# Bond Ladders and Optimal Portfolios*

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**Abstract**

This paper examines portfolios within the framework of a dynamic asset-pricing model when investors can trade equity assets as well as bonds of many different maturities. We specify the model so that investors have demand for both a risky and a safe income stream. We characterize the resulting optimal equilibrium stock and bond portfolios and document that optimal bond investment strategies partly exhibit a ladder structure, if a sufficient number of bonds is available for trade. The main contribution of the paper is to show that complete ladders with all bonds in the economy combined with a market portfolio of equity assets are nearly optimal investment strategies. This paper therefore provides a rationale for bond ladders as a popular bond investment strategy.

Keywords: Bond ladders, reinvestment risk, portfolio choice, bonds, consol.

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# 1 Introduction

This paper provides a rationale for bond ladders as a popular strategy for bond portfolio management. Laddering a bond portfolio requires buying and holding equal amounts of bonds that mature over different periods. When the shortest bond matures, an equal amount of the bond with the longest maturity in the ladder is purchased. Many bond portfolio managers claim that laddering tends to outperform other bond strategies because it reduces both market price risk and reinvestment risk for a bond portfolio in the presence of interest rate uncertainty. Despite the popularity of bond ladders as a strategy for managing investments in fixed-income securities, there is surprising little reference to this subject in the economics and finance literature. In this paper we analyze complex bond portfolios within the framework of a dynamic asset-pricing model. We specify the model so that investors have demand for both a risky and a safe income stream. We characterize the resulting optimal equilibrium stock and bond portfolios and document that optimal bond investment strategies partly exhibit a ladder structure. The main contribution of the paper is to show that complete ladders with all bonds in the economy, combined with a market portfolio of equity assets, are excellent investment strategies in the sense that they are nearly optimal.

For many investors bonds of different maturities constitute an important part of their investment portfolios. When the maturity date of a bond does not coincide with an investor's investment horizon he faces two possible risks. If he must sell a bond with time remaining until maturity because he has some cash demand (e.g., for consumption) he is exposed to market price risk because changes in interest rates may strongly affect the value of the bonds in his portfolio. If bonds in the portfolio mature before the investor needs the invested funds then he is exposed to reinvestment risk, that is, to the risk that he will not be able to reinvest the returned principal at maturity at the same interest rate as that of the initial investment. Instead, he is forced to roll over maturing bonds into new investments at uncertain interest rates. Reinvestment risk also arises if the investor receives periodic payments from a security, such as periodic coupon payments from a bond, long before its maturity date. A popular tool for lessening the impact of both market price and reinvestment risk are bond ladders.

An investor builds a bond ladder by investing an equal amount of money into bonds maturing on different dates. For example, an investor may want to create a ladder of bonds maturing in one, two, three, four, and five years. The strategy is then to invest one-fifth of the money into bonds of each maturity. Once the one-year bond matures the returned principal, and possibly coupon payments from all five bonds, is reinvested into a new five-year bond. At this point the bond portfolio consists again of investments in bonds of each maturity. This bond portfolio strategy delivers much more stable returns over time than investing the entire money into bonds of identical maturity since only a portion of the portfolio matures at any one time.

Many financial advisors advocate the creation of bond ladders to investors, see, for example, a recent article in the journal of the American Association of Individual Investors (Bohlin and Strickland, 2004), or popular financial advice books such as "The Motley Fool's

Money After 40" (Gardner and Gardner, 2004). Morgan Stanley[1] advertises laddered portfolio strategies as a way to save for retirement and college. Thornburg Investment Management (Strickland et al., 2008) stresses that laddered bond portfolios yield consistent returns with reduced market price and reinvestment risk.

Despite the well-documented advantages and resulting popularity of bond ladders as a strategy for managing bond investments, there is to the best of our knowledge no thorough analysis of bond ladders in modern portfolio theory. The large classical portfolio literature, see French (2008) for a history, starting with Markowitz (1952) and Tobin (1958) examines investors' portfolio decisions in one-period models which by their very nature cannot examine bond ladders. The last decade has seen a rapidly evolving literature on optimal asset allocation in stochastic environments. One string of this literature builds on the general dynamic continuous-time framework of Merton (1973) and assumes exogenously specified stochastic processes for stock returns or the interest rate. Recent examples of this literature include Brennan and Xia (2000) and Wachter (2003) among many other works. A second string of literature uses discrete-time factor models to examine optimal asset allocation, see for example Campbell and Viceira (2001, 2002). Most of these papers focus on aspects of the optimal choice of the stock-bond-cash mix but do not examine the details of a stock or bond portfolio. A particular feature of these factor models is that only very few assets are needed for security markets to be complete. For example, the model of Brennan and Xia (2002) can exhibit complete security markets with only four securities, only two of which are bonds. Also Campbell and Viceira (2001) report computational results on portfolios with only 3-month and 10-year bonds. Due to the small number of bonds, the described portfolios in these models do not include bond ladders. Analyzing more bonds in these models would certainly be possible, but additional bonds would be redundant securities since markets are already complete. As a result there would be continua of optimal asset allocations and so any further analysis of particular bond portfolios would depend on quite arbitrary modeling choices. To summarize, neither the classical finance literature of one-period models nor the modern literature on optimal asset allocation in dynamic models can adequately analyze portfolios in the presence of large families of (non-redundant) bonds in a stochastic dynamic framework. An immediate consequence is that neither literature can examine bond ladders and their impact on investors' welfare. These observations motivate the current paper.

We employ a Lucas-style (Lucas, 1978) discrete-time, infinite-horizon general equilibrium model with a finite set of exogenous shocks per period for our analysis of complex bond portfolios because this model offers three advantages. First, when markets are dynamically complete efficient equilibria are stationary. This feature allows for a simple description of equilibrium. Second, general equilibrium restrictions preclude us from making possibly inconsistent assumptions on agents' tastes and asset price processes. Instead, general equilibrium conditions enforce a perfect consistency between tastes, stock dividends and the prices of all securities and thus make the model an excellent expositional tool for our analysis. Third, we can include any desired number of financial securities without causing asset

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[1]Morgan Stanley, Bond Investment Strategies for the Way You Live. (accessed June 2, 2008) http://www.morganstanleyindividual.com/investmentproducts/bonds/strategies/default.asp

redundancy by choosing a sufficiently large number of exogenous shocks. This facet of the model makes it ideally suited for the analysis of portfolios with many stocks and bonds. Into this model we then introduce the classical assumption of equi-cautious HARA utility for all agents. This assumption guarantees that consumption allocations follow a linear sharing rule, see Wilson (1968) and Rubinstein (1974a, 1974b). Linear sharing rules imply that portfolios exhibit the classical property of two-fund monetary separation (Hakansson (1969), Cass and Stiglitz (1970)) if agents can trade a riskless asset, see Rubinstein (1974a, 1974b). In our infinite-horizon economy a consol, that is, a perpetual bond, plays the role of the riskless asset. In the presence of a consol, agents hold the market portfolio of all stocks and have a position in the consol. But if the consol is replaced by a one-period bond ("cash") then two-fund monetary separation fails to hold (generically). The agents no longer hold the market portfolio of stocks. Our analysis of economies with a single bond serves us as a helpful benchmark for our subsequent analysis of portfolios with many bonds.

We begin our analysis of complex bond portfolios with numerical experiments that lead us to several interesting observations. First, although agents' stock portfolios deviate from the market portfolio they rapidly converge to the market portfolio as the number of states and bonds in the economy grows. Second, as the stock portfolios converge to the market portfolio agents' bond portfolios effectively synthesize the consol. The agents use the available bonds with finite maturity to approximately generate the safe income stream that a consol would deliver exactly. Third, the equilibrium portfolios of the bonds with relatively short maturity approximately constitute a bond ladder, that is, we observe an endogenous emergence of bond ladders as a substantial part of optimal portfolios. Fourth, the portfolios of bonds with longest maturity deviate significantly from a ladder structure. Equilibrium positions are implausibly large and the implied trading volume bears no relation to actual bond markets.

The numerical results motivate the further analysis in the paper. We establish sufficient conditions under which the observed separation between the stock and the bond market holds not only approximately but in fact exactly. Specifically, we develop conditions on the underlying stochastic structure of stock dividends guaranteeing that the consol can be perfectly replicated by a few finite-maturity bonds. When this happens the two-fund monetary separation holds in generalized form. Each investor divides her wealth between the market portfolio and the bond portfolio replicating the consol. Our conditions hold for many natural specifications of exogenous shocks but they are nongeneric. Small perturbations of the stochastic structure of stock dividends destroy the exact replication property.

The bond portfolios replicating the consol (approximately or exactly) always exhibit the same qualitative properties once the number of bonds is sufficiently large. The portfolios of short-term bonds display a laddered structure while the holdings of long-term bonds fluctuate considerably and thus appear rather unrealistic. These results motivate the final part of our analysis. We examine how well an investor can do who is restricted to hold the market portfolio of stocks and a ladder of all bonds available in the economy. We find that such a simple investment strategy is an excellent alternative to the equilibrium portfolio. The welfare loss of the simple portfolio is very small and converges to zero as the number of bonds increases. In fact, we find an important role for redundant bonds that do not

increase the span of the traded securities, since adding bonds with a previously unavailable long maturity improves the performance of bond ladder strategies. We also show that the optimal portfolio weights between the bond ladder and the market portfolio deviate from the allocation between a consol and the market portfolio. The reason for this deviation is that while the bond ladder decreases the reinvestment risk it cannot decrease this risk to zero entirely, contrary to the consol.

The remainder of this paper is organized as follows. Section 2 presents the basic dynamic general equilibrium asset market model. Section 3 discusses the classical two-fund separation theory for our dynamic model and examines portfolios with a consol. In Section 4 we present results from extensive numerical experiments which motivate and guide our further analysis of optimal portfolios. In Section 5 we develop sufficient conditions for a small number of bonds of finite maturity to span the consol. Section 6 examines portfolios consisting of an investment in the market portfolio of stocks and bond ladders. We show that as the number of bonds with finite maturities increases the welfare loss from holding such a non-equilibrium portfolio tends to zero. Section 7 concludes the paper with more details on some related literature and a discussion of the results and limitations of our analysis.

# 2    The Asset Market Economy

We examine a standard Lucas asset pricing model (Lucas 1978) with heterogeneous agents (investors) and infinite discrete time, $t = 0, 1, \ldots$. Uncertainty is represented by exogenous shocks $y_t$ that follow a Markov chain with a finite state space $\mathcal{Y} = \{1, 2, \ldots, Y\}$, $Y \geq 3$, and transition matrix $\Pi >> 0$. At time $t = 0$ the economy is in state $y_0$. A date-event is a finite history of shocks, $(y_0, y_1, \ldots, y_t)$.

We assume that there is a finite number of types $\mathcal{H} = \{1, 2, .., H\}$ of infinitely-lived agents. There is a single perishable consumption good. Each agent $h$ has a time-separable expected utility function

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t)\right\},$$

where $c_t$ is consumption at time $t$. All agents have the same discount factor $\beta \in (0, 1)$ and calculate expectations using the transition matrix $\Pi$. We specify functional forms for the utility functions $u_h$ below.

