Bond Portfolios and Two-Fund Separation in the Lucas Asset-Pricing Model

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2007 Summer CAM Conference
University of Notre Dame
June 9, 2007

Modern Portfolio Theory

Asset allocation in stochastic environments

Optimal investment in stocks, bond, and cash

Partial equilibrium analysis:

Exogenously specified stochastic processes of returns and interest rate

Continuous-time literature based on Merton (1973)

Many recent examples, e.g. Brennan and Xia (2000, 2002), Wachter (2003)

Discrete-time factor models

Campbell and Viceira (2001, 2002)

Motivation: This Paper

Popular models are partial equilibrium and not GE models

Few underlying factors, markets are complete with very few assets

Analysis of complex bond portfolios impossible

Our paper: Follow very different approach

Examine investors' portfolios in a dynamic GE model

Lucas asset pricing model with heterogeneous agents and many states of nature

Dynamically complete security markets

Market completeness through presence of many bonds

Summary: One Bond

HARA utility, linear sharing rules

Two-fund separation hinges on maturity of the bond

Consol: two-fund separation

One-period bond: typically no two-fund separation

Consol: riskless asset in an infinite-horizon dynamic model

safe consumption stream over the infinite horizon

uncertain capital value does not affect portfolios

One-period bond: risky asset in an infinite-horizon dynamic model

time-varying interest rates, reinvestment risk

Summary: Many Bonds

Dynamically complete security markets with several zero-coupon bonds

Bonds have maturities of $1, 2, \ldots, K$ periods

Stock portfolios typically do not exhibit two-fund separation

Bond portfolios involve unrealistically large trading volume of long-term bonds

As the number of states and bonds increases:

Stock portfolios approach two-fund separation

Bond portfolios show laddering structure for short maturities

Two-fund separation and bond ladders are approximately optimal

Introduction of redundant bonds is welfare-improving

Overview

- Dynamic GE Model
- HARA Utility Functions and Linear Sharing Rules
- Separation Results for the GE Model with a single bond
- Families of Finite-Maturity Bonds
- Bond Ladders

General Equilibrium Model

Lucas asset pricing model with heterogeneous agents

Dynamically complete asset markets

Markov process of exogenous dividend states, $y \in \mathcal{Y} = \{1, \ldots, Y\}$

Transition matrix $\Pi >> 0$

Finite number of types of infinitely-lived agents, $h \in \mathcal{H} = \{1, \ldots, H\}$

Single perishable consumption good (produced by firms)

Agents have no individual endowments but hold an initial portfolio of firms' stock

Firms distribute output through dividends ("Lucas trees")

Securities

Infinitely-lived stocks with dividends $d^j : \mathcal{Y} \to \mathbb{R}_{++}$ for $j = 1, \ldots, J$

Stocks are in unit net supply

Each agent has initial holding of stocks

Initial model: two types of bonds

Consol with safe payoff $d_y^c = 1$ for all $y \in \mathcal{Y}$

One-period bond with safe payoff next period

Bonds are in zero net supply

Agents hold no initial positions

Utility Function

Time-separable utilities

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t)\right\}$$

Consumption process $c = (c_0, c_1, \ldots)$

$u_h : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ strictly monotone, C^2, and strictly concave

Identical discount factor $\beta \in (0, 1)$ for all agents

Equilibrium

Complete markets: Pareto efficient consumption allocations

Consumption only depends on current dividend state, is independent of history and any other state variables

Consumption "process" is represented by a vector of Y numbers

Negishi approach determines allocations; nonlinear system of equations

Portfolios are constant for Y independent dividend vectors

State-independent portfolio of stocks ψ^h, and consol θ_c^h or bond θ_1^h

Budget equations determine constant portfolios; linear system of equations

Classical Two-Fund Separation

Tobin (1958), Markowitz (1959)

Cass and Stiglitz (1970): single-agent static portfolio demand problem

There is a riskless asset and the agent has HARA utility

Monetary separation: The relative allocation of wealth across risky assets
is invariant to wealth and risk attitude

General Equilibrium

Market-clearing in general equilibrium model

Two-fund separation $\iff \quad \psi_j^h = \psi_{j'}^h \quad \forall\, j, j'$

Rubinstein (1974): Equi-cautious HARA utility leads to linear sharing rules

for all agents in static GE

Generalizes to our dynamic model: $c_y^h = m^h e_y + b^h \quad \forall\, h,\ \forall\, y$

