\bar{c}) = 1.$$

Thus, the survivor function $[1-F(\cdot)] \to 0$ and the denominator of the second term on the right-hand side of (A.2) approaches ∞ when $b \to \bar{b}$. Consequently, the second term goes to zero and the preferred firm's bid equals its cost; *i.e.*, $\bar{b} = \bar{c}$.

The argument holds provided $F\left[\beta_1^{-1}(b)\right] \to F(\bar{c})$ at a faster rate than $\beta_1^{-1'}(b)$ goes to zero. In particular, this is true if $\beta_1^{-1}(\cdot)$ is a monotonically increasing function, which is the case if the regularity conditions of Myerson (1981) are satisfied.

If only one preferred bidder exists at the auction, then the preferred player solves the following problem:

$$\max_b \; (b-c)\left(1-F\left[\beta_2^{-1}\left(\frac{b}{1+\rho}\right)\right]\right)^{n_2}. \tag{A.3}$$

The optimal bid $\bar{b}$ can be solved for directly from the FOC which implies:

$$\bar{b} = \bar{c} + \frac{1-F\left[\beta_2^{-1}\left(\frac{\bar{b}}{1+\rho}\right)\right]}{n_2 f\left[\beta_2^{-1}\left(\frac{\bar{b}}{1+\rho}\right)\right]\frac{1}{1+\rho}\beta_2^{-1'}\left(\frac{\bar{b}}{1+\rho}\right)}. \tag{A.4}$$

Note that a maximum bid of $(1+\rho)\bar{c}$ is suboptimal because the probability of winning the auction is zero. If a firm submits a bid of $\bar{c}$, then it is suboptimal because profits are zero. In determining $\bar{b}$, the firm trades-off higher profits when it wins the auction with an increase in the probability of winning. The optimal $\bar{b}$ will be in the open interval $(\bar{c},\ (1+\rho)\bar{c})$.

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