Agents have no initial endowment of the consumption good. Their initial endowment consists solely of shares in some firms (stocks). The firms distribute their output each period to its owners through dividends. Investors trade shares of firms and other securities in order to transfer wealth across time and states. We assume that there are $J \geq 2$ stocks, $j \in \mathcal{J} \equiv \{1, 2, \ldots, J\}$, traded on financial markets. A stock is an infinitely-lived asset ("Lucas tree") characterized by its state-dependent dividends. We denote the dividend of stock $j \in \mathcal{J}$ by $d^j : \mathcal{Y} \to \mathbb{R}_+$ and assume that the dividend vectors $d^j$ are linearly independent. Agent $h$ has an initial endowment $\psi_j^{h,0}$ of stock $j \in \mathcal{J}$. We assume that all stocks are in unit net supply, that is, $\sum_{h \in \mathcal{H}} \psi_j^{h,0} = 1$ for all $j \in \mathcal{J}$, and so the social endowment in the economy

in state $y$ is the sum of all firms' dividends in that state, $e_y \equiv \sum_{j \in \mathcal{J}} d_y^j$ for all $y \in \mathcal{Y}$. We assume that all stocks have non-constant dividends and that aggregate dividends (i.e. aggregate endowments) are also not constant.

Our model includes the possibility of two types of bonds. One type of bond we analyze is a consol. The consol pays one unit of the consumption good in each period in each state, that is, $d_y^c = 1$ for all $y \in \mathcal{Y}$. We also study finite-lived bonds. There are $K \geq 1$ bonds of maturities $1, 2, \ldots, K$ traded on financial markets. We assume that all finite-lived bonds are zero coupon bonds. (This assumption does not affect any results concerning stock investments since any other bond of similar maturity is equivalent to a sum of zero-coupon bonds.) A bond of maturity $k$ delivers one unit of the consumption good $k$ periods in the future. If at time $t$ an agent owns a bond of maturity $k$ and holds this bond into the next period, it turns into a bond of maturity $k - 1$. Agents do not have any initial endowment of the bonds. All bonds are thus in zero net supply.

As usual a financial markets equilibrium consists of consumption allocations for all agents and prices for stocks and bonds at each date-event such that all asset markets clear and agents maximize their utility subject to their respective budget constraint and a standard transversality condition. For the purpose of this paper we do not need a formal definition of financial markets equilibrium. We thus omit a statement of the formal definition and refer to Judd et al. (2003) and many other papers.

## 2.1 Dynamically Complete Markets

For the remainder of the paper we assume that financial markets are dynamically complete. This assumption holds (generically), for example, if there are as many financial assets as shocks, $Y$. Under this assumption Judd et al. (2003) derive two results that are important for our analysis here. First, equilibrium is Markovian: individual consumption allocations and asset prices depend only on the current state. Second, after one initial round of trading, each agent's portfolio is constant across states and time. These results allow us to express equilibrium with dynamically complete markets in a simple manner. We do not need to express equilibrium values of all variables in the model as a function of the date-event $(y_0, y_1, \ldots, y_t)$ or as functions of a set of sufficient state variables. Instead, we let $c_y^h$ denote consumption of agent $h$ in state $y$. In addition, $q_y^k$ denotes the price of bond $k$ in state $y$, and the price of the consol is $q_y^c$. Similarly, $p_y^j$ denotes the price of stock $j$ in state $y$. All portfolio variables can be expressed without reference to a state $y$. The holdings of household $h$ consist of $\theta_k^h$ bonds of maturity $k$ or $\theta_c^h$ consols, and $\psi_j^h$ units of stock $j$. For ease of reference we summarize the notation for portfolio and price variables.

| $p_y^j$ | price of stock $j$ in state $y$ | $\psi_j^h$ | agent $h$'s holding of stock $j$ |
|---|---|---|---|
| $q_y^k$ | price of maturity $k$ bond in state $y$ | $\theta_k^h$ | agent $h$'s holding of maturity $k$ bond |
| $q_y^c$ | price of the consol in state $y$ | $\theta_c^h$ | agent $h$'s holding of the consol |

Under our assumption of dynamically complete markets we can use a slightly generalized version of the three-step algorithm in Judd et al. (2003) to calculate the equilibrium

values for all variables in the model. First, a Negishi approach yields agents' consumption allocations. Second, given allocations and thus marginal utilities, the agents' Euler equations determine asset prices. And third, given allocations and asset prices, we can solve the agents' budget constraints for the equilibrium portfolios. The first two steps are summarized in Appendix A.1. The analysis of our paper focusses on investors' portfolios and for this purpose we analyze the budget constraints in great detail. We explicitly state them here.

If the economy has a consol but no short-lived bonds then the budget constraint in state $y$ (after time 0) is

$$
\begin{aligned}
c_y^h &= \underbrace{\sum_{j=1}^{J} \psi_j^h \left(d_y^j + p_y^j\right) + \theta_c^h \left(1 + q_y^c\right)}_{\text{beginning-of-period wealth}} - \underbrace{\left(\sum_{j=1}^{J} \psi_j^h p_y^j + \theta_c^h q_y^c\right)}_{\text{end-of-period investment}} \\
&= \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_c^h
\end{aligned}
\tag{1}
$$

The budget constraint greatly simplifies since portfolios are constant over time and the prices of infinitely-lived assets cancel out.

If all bonds are of finite maturity then an agent's budget constraint in state $y$ is

$$
\begin{aligned}
c_y^h &= \underbrace{\sum_{j=1}^{J} \psi_j^h \left(d_y^j + p_y^j\right) + \theta_1^h + \sum_{k=2}^{K} \theta_k^h q_y^{k-1}}_{\text{beginning-of-period wealth}} - \underbrace{\left(\sum_{j=1}^{J} \psi_j^h p_y^j + \sum_{k=1}^{K} \theta_k^h q_y^k\right)}_{\text{end-of-period investment}} \\
&= \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h(1 - q_y^1) + \sum_{k=2}^{K} \theta_k^h(q_y^{k-1} - q_y^k)
\end{aligned}
\tag{2}
$$

Again the prices of stocks cancel out. Only the prices of the (finitely-lived) bonds appear in the simplified budget constraints. There may be trade on financial markets even though portfolios are constant over time. From one period to the next a $k$-period bond turns into a $(k-1)$-period bond and thus agent $h$ needs to rebalance the portfolio whenever $\theta_{k-1}^h \neq \theta_k^h$ to maintain a constant portfolio over time. In addition, the agent needs to reestablish the position in the bond of longest maturity.

## 2.2 HARA Utility and Linear Sharing Rules

The budget equations (1) and (2) enable us to analyze agents' portfolios that deliver the equilibrium consumption allocations. For this analysis a simple description of allocations is clearly helpful. We say that consumption for household $h$ follows a 'linear sharing rule' if there exists real numbers $m^h, b^h$ so that in each shock $y \in \mathcal{Y}$,

$$
c_y^h = m^h e_y + b^h.
\tag{3}
$$

Linear sharing rules partition the consumption vector $c^h$ into a "safe" portion $b^h$ and a "risky" portion $m^h e$. This partition proves important for our analysis of equilibrium portfolios.

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The connection between linear consumption sharing rules (as exposited in Wilson, 1968) and (static) asset market equilibrium was made by Rubinstein (1974a). We follow Rubinstein's approach in our dynamic economy and make the same assumption on investors' utility functions to ensure the emergence of linear sharing rules in equilibrium. Agents need to have equi-cautious HARA utility functions, that is, (per-period) utility functions $u_h$, $h \in \mathcal{H}$, must exhibit linear absolute risk tolerance with identical slopes. We examine three special cases of HARA utility functions: power utility, quadratic utility, and constant absolute risk aversion. We use the following notation for the utility function of household $h$.

power utility functions: $\quad u_h(c) = \begin{cases} \frac{1}{1-\gamma}(c - A^h)^{1-\gamma}, & \gamma > 0, \gamma \neq 1, \quad c > A^h \\ \ln(c - A^h), & \gamma = 1, \qquad\qquad c > A^h \end{cases}$

quadratic utility functions: $\quad u_h(c) = -\frac{1}{2}(B^h - c)^2$

CARA utility functions: $\quad u_h(c) = -\frac{1}{a^h} e^{-a^h c}$

If investors have equi-cautious HARA utility, then equilibrium consumption allocations for all agents follow a linear sharing rule of the form (3) and it holds that $\sum_{h=1}^{H} m^h = 1$ and $\sum_{h=1}^{H} b^h = 0$. Using the Negishi approach from Appendix A.1 we can calculate the sharing rules as functions of Negishi weights, see Appendix A.2. Power utility functions with $A^h = 0$ for all $h \in \mathcal{H}$ constitute the special case of identical CRRA utilities. In this case $b^h = 0$ for all $h \in \mathcal{H}$, that is, each agent $h$ consumes a constant fraction $m^h$ of the aggregate endowment.

While the assumption of HARA utility is certainly a restriction from a theoretical viewpoint this assumption is frequently made in applied work. For example, in a recent paper Brunnermeier and Nagel (2008) use a power utility function in their model of asset allocation that underlies their empirical examination of how investors change their portfolio allocations in response to changes in their wealth level.

# 3 Portfolios with a Consol

Our analysis of complex bond portfolios in the later sections of this paper reveals that equilibrium portfolios in dynamic economies with stocks and a consol (but no finite-maturity bonds) serve as a useful benchmark. This fact motivates the now following description of portfolios in the presence of a consol. Under our assumption of equi-cautious HARA utility functions, portfolios with a consol have the property of two-fund monetary separation.

## 3.1 Two-Fund Monetary Separation

There exists a variety of portfolio separation theorems[2] in the economic literature, but the notion that most people now have in mind when they talk about two-fund separation is

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[2]The literature starts with Tobin's (1958) two-fund result in a mean-variance setting. For textbook overviews see Ingersoll (1987) or Huang and Litzenberger (1988). The standard reference for two-fund and $m$-fund separation in continuous-time models is Merton (1973).

what Cass and Stiglitz (1970) termed two-fund 'monetary' separation. For examples of an application of this notion see Canner et al. (1997) and Elton and Gruber (2000). Two-fund monetary separation states that investors who must allocate their wealth between a number of risky assets and a riskless security should all hold the same mutual fund of risky assets. An investor's risk aversion only affects the proportions of wealth that she invests in the risky mutual fund and the riskless security. But the allocation of wealth across the different risky assets does not depend on the investor's preferences.

Hakansson (1969) and Cass and Stiglitz (1970) showed that the assumption of HARA utility is a necessary and sufficient condition on investors' utility functions for the optimal portfolio in investors' static asset demand problems to satisfy the monetary separation property. Ross (1978) presents conditions on asset return distributions under which two-fund separation holds for static demand problems. In this paper we stay away from analyses that rely on distributional assumptions about asset prices since we focus on equilibrium prices and portfolios, and there is no reason to believe that equilibrium asset prices fall into any of the special families that produce portfolio separation.

We define the notion of two-fund monetary separation for our dynamic general equilibrium model with heterogeneous agents. This form of separation requires the proportions of wealth invested in any two stocks to be the same for all agents in the economy.

**Definition 1** *Suppose an asset with a riskless payoff vector (that is, a one-period bond or a consol) is available for trade. The remaining J assets are risky stocks in unit net supply. We say that portfolios exhibit two-fund monetary separation if*

$$\frac{\psi_j^h \; p_y^j}{\psi_{j'}^h \; p_y^{j'}} = \frac{\psi_j^{h'} \; p_y^j}{\psi_{j'}^{h'} \; p_y^{j'}}$$

*for all stocks $j, j'$ and all agents $h, h' \in \mathcal{H}$ in all states $y \in \mathcal{Y}$.*

All stocks are in unit net supply and so market clearing and the requirement from the definition immediately imply that all agents' portfolios exhibit two-fund separation if and only if each agent has a constant share of each stock in the economy, that is, $\psi_j^h = \psi_{j'}^h$ for all stocks $j, j'$ and all agents $h \in \mathcal{H}$. This constant share typically varies across agents. In the remainder of this paper we identify two-fund monetary separation with this constant-share property. Note that the ratio of wealth invested in any two stocks $j, j'$ equals the ratio $p_y^j/p_y^{j'}$ of their prices and thus depends on the state $y \in \mathcal{Y}$.