Social endowment $e_y = \sum_{j=1}^{J} d_y^j$

Consol vs. One-period Bond

Consol $$m^h\, e_y + b^h \;=\; c_y^h \;=\; \sum_{j=1}^J \psi_j^h d_y^j + \theta_c^h \cdot 1$$

Two-fund separation holds, $\theta_c^h = b^h$ and $\psi_j^h = m^h \; \forall j$

One-period bond $$m^h\, e_y + b^h \;=\; c_y^h \;=\; \sum_{j=1}^J \psi_j^h d_y^j + \theta_1^h \cdot (1 - q_y^1)$$

Generically no two-fund separation when $b^h \neq 0$

Deviations from two-fund separation are quantitatively significant

Many Finite-Maturity Bonds

Infinitely-lived stocks with dividends $d^j : \mathcal{Y} \to \mathbb{R}_{++}$ for $j = 1, \ldots, J$

K zero-coupon bonds of maturities $1, 2, \ldots, K$ in zero net supply

Agent h's bond portfolio, $\theta_1^h, \theta_2^h, \ldots, \theta_K^h$

Agent h's budget constraint (in stationary equilibrium)

$$c_y^h = \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{K} \theta_k^h (q_y^{k-1} - q_y^k)$$

Two-Fund Separation with IID Dividends

Any number J of stocks, two bonds with maturities $k = 1, 2$

IID beliefs over next period's dividend states

Prices of the two bonds are perfectly correlated, $q^2 = \beta q^1$

Portfolio

$$\theta_1^h = b^h, \quad \theta_2^h = \frac{b^h}{(1-\beta)},$$

implements holding b^h of consol and so creates safe consumption stream of size b^h

Two-fund separation for stock portfolio $\psi_j^h = m^h \;\; \forall j = 1, \ldots, J$

Spanning the Consol

Finite-maturity bonds span consol $\Longrightarrow$ two-fund separation

$$c_y^h = m^h e_y + b^h = \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{K} \theta_k^h (q_y^{k-1} - q_y^k)$$

Spanning means relationship between bond price vectors $q^1, q^2, \ldots, q^K$

Sufficient conditions for spanning

Key ingredient is Markov transition matrix Π of exogenous shocks

Examples with Many Bonds

J independent stocks with independent high and low dividends

stock	1	2	3	4	5	6	7
high d	1.02	1.23	1.05	1.2	1.09	1.14	1.1
low d	0.98	0.77	0.95	0.8	0.91	0.86	0.9
pers.	0.55	0.81	0.61	0.74	0.66	0.7	0.68

Model with J independent stocks: $Y = 2^J$ states

With different persistence, $K = Y - J$ bonds

$H = 2$ agents with power utilities, $\frac{1}{1-\gamma}(c - A^h)^{1-\gamma}$, $\beta = 0.95$

Sharing rule for agent 1, $c_y^1 = 0.3 \cdot e_y + 0.2$

$(J,K)=$	$(3,5)$	$(4,12)$	$(J,K)=$	$(5,27)$	$(J,K)=$	$(6,58)$
ψ_1^1	0.431	0.30	ψ_1^1	0.30	ψ_1^1	0.30
ψ_2^1	0.351	0.30	ψ_2^1	0.30	ψ_2^1	0.30
ψ_3^1	0.387	0.30	ψ_3^1	0.30	ψ_3^1	0.30
ψ_4^1		0.30	ψ_4^1	0.30	ψ_4^1	0.30
			ψ_5^1	0.30	ψ_5^1	0.30
					ψ_6^1	0.30
θ_1^1	0.152	0.20	θ_1^1	0.20	θ_1^1	0.20
θ_2^1	-0.184	0.20	θ_2^1		θ_2^1	
θ_3^1	2.337	0.20	θ_8^1	$\vdots$	θ_3^1	$\vdots$
θ_4^1	-7.498	0.20	θ_{11}^1		θ_{26}^1	
θ_5^1	8.074	0.20	θ_{12}^1	0.20	θ_{27}^1	0.20
θ_7^1		-0.66	θ_{20}^1	-5.2	θ_{50}^1	1179
θ_8^1		6.33	θ_{25}^1	556	θ_{56}^1	10177
θ_{11}^1		-86.58	θ_{26}^1	-423	θ_{57}^1	-4627
θ_{12}^1		46.58	θ_{27}^1	146	θ_{58}^1	998