## 3.2 The Consol is the Riskless Asset

As mentioned earlier, Rubinstein (1974a) introduced the assumption of equi-cautious HARA utility functions for all investors which yields linear sharing rules of consumption. In fact, the principal motivation for having linear sharing rules is that these in turn result in investors' portfolios that satisfy two-fund monetary separation. We can interpret Rubinstein's results essentially as a generalization of the conditions of Hakansson (1969) and Cass and Stiglitz (1970) for static asset demand problem to a static general equilibrium model.

We obtain the same connection between linear sharing rules and two-fund monetary separation in our dynamic model if agents can trade a consol.[3] The consol serves as the riskless asset in an infinite-horizon economy, see, for example, Rubinstein (1974b), Connor and Korajczyk (1989), Bossaerts and Green (1989).

**Theorem 1 (Two-Fund Monetary Separation)** *Consider an economy with $J \leq Y - 1$ stocks and a consol. If all agents have equi-cautious HARA utilities then their portfolios exhibit two-fund monetary separation in all efficient equilibria.*

**Proof:** The statement of the theorem follows directly from the budget constraint (1). Agents' sharing rules are linear, $c_y^h = m^h e_y + b^h$ for all $h \in \mathcal{H}$, $y \in \mathcal{Y}$, and so the budget constraints immediately yield $\theta_c^h = b^h$ and $\psi_j^h = m^h$ for all $j = 1, \ldots, J$. $\square$

The consol is the riskless asset in an infinite-horizon dynamic economy. An agent establishes a position in the consol at time 0 once and forever. Fluctuations in the price of the consol therefore do no affect the agent just like she is unaffected by stock price fluctuations. This fact allows her to hold a portfolio exhibiting two-fund monetary separation. We can read off agents' portfolios from their linear sharing rules. The holding $b^h$ of the consol delivers the safe portion of the consumption allocation, $m^h e + b^h$, and the holding $m^h$ of the market portfolio of all stocks delivers the risky portion $m^h e$ of the allocation. Recall that in the special case of CRRA utility functions, $A^h = 0$ for all $h \in \mathcal{H}$, and so the agents do not trade the consol. This is a corollary to the theorem: Whenever the intercept terms of the sharing rules are zero then agents do not trade the consol and the stock markets are dynamically complete even without a bond market. However, under the additional condition $\sum_{h \in \mathcal{H}} A^h \neq 0$ sharing rules have a nonzero intercept for a generic set of agents' initial stock portfolios. That is, with the exception of a set of initial portfolios that has measure zero and is closed, sharing rules will have nonzero intercepts. (See Schmedders, 2007.)

The fact that sharing rules have generically nonzero intercepts immediately implies that a one-period bond cannot serve as the riskless asset. Even when markets are complete, there will be no two-fund separation. The economic intuition for this fact follows directly from the budget equations in an economy with $Y = J + 1$ states, $J$ stocks and a one-period bond,

$$m^h \cdot e_y + b^h = \eta^h \cdot e_y + \theta_1^h (1 - q_y^1) . \tag{4}$$

Contrary to the budget equations for an economy with a consol the bond price $q_y^1$ now appears. Investors have to reestablish their position in the short-lived bond in every period. As a result they face reinvestment risk due to fluctuating equilibrium interest rates of the short-term bond. Fluctuations in the price of the one-period bond generically prohibit a

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[3]Due to linear sharing rules an asset market economy with stocks and a consol has efficient equilibria even when markets are nominally incomplete, that is, when the number of assets $J + 1$ is smaller than the number of states, $Y$. For analog static and two-period versions of this result see Rubinstein (1974a) and Detemple and Gottardi (1998), respectively. For a multi-period version see Rubinstein (1974b). The agents' portfolios are unique as long as $J + 1 \leq Y$ since the vectors $d^c$ and $d^j$, $j \in \mathcal{J}$, are linearly independent.

solution to equations (4), see Schmedders (2007). The reinvestment risk affects agents' bond and thus stock portfolios and leads to a change of the portfolio weights that implement equilibrium consumption.

Obviously the agents' portfolios do satisfy a generalized separation property. Consumption follows a linear sharing rule and so an agent's portfolio effectively consists of one fund generating the safe payoff stream of a consol and the second fund generating a payoff identical to aggregate dividends. Both funds have non-zero positions of stocks and of the bond. Agent $h$ holds $b^h$ units of the first fund and $m^h$ units of the second fund. However, this generalized definition is not the notion people have in mind when they discuss two-fund separation. Instead they think of monetary separation, see, for example, the discussions in Canner et al. (1997) and Elton and Gruber (2000). And it is also exactly this notion of two-fund monetary separation that appears in the analysis of complex bond portfolios below.

## 3.3 Illustrative Example: Consol vs. Short-lived Bonds

We complete our discussion of equilibrium portfolios with an example. Consider an economy with $H = 3$ agents who have CARA utility functions with coefficients of absolute risk aversion of 1, 2, and 3, respectively. The agents' discount factor is $\beta = 0.95$. There are two independent stocks with identical 'high' and 'low' dividends of 1.02 and 0.98, respectively. The dividends of the first stock have a persistence probability of 0.8, that is, if the current dividend level is high (low), then the probability of having a high (low) dividend in the next period is 0.8. The corresponding probability of the second stock equals 0.6. As a result of this dividend structure the economy has $S = 4$ exogenous states of nature. The dividend vectors are

$$d^1 = (1.02, 1.02, 0.98, 0.98)^\top \text{ and } d^2 = (1.02, 0.98, 1.02, 0.98)^\top.$$

The Markov transition matrix for the exogenous dividend process is

$$\Pi = \begin{bmatrix} 0.48 & 0.32 & 0.12 & 0.08 \\ 0.32 & 0.48 & 0.08 & 0.12 \\ 0.12 & 0.08 & 0.48 & 0.32 \\ 0.08 & 0.12 & 0.32 & 0.48 \end{bmatrix}.$$

At time $t = 0$ the economy is in state $y_0 = 1$. The agents' initial holdings of the two stocks are identical, so $\psi_j^{h,0} = \frac{1}{3}$ for $h = 1, 2, 3$, $j = 1, 2$.

Applying the algorithm of Appendix A.1 we can easily calculate consumption allocations and price any asset in this model. We do not need these values here but, for completion, state them in Appendix A.3.

If in addition to the two stocks a consol is available for trade then the economy satisfies the conditions of Theorem 1 and agents' portfolios exhibit two-fund monetary separation,

$$(\psi_1^1, \psi_2^1, \theta_c^1) = (\frac{6}{11}, \frac{6}{11}, -0.425),$$

$$(\psi_1^2, \psi_2^2, \theta_c^2) \;\; = \;\; (\frac{3}{11}, \frac{3}{11}, \;\; 0.121),$$

$$(\psi_1^3, \psi_2^3, \theta_c^3) \;\; = \;\; (\frac{2}{11}, \frac{2}{11}, \;\; 0.304).$$

We contrast the portfolios in an economy with a consol with the portfolios in an economy with short-lived bonds. We need two bonds to complete markets. In addition to the natural choice of having a one- and a two-period bond available for trade we also report portfolios for cases where the second bond has a longer maturity $k_2$. Table I shows portfolios for all three agents and Table II reports the corresponding end-of-period investments in the four assets at asset prices in state 1.

| bonds | | Agent 1 | | | | Agent 2 | | | | Agent 3 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_1$ | $k_2$ | $\psi_1^1$ | $\psi_2^1$ | $\theta_1^1$ | $\theta_{k_2}^1$ | $\psi_1^2$ | $\psi_2^2$ | $\theta_1^2$ | $\theta_{k_2}^2$ | $\psi_1^3$ | $\psi_2^3$ | $\theta_1^3$ | $\theta_{k_2}^3$ |
| 1 | 2 | 0.467 | 0.191 | −1.029 | 1.249 | 0.295 | 0.374 | 0.294 | −0.357 | 0.238 | 0.435 | 0.735 | −0.892 |
| 1 | 5 | 0.603 | 1.878 | 0.835 | −45.582 | 0.256 | −0.107 | −0.238 | 12.977 | 0.141 | −0.771 | −0.598 | 32.605 |
| 1 | 10 | 0.519 | 0.395 | −0.647 | −6.830 | 0.280 | 0.316 | 0.184 | 1.945 | 0.201 | 0.290 | 0.463 | 4.885 |
| 1 | 25 | 0.518 | 0.381 | −0.660 | −13.739 | 0.281 | 0.319 | 0.188 | 3.912 | 0.201 | 0.299 | 0.472 | 9.827 |
| 1 | 50 | 0.518 | 0.381 | −0.660 | −49.528 | 0.281 | 0.319 | 0.188 | 14.101 | 0.201 | 0.299 | 0.472 | 35.427 |

Table I: Equilibrium Portfolios with Two Bonds

| bonds | | Agent 1 | | | | Agent 2 | | | | Agent 3 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_1$ | $k_2$ | $\psi_1^1$ | $\psi_2^1$ | $\theta_1^1$ | $\theta_{k_2}^1$ | $\psi_1^2$ | $\psi_2^2$ | $\theta_1^2$ | $\theta_{k_2}^2$ | $\psi_1^3$ | $\psi_2^3$ | $\theta_1^3$ | $\theta_{k_2}^3$ |
| 1 | 2 | 9.07 | 3.71 | −0.99 | 1.15 | 5.73 | 7.25 | 0.28 | −0.33 | 4.63 | 8.44 | 0.71 | −0.82 |
| 1 | 5 | 11.71 | 36.45 | 0.80 | −36.02 | 4.98 | −2.07 | −0.23 | 10.26 | 2.74 | −14.97 | −0.58 | 25.77 |
| 1 | 10 | 10.08 | 7.66 | −0.62 | −4.18 | 5.45 | 6.12 | 0.18 | 1.19 | 3.90 | 5.62 | 0.45 | 2.99 |
| 1 | 25 | 10.07 | 7.40 | −0.64 | −3.90 | 5.45 | 6.20 | 0.18 | 1.11 | 3.91 | 5.81 | 0.45 | 2.79 |
| 1 | 50 | 10.07 | 7.40 | −0.64 | −3.90 | 5.45 | 6.20 | 0.18 | 1.11 | 3.91 | 5.81 | 0.45 | 2.79 |

Table II: End-of-period Investment across Assets in State 1

Two main observations stand out. First, agents' portfolios clearly do not resemble two-fund monetary separation. The stock portfolios are not the market portfolio. Secondly, the equilibrium portfolios greatly depend on the set of bonds available to the investors. Any arbitrary choice of bond maturities in the model will strongly affect both the equilibrium holdings and the end-of-period wealth invested in stocks and bonds. But note that economies with a one-period bond and a 25-period or 50-period bond, respectively, show (almost) identical positions in stocks and the one-period bond. Moreover, the wealth invested in the long-maturity bonds is (almost) identical. Over a horizon of 25 or 50 periods the distribution of the exogenous state of the economy at the time of maturity of the bonds is very close to the stationary distribution of the underlying Markov chain of exogenous states. Therefore,

these bonds are nearly perfect substitutes, the 50-period bond is approximately a 25-period bond with additional 25 periods of discounting. This substitutability of long-maturity bonds turns out to be significant in our analysis below.