Deviations from Two-Fund Separation

J	K	Δ^S	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
4	12	4.5 (−9)	1.3 (−9)	3.5 (−8)	2.0 (−6)	1.1 (−4)	3.7 (−3)
5	27	3.5 (−33)	6.3 (−34)	8.3 (−31)	8.3 (−28)	4.6 (−25)	1.6 (−22)
6	58	9.6 (−88)	4.2 (−85)	3.1 (−81)	1.1 (−77)	2.1 (−74)	3.0 (−71)
7	121	2.0 (−222)	4.9 (−214)	1.8 (−209)	3.0 (−205)	3.2 (−201)	2.4 (−197)

As K increases,

stock portfolios converge to holdings satisfying two-fund separation

holdings of bonds with short maturity are approximately b^h for agent h

More Deviations

k	$(5, 27)$	$(6, 58)$	$(7, 121)$
6	3.5 (−20)	3.0 (−68)	1.4 (−193)
7	5.3 (−18)	2.4 (−65)	6.3 (−190)
10	3.0 (−12)	2.9 (−57)	2.0 (−179)
11	1.5 (−10)	9.9 (−55)	4.5 (−176)
12	5.4 (−9)	2.9 (−52)	8.9 (−173)
20	5.37	7.5 (−35)	3.5 (−148)
25	555.6	1.1 (−25)	3.9 (−134)
26	423.4	5.3 (−24)	2.0 (−131)
27	145.8	2.4 (−22)	9.1 (−129)
40	–	3.7 (−5)	1.0 (−96)
50	–	1179.3	4.3 (−75)
56	–	10178	3.0 (−63)
57	–	4627.2	2.3 (−61)
58	–	998.2	1.7 (−59)

Bond Ladders

Consider very simple portfolios

Stock portfolios must exhibit two-fund separation

Bond portfolios must have ladder structure

$\psi_j^h = \hat{m}^h, \ \forall j = 1, \ldots, J$ (two-fund separation)

$\theta_k^h = \hat{b}^h, \ \forall k = 1, \ldots, B$ (bond ladder)

Welfare loss of such portfolios ?

Welfare Comparison of Three Portfolios

Consumption stream c^h yields lifetime utility $V^h(c^h)$

Consumption equivalent C^h defined by $\sum_{t=0}^{\infty} \beta^t u^h(C^h) = V^h(c^h)$

$C^{h,0}$ = CE for consumption stream from initial portfolio

$C^{h,*}$ = CE for equilibrium consumption stream

$C^{h,B}$ = CE for portfolio with bond ladder $(\hat{m}^h, \hat{b}^h)$ of size B

Welfare loss from bond ladder

$$\Delta C^h = 1 - \frac{C^{h,B} - C^{h,0}}{C^{h,*} - C^{h,0}} = \frac{C^{h,*} - C^{h,B}}{C^{h,*} - C^{h,0}}$$

Equilibrium Portfolio vs. Bond Ladder

Economy with $J = 4$ independent stocks, so $Y = 2^4$ states

stock	1	2	3	4
high d	1.05	1.08	1.12	1.15
low d	0.95	0.92	0.88	0.85

Persistence probability of both states is 0.6 for all stocks

Welfare Losses of Bond Ladders

$B \backslash \gamma$	1	3	5	10
1	1.4 (−4)	1.4 (−3)	7.2 (−3)	4.8 (−2)
2	5.0 (−6)	3.0 (−3)	1.3 (−2)	7.5 (−2)
5	2.4 (−10)	3.2 (−3)	1.4 (−2)	8.1 (−2)
10	6.3 (−13)	2.6 (−3)	1.2 (−2)	7.5 (−2)
30	≈ 0	7.7 (−4)	5.1 (−3)	4.7 (−2)
50	≈ 0	1.6 (−4)	1.4 (−3)	1.9 (−2)
100	≈ 0	1.2 (−6)	1.3 (−5)	2.8 (−4)

Summary

Portfolio analysis in Lucas asset-pricing model with many states and bonds

Equilibrium portfolios are economically unintuitive

Simple portfolios with two-fund separation and bond ladders
  are approximately optimal

Such portfolios benefit from the introduction of redundant bonds