We believe that the most natural choice of bonds in our model is to have bonds with consecutive maturities, but in the literature often other combinations are chosen. In our model with non-consecutive bond maturities an agent would be artificially forced to sell bonds whenever a bond changed its remaining maturity to a level that is not permitted by the model. For example, after one period a 5-period bond turns into a 4-period bond. The agent would then be forced to sell this bond and thus would face considerable market price risk for this transaction. Clearly this risk would influence the optimal portfolio decisions. To avoid such unnatural restrictions on agents' portfolio choices, we only consider economies with the property that if a bond of maturity $k$ is present, then also bonds of maturity $k-1, k-2, \ldots, 1$ are available to investors.

# 4  Portfolios with Many Finite-Maturity Bonds

We have seen that, instead of the short-lived bond, the consol is the ideal asset to generate a riskless consumption stream in our dynamic economy. But real-world investors do not have access to a consol[4] and instead can only trade finite-maturity bonds. While modern markets offer investors the opportunity to trade bonds with many different finite maturities, these bonds expose investors to reinvestment risk if their investment horizon exceeds the available maturity levels. As a result investors who demand a portion of their consumption stream to be safe face the problem to generate such a constant stream without the help of a riskless asset. We now examine this problem in our dynamic framework. In our model we can include any number of independent bonds by choosing a sufficiently large number of states. We begin our analysis of complex bond portfolios with some numerical experiments. The purpose of these experiments is to learn details about the structure of equilibrium portfolios that then guide our subsequent analysis.

We consider economies with $H = 2$ agents with power utility functions. Setting $A^1 = -A^2 = b$ results in the linear sharing rules[5] $c^1 = m^1 \cdot e + b \cdot 1_Y$ and $c^2 = (1 - m^1) \cdot e - b \cdot 1_Y$, where $1_Y$ denotes the vector of all ones. We set $m^1 = \frac{1}{2} - b$ so that both agents consume on average half of the endowment. For the subsequent examples we use $b = 0.2$, $\gamma = 5$ and $\beta = 0.95$. The agents' sharing rules are then

$$c^1 = 0.3 \cdot e + 0.2 \cdot 1_Y \ \text{ and } \ c^2 = 0.7 \cdot e - 0.2 \cdot 1_Y.$$

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[4]With the exception of some perpetual bonds issued by the British treasury in the 19th century, infinite-horizon bonds do not exist and are no longer issued – see Calvo and Guidotti (1992) for a theory of government debt structure that explains why modern governments do not issue consols.

[5]To simplify the analysis we do not compute linear sharing rules for some given initial portfolios but instead take sharing rules as given and assume that the initial endowment is consistent with the sharing rules. There is a many-to-one relationship between endowments and consumption allocations, and it is more convenient to fix consumption rules.

We consider economies with $J \in \{3, 4, 5, 6, 7\}$ independent stocks. Each stock $j \in \mathcal{J}$ in the economy has only two dividend states, a "high" and a "low" state, resulting in a total of $2^J$ possible states in the economy. We define the persistence parameters $\xi^j$ for each stock $j$ and denote the dividend's $2 \times 2$ transition matrix by

$$\Xi = \begin{bmatrix} \frac{1}{2}(1 + \xi^j) & \frac{1}{2}(1 - \xi^j) \\ \frac{1}{2}(1 - \xi^j) & \frac{1}{2}(1 + \xi^j) \end{bmatrix}$$

with $\xi^j \in (0, 1)$. The Markov transition matrix $\Pi = \bigotimes_{j=1}^{J} \Xi$ for the entire economy is a Kronecker product of the individual transition matrices, see Appendix B.2. Table III reports the parameter values for our examples. These parameter values cover a reasonable range

| stock | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| high $d$ | 1.02 | 1.23 | 1.05 | 1.2 | 1.09 | 1.14 | 1.1 |
| low $d$ | 0.98 | 0.77 | 0.95 | 0.8 | 0.91 | 0.86 | 0.9 |
| $\xi$ | 0.1 | 0.62 | 0.22 | 0.48 | 0.32 | 0.4 | 0.36 |
| $\frac{1}{2}(1 + \xi)$ | 0.55 | 0.81 | 0.61 | 0.74 | 0.66 | 0.7 | 0.68 |

Table III: Stock Characteristics

of persistence and variance in stock dividends. The varying dividend values and persistence probabilities are chosen so that the examples display generic behavior. (We calculated hundreds of examples showing qualitatively similar behavior.) To keep the expected social endowment at 1 we always normalize the dividend vectors. For this reason we multiply the dividend vectors by $1/J$ for the economy with $J$ stocks. However, as we show below this normalization is unnecessary.

The economy has $J$ stocks, $Y = 2^J$ states of nature and $K = 2^J - J$ bonds. For example, for $J = 5$ stocks and $Y = 32$ states our model has 27 bonds of maturities $1, 2, \ldots, 27$. All 32 assets are independent and thus markets are dynamically complete. Table IV reports the stock portfolio for Agent 1. Table V reports the agent's entire bond portfolio for $J \in \{3, 4\}$ and positions of selected bonds for $J \in \{5, 6, 7\}$. (There are just too many bonds to report complete portfolios.)

We make several observations about the agents' portfolios. First, consider the stock portfolios of Agent 1 in Table IV. For $J = 3$ stocks and $Y = 8$ states the stock portfolio deviates significantly from the market portfolio. But for $J \in \{4, 5, 6, 7\}$ Agent 1's stock holdings are very close to the slope $m^1 = 0.3$ of the linear sharing rule. In fact, the stock positions match at least the first 7 digits (to keep the table small we report fewer than 7 digits) of $m^1$. In other words, the agent's stock portfolios come extremely close to being the market portfolio. Secondly, consider the bond portfolios in Table V. For $J = 3$ the positions in the 5 bonds exhibit no meaningful structure. For $J = 4$ Agent 1's holdings of the bonds of maturity $1, 2, \ldots, 5$, match the intercept term $b^1$ of the linear sharing rule for

| $(J,K) =$ | $(3,5)$ | $(4,12)$ | $(5,27)$ | $(6,58)$ | $(7,121)$ |
|---|---|---|---|---|---|
| $\psi_1^1$ | 0.431 | 0.300 | 0.300 | 0.300 | 0.300 |
| $\psi_2^1$ | 0.351 | 0.300 | 0.300 | 0.300 | 0.300 |
| $\psi_3^1$ | 0.387 | 0.300 | 0.300 | 0.300 | 0.300 |
| $\psi_4^1$ | | 0.300 | 0.300 | 0.300 | 0.300 |
| $\psi_5^1$ | | | 0.300 | 0.300 | 0.300 |
| $\psi_6^1$ | | | | 0.300 | 0.300 |
| $\psi_7^1$ | | | | | 0.300 |

Table IV: Stock Portfolio of Agent 1

| $(J,K) =$ | $(3,5)$ | $(4,12)$ | $(5,27)$ | | $(6,58)$ | | $(7,121)$ | |
|---|---|---|---|---|---|---|---|---|
| $k$ | $\theta_k^1$ | $\theta_k^1$ | $k$ | $\theta_k^1$ | $k$ | $\theta_k^1$ | $k$ | $\theta_k^1$ |
| 1 | 0.152 | 0.20 | 1 | 0.20 | 1 | 0.20 | 1 | 0.20 |
| 2 | $-0.184$ | 0.20 | 2 | 0.20 | 5 | 0.20 | 10 | 0.20 |
| 3 | 2.337 | 0.20 | 8 | 0.20 | 10 | 0.20 | 30 | 0.20 |
| 4 | $-7.498$ | 0.20 | 9 | 0.20 | 15 | 0.20 | 50 | 0.20 |
| 5 | 8.074 | 0.20 | 10 | 0.20 | 20 | 0.20 | 58 | 0.20 |
| 6 | | 0.27 | 11 | 0.20 | 25 | 0.20 | 95 | 0.20 |
| 7 | | $-0.66$ | 12 | 0.20 | 27 | 0.20 | 100 | 0.18 |
| 8 | | 6.33 | 15 | 0.20 | 40 | 0.20 | 110 | $-29675$ |
| 9 | | $-26.23$ | 20 | $-5.2$ | 50 | 1179 | 115 | 504548 |
| 10 | | 66.16 | 25 | 556 | 56 | 10177 | 119 | 157907 |
| 11 | | $-86.58$ | 26 | $-423$ | 57 | $-4627$ | 120 | $-50013$ |
| 12 | | 46.58 | 27 | 146 | 58 | 998 | 121 | 7670 |

Table V: Bond Portfolio of Agent 1

the first two digits. For $J = 5$ there is a corresponding match already for more than the first 15 bonds. As $J$, $Y$, and $K$ increase further the pattern of an increasing number of bond positions approximately matching $b^1$ continues. So the agent's bond positions for relatively short-term bonds come extremely close to a bond ladder in which the holding of each bond (approximately) matches the level $b^1$ of the riskless consumption stream. This pattern falls apart for bonds with long maturity. The longer the maturity of the bonds the greater the deviations of holdings from $b^h$ (with the exception of just the holdings of bonds with longest maturities). In addition, once holdings deviate significantly from $b^h$ they alternate in sign[6].

To underscore our observations we next report the deviations of the stock holdings from

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[6]The nature of these bond holdings, namely the very large positions of alternating signs, may remind some readers of the unrelated literature on the optimal maturity structure of noncontingent government debt, see, for example, Angeletos (2002) and Buera and Nicolini (2004). Buera and Nicolini report very high debt positions from numerical calculations of their model with four bonds. The reason for their highly sensitive large debt positions is the close correlation of bond prices.

the slope of the linear sharing rule and the deviations of the bond holdings from the intercept term of the sharing rule. As a measure for these deviations define

$$\Delta^S \equiv \max_{j=1,2,\ldots,J} |\psi_j^1 - m^1|$$

to be the maximal deviation of agents' equilibrium stock holdings from the appropriate holding of the market portfolio where we maximize the difference across all stocks. (Due to market clearing it suffices to calculate the difference for the first agent.) Similarly, we define

$$\Delta^k \equiv |\theta_k^1 - b^1|$$

to be the maximal difference between agents' holdings in bond $k$ and the intercept of their linear sharing rules. Table VI reports deviations in stock holdings and the first five bonds, and Table VII reports deviations[7] in bond positions for some selected longer maturity bonds.

| $J$ | $K$ | $\Delta^S$ | $\Delta^1$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---|---|---|---|---|---|---|---|
| 4 | 12 | 4.5 $(-9)$ | 1.3 $(-9)$ | 3.5 $(-8)$ | 2.0 $(-6)$ | 1.1 $(-4)$ | 3.7 $(-3)$ |
| 5 | 27 | 3.5 $(-33)$ | 6.3 $(-34)$ | 8.3 $(-31)$ | 8.3 $(-28)$ | 4.6 $(-25)$ | 1.6 $(-22)$ |
| 6 | 58 | 9.6 $(-88)$ | 4.2 $(-85)$ | 3.1 $(-81)$ | 1.1 $(-77)$ | 2.1 $(-74)$ | 3.0 $(-71)$ |
| 7 | 121 | 2.0 $(-222)$ | 4.9 $(-214)$ | 1.8 $(-209)$ | 3.0 $(-205)$ | 3.2 $(-201)$ | 2.4 $(-197)$ |

Table VI: Deviations in Stock Holdings from $m^h$ and Bond Holdings from $b^h$

The results are clear. First, Table VI shows just how close equilibrium stock portfolios are to the fraction $m^h$ of the market portfolio. Stock positions are practically identical to $m^h$ when there are many states and bonds. Secondly, both Tables VI and VII highlight that the deviations in the bond holdings are also negligible for bonds that are of short maturity relative to the maximally available maturity $K$. For example, in the model with $Y = 64$ states and $K = 58$ bonds the holdings for the first 40 bonds are close to the intercept $b^h$ of the linear sharing rules. The agent's portfolio of these 40 bonds is practically a bond ladder. The deviations from the constant $b^h$ increase in the maturity $k$ of the bonds and eventually get huge. They peak for maturity levels just short of the longest maturity $K$. Moreover, the deviations explode as the number of states and bonds increases, see Tables V and VII.

Recall that the agents' portfolios are the solutions to their budget constraints,

$$c^h = m^h e + b^h 1_Y = \sum_{j=1}^{J} \psi_j^h d^j + \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k)$$

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[7]Computing the results in Tables IV, V, VI, and VII requires us to solve the agents' budget equations (2). Although these equations are linear, solving them numerically is very difficult. The prices of bonds with very long maturity $k$ are nearly perfectly correlated. This fact makes the equilibrium equations nearly singular and thus difficult to solve. One cannot solve them using a regular linear equation solver on a computer using 16 decimal digits of precision. To handle this difficulty, we used Mathematica with up to 1024 decimal digits of precision.

| $k$ | $(5, 27)$ | $(6, 58)$ | $(7, 121)$ |
|---|---|---|---|
| 6 | 3.5 $(-20)$ | 3.0 $(-68)$ | 1.4 $(-193)$ |
| 7 | 5.3 $(-18)$ | 2.4 $(-65)$ | 6.3 $(-190)$ |
| 10 | 3.0 $(-12)$ | 2.9 $(-57)$ | 2.0 $(-179)$ |
| 11 | 1.5 $(-10)$ | 9.9 $(-55)$ | 4.5 $(-176)$ |
| 12 | 5.4 $(-9)$ | 2.9 $(-52)$ | 8.9 $(-173)$ |
| 20 | 5.37 | 7.5 $(-35)$ | 3.5 $(-148)$ |
| 25 | 555.6 | 1.1 $(-25)$ | 3.9 $(-134)$ |
| 26 | 423.4 | 5.3 $(-24)$ | 2.0 $(-131)$ |
| 27 | 145.8 | 2.4 $(-22)$ | 9.1 $(-129)$ |
| 40 | – | 3.7 $(-5)$ | 1.0 $(-96)$ |
| 50 | – | 1179.3 | 4.3 $(-75)$ |
| 56 | – | 10178 | 3.0 $(-63)$ |
| 57 | – | 4627.2 | 2.3 $(-61)$ |
| 58 | – | 998.2 | 1.7 $(-59)$ |

Table VII: Deviations in Bond Holdings from $b^h$

where $1_Y$ denotes again the vector of all ones. The computed portfolios show that with many states and bonds in equilibrium it holds that

$$m^h e \approx \sum_{j=1}^{J} \psi_j^h d^j \ \text{ with } \ \psi_j^h \approx m^h, \ \forall \, j \in \mathcal{J}, \ \text{ and } \ b^h 1_Y \approx \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k).$$

We observe that a natural *generalization of two-fund monetary separation emerges!* Stock holdings are approximately the market portfolio of stocks. The purpose of the bond portfolios is to synthesize the consol and to generate the safe portion of the consumption stream.

Furthermore the emerging bond ladder of bonds up to some maturity $B < K$, that is, $\theta_k^h \approx b^h$ for $k = 1, 2, \ldots, B$, implies that

$$0 \approx b^h 1_Y - \left( \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k) \right) \approx \sum_{k=B}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K.$$

The bond price vectors of the long-maturity bonds $B, B+1, \ldots, K$ are nearly dependent. This result comes as no surprise since already any two bonds of very long maturity are nearly perfect substitutes as we observed in the introductory example in Section 3.3.

We tried many different examples and always observed the same results as those reported here. Also, the results are surprisingly invariant to the size of the stock dividends and the utility parameter $\gamma$, a fact that we address in more detail below. We now summarize the most important robust observations from our numerical experiments.

EQUILIBRIUM PORTFOLIOS IN MODELS WITH MANY STATES AND BONDS HAVE THE FOLLOWING PROPERTIES.

1. THE PORTFOLIOS APPROXIMATELY EXHIBIT GENERALIZED TWO-FUND MONETARY SEPARATION.

   (A) STOCK PORTFOLIOS ARE APPROXIMATELY THE MARKET PORTFOLIO OF ALL STOCKS. EACH AGENT $h$ HOLDS APPROXIMATELY A CONSTANT AMOUNT $m^h$ OF EACH STOCK.

   (B) BOND PORTFOLIOS APPROXIMATELY YIELD THE SAME PAYOFF AS A CONSOL HOLDING.

   (C) STOCK PORTFOLIOS GENERATE ALMOST EXACTLY THE RISKY PORTION $m^h e$ OF THE CONSUMPTION ALLOCATION. BOND PORTFOLIOS GENERATE ALMOST EXACTLY THE SAFE PORTION $b^h 1_Y$ OF THE ALLOCATION.

2. THE HOLDINGS OF BONDS OF SHORT MATURITY (RELATIVE TO THE LONGEST AVAILABLE MATURITY $K$) APPROXIMATELY CONSTITUTE A BOND LADDER.

3. HOLDINGS OF LONG BONDS ARE HIGHLY VOLATILE, IMPLYING THAT INVESTORS ARE MAKING DRAMATICALLY LARGE TRADES IN LONG BONDS IN EACH PERIOD.

These results raise two sets of questions. First, the finite-maturity bonds approximately span the consol. Do there exist specifications of our dynamic economy in which finite-maturity bonds can span the consol exactly? And if so, what do portfolios look like in such economies? Secondly, bond ladders emerge as the holdings of short-lived bonds but holdings of long-lived bonds look rather unnatural. How well can investors do if they are restricted to only hold bond ladders of all bonds available for trade? And what do optimal portfolios look like under this restriction? The now following Section 5 provides answers to the first set of questions. The then following Section 6 addresses the second set of questions.

# 5 Multiple Finite-Maturity Bonds Span the Consol

For an agent's stock holdings to be the market portfolio there must exist stock weights $\psi_j^h \equiv \eta^h$ for all $j \in \mathcal{J}$ such that

$$m^h \cdot e + b^h 1_Y = c^h = \eta^h \cdot e + \theta_1^h (1_Y - q^1) + \sum_{k=2}^{K} \theta_k^h (q^{k-1} - q^k). \tag{5}$$

Rearranging (5) yields

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + \sum_{k=1}^{K-1} (\theta_k^h - \theta_{k+1}^h) q^k + \theta_K^h q^K = 0. \tag{6}$$

Equation (6) implies that the $K + 2$ vectors $e, 1_Y$ and $q^1, \ldots, q^K$ in $\mathbb{R}^Y$ have to be linearly dependent. It appears as if whenever the number of states $Y$ exceeds $K + 2$ then this condition cannot be satisfied. For example, if the total number of stocks and bonds $J + K$ equals the number of states $Y$, and there are $J \geq 3$ stocks then the system (6) has more equations than unknowns. And in fact, using a standard genericity argument we can show that equation (6) does not have a solution unless parameters lie in some measure zero space.

Although agents' portfolios typically do not exhibit two-fund monetary separation in economies with only finite-maturity bonds we can develop special (non-generic) but economically reasonable conditions that do lead to portfolio separation in such economies.

## 5.1 Equilibrium Portfolios with IID Dividends

We first examine economies with i.i.d. dividends. The case of no persistence in dividends may be economically unrealistic but serves as a useful benchmark.

**Proposition 1** *Consider an economy with $J$ stocks, a one-period and a two-period bond and $Y \geq J + 2$ dividend states. Suppose further that the Markov transition probabilities are state-independent, so all rows of the transition matrix $\Pi$ are identical. If all agents have equicautious HARA utility functions then agents' portfolios exactly exhibit generalized two-fund monetary separation in an efficient equilibrium.*

**Proof:** Under the assumption that all states are i.i.d. the Euler equations (16, 17 in Appendix A.1) imply that the price of the two-period bond satisfies $q^2 = \beta q^1$, that is, the prices of the two bonds are perfectly correlated. Then condition (6) becomes

$$(m^h - \eta^h) \cdot e + (b^h - \theta_1^h) \cdot 1_Y + (\theta_1^h - (1 - \beta)\theta_2^h)q^1 = 0.$$

These equations have the unique solution $\eta^h = m^h$, $\theta_1^h = b^h$, $\theta_2^h = \frac{b^h}{1-\beta}$. $\square$

For i.i.d. dividend transition probabilities the solution to agents' budget equations satisfies

$$b^h 1_Y = \theta_1^h (1_Y - q^1) + \theta_2^h (q^1 - q^2).$$

Two bonds are sufficient to span the consol. Just like for economies with a consol markets are dynamically complete even when they are nominally incomplete, that is, when $Y > J + 2$.

## 5.2 Spanning the Consol

We next generalize the exact two-fund monetary separation for economies with i.i.d. dividends to more general Markov chains of dividends. The key sufficient condition for spanning the consol through a few finite-maturity bonds is a restriction on the transition matrix $\Pi$ that we derive with some simple linear algebra.

If $L$ bonds of maturity $k = 1, 2, \ldots, L$ span the consol then there must be a portfolio $(\theta_1, \ldots, \theta_L)$ of these bonds such that

$$1_Y = \theta_1(1_Y - q^1) + \sum_{k=2}^{L} \theta_k(q^{k-1} - q^k). \tag{7}$$

This system of equations is equivalent to

$$(1 - \theta_1) \cdot 1_Y + \sum_{k=1}^{L-1}(\theta_k - \theta_{k+1})q^k + \theta_L q^L = 0. \tag{8}$$

Multiplying the equation for state $y$ by the price of consumption $u_1'(c_y^1)$ in that state (see Appendix A.1) and then factoring the vector of prices $P = \left(u_1'(c_y^1)\right)_{y \in \mathcal{Y}}$ we obtain

$$\left((1 - \theta_1)I_Y + \sum_{k=1}^{L-1}(\theta_k - \theta_{k+1})(\beta\Pi)^k + \theta_L(\beta\Pi)^L\right)P = 0, \tag{9}$$

where $I_Y$ denotes the $Y \times Y$ identity matrix. A sufficient condition for these equations to have a solution $(\theta_1, \ldots, \theta_L)$ is that the following matrix equation has a solution,[8]

$$(1 - \theta_1)I_Y + \sum_{k=1}^{L-1}\left((\theta_k - \theta_{k+1})\beta^k\right)\Pi^k + (\theta_L\beta^L)\Pi^L = 0. \tag{10}$$

The derivation of equations (10) from equations (7) reduces the spanning issues to properties of $\Pi$, independent of the actual prices $P$, the initial endowments, and the dynamic evolution of the distribution of wealth. Therefore, when equations (10) are satisfied dividends and preferences do not matter for spanning the consol through finite-maturity bonds. Instead the issue is how many powers of $\Pi$ are needed to span $I = \Pi^0$. In the case where $\Pi$ is diagonalizable, i.e. there is a diagonal matrix $D$ and an invertible matrix A such that $\Pi = A^{-1}DA$ (our examples below show that to be a reasonable assumption), the following technical Lemma ensures that the number of distinct eigenvalues is the minimal number to accomplish that span. Appendix B.1 contains the proof.

**Lemma 1** *Suppose the $Y \times Y$ transition matrix $\Pi \gg 0$ governing the Markov chain of exogenous states in the economy has only real eigenvalues. Further assume that $\Pi$ is diagonalizable and has $L$ ($\leq Y$) distinct eigenvalues. Then the following statements are true.*

*(1) If all eigenvalues are nonzero then the matrix equation $I_Y + \sum_{k=1}^{L} a_k \Pi^k = 0$ has a unique solution $(a_1^*, \ldots, a_L^*)$. Moreover, $\sum_{k=1}^{L} a_k^* = -1$.*

*(2) If zero is an eigenvalue of $\Pi$ then the matrix equation $\sum_{k=1}^{L} a_k \Pi^k = 0$ has a nontrivial solution. Moreover, any solution $(a_1^*, \ldots, a_L^*)$ satisfies $\sum_{k=1}^{L} a_k^* = 0$.*

---

[8]It is straightforward to create examples where the sufficient condition (10) is not necessary by choosing dividends such that the aggregate endowment in the economy is identical across several different states. But for a generic set of parameters condition (10) is also necessary.

The lemma implies the following sufficient condition for spanning.

**Theorem 2** *Suppose the $Y \times Y$ transition matrix $\Pi \gg 0$ governing the Markov chain of exogenous states in the economy has only real eigenvalues. Further assume that $\Pi$ is diagonalizable and has $L \ (\leq Y)$ distinct eigenvalues. Then the consol is spanned by bonds of maturities $k = 1, 2, \ldots, L$.*

In Section 5.3 below we describe economically motivated examples that make nice use of this condition. Before doing so, we use the condition of Theorem 2 to characterize the bond portfolio that spans the consol.

**Corollary 1 (Corollary to Theorem 2)** *Suppose the transition matrix $\Pi$ satisfies the assumptions of Theorem 2. Suppose further that all agents have equi-cautious HARA utilities. If there are bonds of maturities $k = 1, 2, \ldots, L$ in the economy then there is an efficient equilibrium in which agents' portfolios satisfy generalized monetary separation. Moreover, the bond portfolios in this equilibrium satisfy the following properties.*

(a) *If the transition matrix $\Pi$ has only nonzero eigenvalues, then agent $h$'s holdings of the bonds of maturity $j = 1, 2, \ldots, L$ are*

$$\theta_j^h = \frac{b^h}{M_a} \left( \sum_{k=j}^{L} \beta^{L-k} a_k^* \right)$$

*where $M_a = \beta^L + \sum_{k=1}^{L} \beta^{L-k} a_k^*$ and $(a_1^*, a_2^*, \ldots, a_L^*)$ is the unique solution to the matrix equation $I_Y + \sum_{k=1}^{L} a_k \, \Pi^k = 0$.*

(b) *If the transition matrix $\Pi$ has a zero eigenvalue, then agent $h$ holds $\theta_1^h = b^h$ and has holdings of the bonds of maturity $j = 1, 2, \ldots, L$ of*

$$\theta_j^h = \frac{b^h}{M_b} \left( \sum_{k=j}^{L} \beta^{L-k} a_k^* \right)$$

*where $M_b = \sum_{k=1}^{L} \beta^{L-k} a_k^*$ and $(a_1^*, a_2^*, \ldots, a_L^*)$ is a nontrivial solution to the matrix equation $\sum_{k=1}^{L} a_k \, \Pi^k = 0$.*

Appendix B.1 contains the proof of this corollary. A close examination of the statements of Corollary 1 yields a number of interesting observations.

1. Proposition 1 is a simple consequence of Corollary 1, Part (b). With i.i.d. beliefs the Markov transition matrix $\Pi$ has only $L = 2$ distinct eigenvalues, namely 1 and 0. Case (b) then states that 2 bonds are sufficient to span the consol. Moreover, since $\Pi = \Pi^2$ the pair $a^* = (a_1^*, a_2^*) = (-1, 1)$ is a solution to the matrix equation of Lemma 1, Part (2). This leads to $M_b = 1 - \beta$ and to holdings of $\theta_1^h = b^h$ and $\theta_1^h = \frac{b^h}{1-\beta}$.

2. Another extreme case is a transition matrix $\Pi$ with the maximal number of $L = Y$ distinct eigenvalues. In that case the sufficient condition of Theorem 2 and Corollary 1 states that the number of bonds needed to span the consol is exactly the number of states $Y$. Then the economy with $J$ stocks would have a total of $J + Y$ assets, which exceeds the number of states $Y$ and optimal portfolios will be indeterminate. The portfolio exhibiting two-fund separation is then just one point in the manifold of equilibrium portfolios.

3. As $\beta \to 1$ it follows that $M_a \to 0$ and $M_b \to 0$. It can also be easily seen that $\left( \sum_{k=j}^{L} \beta^{L-k} a_k^* \right) \nrightarrow 0$ for all $j$. Thus, $|\theta_j| \to \infty$ for all $j$ in case (a) and all $j \geq 2$ in case (b). That is, as the discount factor tends to 1 the bond holdings spanning the consol become unboundedly large.

## 5.3 Identical Persistence Across Stocks and States

We complete our discussion of finite-maturity bonds spanning the consol by examining economically reasonable assumptions that imply the technical condition of Theorem 2. Suppose that each stock in the economy has $D$ different dividend states and that dividends are independent across stocks. (The latter condition may require a decomposition of stock payoffs into different independent factors.) Since the individual dividend processes are independent there is a total of $Y = D^J$ possible states in this economy. The dividends may vary across stocks, but the stocks' $D \times D$ dividend transition matrices, $\Xi$, are identical.[9] We assume that $\Xi$ has only real nonzero eigenvalues, is diagonalizable, and has $l$ distinct eigenvalues. The Markov transition matrix $\Pi$ for the economy is then the $J$-fold Kronecker product (see Appendix B.2) of the individual transition matrix for the dividend states of an individual stock, $\Pi = \Xi \bigotimes \Xi \bigotimes \cdots \bigotimes \Xi = \bigotimes_{j=1}^{J} \Xi$.

**Theorem 3** *Consider an economy with $J$ independent stocks that each have $D$ (stock-dependent) dividend states with identical diagonalizable transition matrices $\Xi$ having only real nonzero eigenvalues. The matrix $\Xi$ has $l$ distinct eigenvalues. Then bonds of maturities $k = 1, 2, \ldots, L$ span the consol, where $L = \binom{J+l-1}{l-1}$. In the presence of these $L$ bonds, and if all agents have equi-cautious HARA utilities, there exists an efficient equilibrium in which agents' portfolios satisfy two-fund separation.*

**Proof:** Lemma 2 in Appendix B.2 states that the matrix $\Pi = \bigotimes_{j=1}^{J} \Xi$ has only real nonzero eigenvalues, $L = \binom{J+l-1}{l-1}$ of which are distinct, and is diagonalizable. Theorem 2 and Corollary 1 then imply the statements of the theorem. $\square$

---

[9]Actually, it would be sufficient for all the individual transition matrices to have the same eigenvalues. The matrices do not have to be identical.

We illustrate Theorem 3 with an example. Each stock has only two dividend states, a "high" and a "low" state. We denote the dividend's $2 \times 2$ transition matrix by

$$\Xi = \begin{bmatrix} \frac{1}{2}(1+\xi_H) & \frac{1}{2}(1-\xi_H) \\ \frac{1}{2}(1-\xi_L) & \frac{1}{2}(1+\xi_L) \end{bmatrix}$$

with $\xi_H, \xi_L \in (0,1)$. This matrix $\Xi$ has $D = 2$ distinct eigenvalues, $1$ and $\xi = (\xi_H + \xi_L)/2 < 1$. The Markov transition matrix $\Pi = \bigotimes_{j=1}^{J} \Xi$ for the entire economy has only real nonzero eigenvalues, $J + 1$ of which are distinct. The eigenvalues are $1, \xi, \xi^2, \ldots, \xi^J$. (See Appendix B.2.) In this economy $J + 1$ bonds span the consol. The formulas of Corollary 1, Part (a), yield closed-form solutions for the individual bond holdings, but these formulas are difficult to assess. Here we report numerical solutions. Table VIII displays the portfolios of finite-maturity bonds that span one unit of the consol for $\xi = 0.2$ and $\beta = 0.95$. (For a robustness check we also report results for $\xi = 0.2, \beta = 0.99$, for $\xi = 0.5, \beta = 0.95$, and for $\xi = 0.5, \beta = 0.99$ in Appendix C.1.)

| $J$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $\theta_1$ | 1.176 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_2$ | −4.569 | 1.220 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_3$ | 25.667 | −4.800 | 1.229 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_4$ | | 25.863 | −4.847 | 1.231 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_5$ | | | 25.903 | −4.856 | 1.231 | 0.998 | 1.000 | 1.000 | 1.000 |
| $\theta_6$ | | | | 25.911 | −4.858 | 1.231 | 0.998 | 1.000 | 1.000 |
| $\theta_7$ | | | | | 25.912 | −4.858 | 1.232 | 0.998 | 1.000 |
| $\theta_8$ | | | | | | 25.912 | −4.859 | 1.232 | 0.998 |
| $\theta_9$ | | | | | | | 25.912 | −4.859 | 1.232 |
| $\theta_{10}$ | | | | | | | | 25.912 | −4.859 |
| $\theta_{11}$ | | | | | | | | | 25.912 |

Table VIII: Bond Portfolio Spanning one Unit of the Consol, $\xi = 0.2$, $\beta = 0.95$

The bond portfolios that exactly span the consol exhibit the same qualitative properties as those bond portfolios approximately spanning the consol in Section 4. Again we observe the endogenous emergence of a laddered portfolio of short-maturity bonds as the number of bonds increases.[10] The weights for the few bonds with longest maturity again fluctuate significantly. (Moreover, see Appendix C.1, as the eigenvalue stemming from the persistence parameters grows, these positions become even larger. The same is true when the discount

---

[10]The weight for the one-period bond converges quickly to 1 as the number of bonds, $J + 1$, grows. The same is true for the other bond weights. The weights are given by the formula of Corollary 1, Part (a). Note that for $j = 1$ the denominator exceeds the numerator by $\beta^{J+1}$ and so, as $J$ grows, the ratio on the right-hand side tends to 1. We can make similar arguments for the other bond positions.

factor increases.) As we pointed out before, the reason for the form of the portfolio is that the bond price vectors $q^k$ become more and more collinear as $k$ grows. The spanning condition then requires increasingly larger (in absolute value) weights on these vectors that also have to alternate in sign.

# 6 Nearly Optimal Portfolios with Bond Ladders

The theoretically derived portfolios exhibiting the exact generalization of two-fund monetary separation in Section 5 look just like the numerically computed portfolios in Section 4 which displayed this property approximately. Equilibrium portfolios consist of the market portfolio of stocks and a bond portfolio generating a safe income stream. In the presence of sufficiently many bonds the holdings of the short-term bonds are almost equal to the safe portion $b^h$ of the income stream. However, the holdings of long bonds always differ substantially from a constant portfolio and the implied asset trading volume bears no relation to actual security markets.

We now show that very simple and economically much more reasonable non-equilibrium portfolio strategies, namely portfolios consisting of the market portfolio of stocks and a bond ladder of all bonds available for trade, come very close to implementing equilibrium utility, in particular if the number of finite-maturity bonds available for trade is sufficiently large.

## 6.1 A Limit Result for Bond Ladders

The next theorem states that a portfolio with constant stock holdings and constant bond holdings (consistent with the linear sharing rules) yields the equilibrium consumption allocation in the limit as the number of bonds tends to infinity.

**Theorem 4** *Assume there are $Y$ states, $J$ stocks and that investors have equi-cautious HARA utility functions. Suppose the economy has $B$ finite-maturity bonds and that allocations in an efficient equilibrium follow the linear sharing rules $c^h = m^h e + b^h \cdot 1_Y$, $h \in \mathcal{H}$. Define portfolios of $\psi_j^h = m^h$, $\forall j \in \mathcal{J}$, and $\theta_k^h = b^h$, $\forall k$. Then in the limit as $B$ increases*

$$\lim_{B \to \infty} \left( \sum_{j=1}^{J} \psi_j^h d_y^j + \theta_1^h (1 - q_y^1) + \sum_{k=2}^{B} \theta_k^h (q_y^{k-1} - q_y^k) \right) = c_y^h.$$

**Proof:** Asset prices for bonds and stocks will not depend on $B$ since we are assuming that $B$ is large enough so that the equilibrium implements the consumption sharing rules $c^h = m^h e + b^h \cdot 1_Y$ for all $B$. The budget constraint (2) yields the consumption allocation that is implied by a portfolio with $\psi_j^h = m^h$ $\forall j = 1, \ldots, J$, $\theta_k^h = b^h$ $\forall k = 1, \ldots, B$, namely

$$
\begin{aligned}
c_y^h &= \sum_{j=1}^{J} m^h d_y^j + b^h (1 - q_y^1) + \sum_{k=2}^{B} b^h (q_y^{k-1} - q_y^k) \\
&= m^h e_y + b^h - b^h q_y^B.
\end{aligned}
$$

The price $q_y^B$ of bond $B$ is given by equation (18), see Appendix A.1. Because $\beta < 1$, $q_y^B \to 0$ as $B \to \infty$. Thus, $c_y^h \to m^h e_y + b^h$ and the statement of the theorem[11] follows. $\square$

Theorem 4 states that if we have a large number of finite-maturity bonds then the portfolio consisting of the market portfolio of stocks and the bond ladder comes arbitrarily close to implementing the equilibrium sharing rule. But real markets do not offer bonds with arbitrarily large maturities. We now check how close portfolios with ladders of a finite number of bonds of maturities $1, 2, \ldots, B$ come in generating efficient equilibrium outcomes. For this purpose we calculate the agents' welfare losses from using such a portfolio as opposed to the optimal portfolio.

## 6.2 Welfare Measure for Portfolios

Define a utility vector $v^h$ by $v_y^h = u^h(c_y^h)$ for a consumption vector $c^h$, where $c_y^h$ is the consumption of agent $h$ in state $y \in \mathcal{Y}$. Next define

$$V_{y_0}^h(c^h) = \sum_{t=0}^{\infty} \left(\beta^t \Pi^t\right)_{y_0 \cdot} v^h = \left([I - \beta\Pi]^{-1}\right)_{y_0 \cdot} v^h$$

to be the total discounted expected utility value over the infinite horizon when the economy starts in state $y_0$. Now we can define $C_{y_0}^{h,*}$ to be the consumption equivalent of agent $h$'s equilibrium allocation $m^h e + b^h$, which is defined by

$$\sum_{t=0}^{\infty} \beta^t u^h(C_{y_0}^{h,*}) = V_{y_0}^h(m^h e + b^h \cdot 1_Y) \iff C_{y_0}^{h,*} = (u^h)^{-1}\left((1-\beta)V_{y_0}^h(m^h e + b^h \cdot 1_Y)\right).$$

Similarly, we define a consumption equivalent $C_{y_0}^{h,B}$ for the consumption process that agent $h$ can achieve by holding the market portfolio of all stocks and a laddered portfolio of bonds of maturity $1, 2, \ldots, B$. Recall from the proof of Theorem 4 that such a portfolio with stock weights $m$ and bond weights $b$ supports the allocation $me + b \cdot 1_Y - bq_y^B$. The agent chooses the optimal weights $\hat{m}^h$ and $\hat{b}^h$ subject to the infinite-horizon budget constraint,

$$\max_{\{m,b\}} V_{y_0}^h(me + b \cdot 1_Y - bq_y^B) \quad \text{s.t.} \quad \left([I_S - \beta\Pi]^{-1}(P \otimes ((me + b \cdot 1_Y - bq_y^B) - c^h))\right)_{y_0} = 0$$

where $c^h \; (= m^h e + b^h \cdot 1_Y)$ denotes equilibrium consumption. The prices in the budget constraint are given by the equilibrium prices. We denote the consumption equivalent from this portfolio, which is optimal given the restrictions imposed on the agent, by

$$C_{y_0}^{h,B} = (u^h)^{-1}\left((1-\beta)V_{y_0}^h(\hat{m}^h e + \hat{b}^h \cdot 1_Y - \hat{b}^h q_y^B))\right).$$

---

[11] Note that as $B$ increases the number of assets $J + B$ will eventually exceed the fixed number of states, $Y$, and so the bond price vectors will be linearly dependent. As a result optimal portfolios will be indeterminate. The theorem only examines one particular portfolio, namely one consisting of a portion of the market portfolio and a bond ladder. To avoid indeterminate optimal portfolios we could increase the number of states in the limit process in order to keep the number of states and assets identical.

For the welfare comparison of the portfolio with a bond ladder to an agent's equilibrium portfolio we compute the welfare gain of each of these two portfolios relative to the welfare of the agent's initial endowment of stocks. For this purpose we also define a consumption equivalent $C_{y0}^{h,0}$ for the consumption vector that would result from constant initial stock holdings $\psi^{h,0} \equiv \psi_j^{h,0}$ for all $j \in \mathcal{J}$. Since in our examples we take sharing rules as given we need to calculate supporting initial stock endowments $\psi^{h,0}$ by solving the budget equations

$$\left([I_S - \beta\Pi]^{-1}(P \otimes ((m^h e + b^h \cdot 1_Y) - \psi^{h,0}e)))\right)_{y0} = 0, \ h = 1, \ldots, H.$$

Again the prices in the budget equation are the equilibrium prices. We denote the consumption equivalent from this initial portfolio by

$$C_{y0}^{h,0} = (u^h)^{-1}\left((1 - \beta)V_{y0}^h(\psi^{h,0}e)\right).$$

The welfare loss of the portfolio with constant bond holdings $\hat{b}^h$ relative to the optimal portfolio is then given by

$$\Delta C_{y0}^h = 1 - \frac{C_{y0}^{h,B} - C_{y0}^{h,0}}{C_{y0}^{h,*} - C_{y0}^{h,0}} = \frac{C_{y0}^{h,*} - C_{y0}^{h,B}}{C_{y0}^{h,*} - C_{y0}^{h,0}}.$$

## 6.3  Portfolios with Bond Ladders

We calculate welfare losses for portfolios with bond ladders and choose some of the same model specifications as before. We use the power utility functions from Section 4 with the resulting linear sharing rules

$$c^1 = \left(\frac{1}{2} - b\right) \cdot e + b \cdot 1_Y \ \text{ and } \ c^2 = \left(\frac{1}{2} + b\right) \cdot e - b \cdot 1_Y.$$

As before, we normalize stock dividends so that the expected aggregate endowment equals 1 and both agents consume on average half of the endowment. The dividend vectors of the $J = 4$ independent stocks are as follows,

| stock | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| high $d$ | 1.05 | 1.08 | 1.12 | 1.15 |
| low $d$ | 0.95 | 0.92 | 0.88 | 0.85 |

.

The economy starts in state $y_0 = 7$ (since $c_7^1 = c_7^2 = 0.5$). The transition probabilities for all four stocks are those of Section 5.3, that is, all four stocks have identical $2 \times 2$ transition matrices. Markets are complete with $J + 1 = 5$ bonds. For our first set of examples we set $\xi = 0.2$ and so the persistence probability for a stock's dividend state is 0.6. The discount factor is $\beta = 0.95$. The equilibrium portfolios for this economy then follow directly from Theorem 3 and the column for $J = 4$ in Table VIII. We vary the utility parameters $b$ and $\gamma$. Table IX reports the maximal welfare loss (always rounded upwards) across agents, $\Delta C = \max_{h \in \{1,2\}} \Delta C_7^h$. (We performed these welfare calculations with standard double

| $\gamma$ | 1 | | 3 | | 5 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| $B \backslash b$ | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 |
| 1 | 1.4 $(-4)$ | 1.4 $(-4)$ | 1.4 $(-3)$ | 1.4 $(-3)$ | 7.2 $(-3)$ | 7.2 $(-3)$ | 4.3 $(-2)$ | 4.8 $(-2)$ |
| 2 | 5.0 $(-6)$ | 5.0 $(-6)$ | 3.0 $(-3)$ | 3.0 $(-3)$ | 1.3 $(-2)$ | 1.3 $(-2)$ | 6.7 $(-2)$ | 7.5 $(-2)$ |
| 5 | 2.4 $(-10)$ | 2.4 $(-10)$ | 3.2 $(-3)$ | 3.2 $(-3)$ | 1.4 $(-2)$ | 1.4 $(-2)$ | 7.1 $(-2)$ | 8.1 $(-2)$ |
| 10 | 8.3 $(-13)$ | 6.3 $(-13)$ | 2.6 $(-3)$ | 2.6 $(-3)$ | 1.2 $(-2)$ | 1.2 $(-2)$ | 6.7 $(-2)$ | 7.5 $(-2)$ |
| 30 | $\approx 0$ | $\approx 0$ | 7.7 $(-4)$ | 7.7 $(-4)$ | 5.1 $(-3)$ | 5.1 $(-3)$ | 4.2 $(-2)$ | 4.7 $(-2)$ |
| 50 | $\approx 0$ | $\approx 0$ | 1.6 $(-4)$ | 1.6 $(-4)$ | 1.4 $(-3)$ | 1.4 $(-3)$ | 1.7 $(-2)$ | 1.9 $(-2)$ |
| 100 | $\approx 0$ | $\approx 0$ | 1.2 $(-6)$ | 1.2 $(-6)$ | 1.3 $(-5)$ | 1.3 $(-5)$ | 2.7 $(-4)$ | 2.8 $(-4)$ |

Table IX: Welfare Loss from Bond Ladder ($\xi = 0.2$)

precision. Numbers that are too close to computer machine precision to be meaningful are not reported and instead replaced by "$\approx 0$".)

As expected the relative welfare losses decrease to zero as the number $B$ of bonds increases. However, the losses do not decrease monotonically to zero. Recall that the equilibrium portfolios exhibit holdings close to $b$ for the one-period bond but already very different holdings for bonds of other short maturity. A trivial bond ladder of length 1 prescribes a bond holding that is not too far off from the equilibrium holding of approximately $b$. On the contrary, a bond ladder of length 5, for example, forces a portfolio upon an agent that is very different from the equilibrium portfolio in the holdings of these bonds. At the same time the ladder consists of too few bonds for the limiting behavior of Theorem 4 to set in. These facts result in the increased welfare losses for $B = 5$ and $\gamma \geq 3$. So the agent would prefer to just hold the one-period bond instead of the ladder with five bonds. Once the ladder gets long enough the welfare losses decrease monotonically to zero. Observe that welfare losses continue to decrease even after sufficiently many bonds are present to ensure market completeness. With $J = 4$ stocks and $Y = 16$ states only 12 bonds are needed to complete the markets. The addition of more long-term bonds improves the performance of bond ladder strategies even though the new bonds do not improve the span of the traded assets. The longer the time to maturity of the longest bond the smaller are both its prices across states and the standard deviation of these prices. The decreasing reinvestment risk results in smaller welfare losses of the bond ladder. Thus redundant bonds play an important role in improving investors' welfare.

Table X reports the restricted portfolio weights $(\hat{m}^1, \hat{b}^1)$ for agent 1. The last row in the table shows the coefficients of the linear sharing rule, which correspond to the holdings of stocks and the consol in an economy with a consol. The agent's holdings deviate considerably from these coefficients even when the welfare loss is already very small. For example, if $\gamma = 5$, $b = 0.3$ and $B = 50$, the holdings are $(\hat{m}^1, \hat{b}^1) = (.290, .229)$ instead of $(m^1, b) = (.2, .3)$ even though the welfare loss is less than 0.14%. This deviation is caused by the reinvestment risk in the longest bond. So, even though a ladder of, for example, 50 bonds comes very close

| $\gamma$ | 1 | | 3 | | 5 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| $B\backslash b$ | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 |
| 1 | (.497, .062) | (.482, .370) | (.499, .021) | (.496, .129) | (.500, .013) | (.500, .077) | (.501, .006) | (.507, .036) |
| 2 | (.495, .052) | (.470, .311) | (.498, .019) | (.492, .111) | (.500, .012) | (.498, .067) | (.501, .005) | (.505, .031) |
| 5 | (.489, .050) | (.433, .300) | (.496, .020) | (.476, .118) | (.498, .012) | (.488, .073) | (.500, .006) | (.501, .034) |
| 10 | (.480, .050) | (.380, .300) | (.491, .023) | (.447, .137) | (.495, .015) | (.469, .088) | (.498, .007) | (.491, .042) |
| 30 | (.461, .050) | (.265, .300) | (.473, .035) | (.336, .210) | (.479, .027) | (.376, .161) | (.488, .016) | (.432, .095) |
| 50 | (.454, .050) | (.223, .300) | (.460, .043) | (.260, .260) | (.465, .038) | (.290, .229) | (.474, .029) | (.346, .172) |
| 100 | (.450, .050) | (.202, .300) | (.451, .049) | (.205, .297) | (.451, .049) | (.209, .293) | (.453, .047) | (.218, .284) |
| $(m^1, b)$ | (.45 , .05 ) | (.2 , .3 ) | (.45 , .05 ) | (.2 , .3 ) | (.45 , .05 ) | (.2 , .3 ) | (.45 , .05 ) | (.2 , .3 ) |

Table X: $(\hat{m}^1, \hat{b}^1)$ for Table IX

to implementing the equilibrium allocation it uses portfolio weights different from the stock and consol weights to do so.

We recalculated all numbers in Tables IX and X for various sets of parameters. For completion we report in Appendix C.2 results for a larger level of the persistence parameter ($\xi = 0.5$). The results do not change qualitatively. Similarly, changing the discount factor does not result in qualitatively different results.

While we do not explicitly model transaction costs we can motivate the construction of bond ladders as a sensible investment approach in the face of transaction costs. As we have seen, equilibrium investment strategies imply enormous trading volume in the bond markets which would be very costly in the presence of even small transaction costs. On the contrary, bond ladders minimize transaction costs since the only transaction costs are those borne at the time the bonds are issued.[12]

# 7 Concluding Discussion

We conclude this paper with a reexamination of the asset allocation puzzle in light of our results. Finally, we argue that some limitations of our analysis, which are common in the literature, do not diminish the relevance of our key results.

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[12]In fact we may argue that transaction costs are lowest for newly issued so-called "on-the-run" bonds. A large portion of previously issued "off-the-run" bonds is often locked away in investors' portfolios which results in decreased liquidity of these bonds. For example, Amihud and Mendelson (1991) explain that bonds with lower liquidity have higher transaction costs. The increase of transaction costs for previously issued bonds clearly makes bond ladders even more sensible relative to equilibrium portfolios.

## 7.1   On the Asset Allocation Puzzle

Our analysis of investors' portfolios allows us to contribute to a recent discussion on the two-fund paradigm. In our discussion of two-fund separation in Section 3.1 we mentioned that various notions of this concept exist but that the notion that most people now have in mind when they talk about two-fund separation is monetary separation as defined in a static demand context by Cass and Stiglitz (1970). That is, people typically refer to the separation of investors' portfolios into the riskless asset and a common mutual fund of risky assets. To this day, and despite the early criticism of Merton (1973), this and other static results are often applied to dynamic contexts. Canner et al. (1997) document recommendations from different investment advisers who all encourage conservative investors to hold a higher ratio of bonds to stocks than aggressive investors.[13] They point out that this financial planning advice violates the two-fund monetary separation theorem and call this observation the "asset allocation puzzle." This apparent puzzle received considerable attention. Brennan and Xia (2000), Campbell and Viceira (2001), and Bajeux-Besnainou et al. (2001), among other papers, offer a resolution of this puzzle. In their models the two-fund separation theorem does not hold and the optimal ratio of bonds to stocks increases with an investor's risk aversion, which coincides with the recommendations of typical investment advisors. All three papers argue that the investment horizon is important and stress that the application of the classical static results to a dynamic problem can likely lead to misleading results.

We can give a different resolution of the asset allocation puzzle based on the results of our analysis. Cash is not a riskless asset in a dynamic world. Moreover, if the investment horizon exceeds the longest available bond maturity then investors do not have access to a safe asset. In the absence of a safe asset we cannot expect portfolios to exhibit the classical notion of two-fund monetary separation (in its narrow static sense). In our model with many bonds, instead, all investors, independently of their wealth and risk aversion, use the available finite-maturity bonds to generate the safe portion of their consumption stream. Investors with a higher demand for a safe consumption stream, such as more risk-averse investors, take larger positions in all bonds. For example, investors holding nearly optimal portfolios consisting of the market portfolio and a bond ladder would make larger investments in bonds of all maturities the larger their demand for a safe income stream. Therefore, we argue that we should view bonds as part of the portfolio that generates a safe stream of income even though their prices fluctuate over time. Only the stocks should be viewed as risky assets. And with this view two-fund monetary separation reemerges. All investors invest some portion of their wealth in the market portfolio of stocks and the remaining portion in a portfolio of all available bonds that (approximately) delivers a safe income stream. This interpretation of

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[13]Canner et al. consider portfolios consisting of stocks, bonds, and cash, with cash being treated as the riskless asset. They document that investment advisors recommend conservative (and even moderately risk averse) investors to hold a significant fraction of their wealth – beyond what liquidity needs would require – in cash assets. In addition, the recommended relative portions of stocks and bonds depend strongly on investors' risk attitudes. Advisors treat bonds as risky relative to cash, so that the risky portfolio consists of both stocks and bonds. The fact that the recommended ratio of these assets depends on the investor's risk aversion violates two-fund (monetary) separation.

our results thus reconciles the fact that bond investments are increasing in investors' risk aversion with the two-fund separation paradigm.

## 7.2   Limitations and Implications

Similar to many other analyses of bonds in the literature we do not account for all characteristics of bonds that a sensible investor needs to be aware of. First, we assume that the bonds in our model have no credit risk. We thus completely abstract from the possibility that the bond issuer may default. Second, all bonds in the model have no call provision. Strickland et al. (2008) emphasizes that a laddered bond portfolio should ideally consists only of non-callable bonds. Third, we abstract from tax consequences of bond investments and thus do not distinguish between taxable, tax-deferred, or tax-exempt bonds.

In addition, and again similar to much other work, we do not account for inflation. As a result the bonds in our model should be interpreted as inflation-protected bonds. Such bonds exists, for example, the U.S. Treasury has been issuing Treasury Inflation-Protected Securities (TIPS) since 1997. Hammond (2002) advocates that investors should buy such inflation-protected bonds. Our model implies that there should be TIPS with long maturities since those are the key to a bond ladder's effectiveness.[14]

Most modern work on portfolio choice examines pure asset demand instead of equilibrium portfolios. Asset price or return processes are exogenously given and are not determined by equilibrium conditions. Instead, we employ a general equilibrium model in order to enforce a consistency between investors' preferences, dividends and the prices of all securities. We regard our general equilibrium model to be an excellent expositional tool for our analysis. It would certainly be possible to do a similar analysis with exogenously specified non-equilibrium price processes in our model with many states and bonds.

We believe that neither our choice of model nor the limitations of our analysis diminish the relevance of our bond ladder results. Any sensible analysis of bonds with many different maturities, whether in the presence or absence of inflation, whether in face of equilibrium prices or exogenously specified price processes, will also imply that long-term bonds are nearly perfect substitutes. Naturally, 'optimal' portfolios of such bonds will likely exhibit the implausibly large long and short position of the nearly dependent bonds. We expect that in these circumstances bond ladders will again serve as both simple and nearly optimal bond investment strategies for investors who want to generate a safe income stream. The

---

[14]Hammond (2002) writes:

> *In fact, it might be more appropriate to think of inflation bonds, not as one of the portfolio's risky assets, but rather as the closest we can get to the theoretical riskless asset.*

Our analysis refines this statement. It is not an inflation-indexed bond of fixed maturity that gets closest to the theoretical riskless asset, but actually a bond ladder of inflation-indexed bonds of varying maturities. Moreover, the longer the maturity of the longest-maturity bond the better the bond ladder replicates the theoretical riskless asset. This finding has clear policy implications: It is beneficial for investors to have access to inflation-indexed bonds with very long maturities. Our results support the U.S. federal government's renewed commitment to inflation-indexed bonds (Hammond, 2002).

introduction of redundant bonds that increase the set of available maturities further reduces the reinvestment risk of ladders and thus helps investors to generate a stream of safe payoffs. In sum, the features of our analysis that make bond ladders an attractive investment strategy are robust to sensible variations of the modeling framework. It is, therefore, no surprise that we observe laddered bond portfolios as a popular investment strategy on financial markets.

# Appendix

# A   Equilibrium in Dynamically Complete Markets

## A.1   Equilibrium Formulas

We use the Negishi approach (Negishi (1960)) of Judd et al. (2003) to characterize efficient equilibria in our model. Efficient equilibria exhibit time-homogeneous consumption processes and asset prices, that is, consumption allocations and asset prices only depend on the last shock $y$. Define the vector $P = \left(u_1'(c_y^1)\right)_{y \in \mathcal{Y}} \in \mathbb{R}_{++}^S$ to be the vector of prices for consumption across states $y \in \mathcal{Y}$. We denote the $S \times S$ identity matrix by $I_S$, Negishi weights by $\lambda^h$, $h = 2, \ldots, H$, and use $\otimes$ to denote element-wise multiplication of vectors.

If the economy starts in the state $y_0 \in \mathcal{Y}$ at period $t = 0$, then the Negishi weights and consumption vectors must satisfy the following equations.

$$u_1'(c_y^1) - \lambda^h u_h'(c_y^h) = 0, \ h = 2, \ldots, H, \ y \in \mathcal{Y}, \tag{11}$$

$$\left([I_S - \beta\Pi]^{-1}(P \otimes (c^h - \sum_{j=1}^{J} \psi_j^{h,0} d^j))\right)_{y_0} = 0, \ h = 2, \ldots, H, \tag{12}$$

$$\sum_{h=1}^{H} c_y^h - e_y = 0, \ y \in \mathcal{Y}. \tag{13}$$

Once we have computed the consumption vectors we can give closed-form solutions for asset prices and portfolio holdings. The price vector of a stock $j$ is given by

$$q^j \otimes P = [I_S - \beta\Pi]^{-1}\beta\Pi(P \otimes d^j). \tag{14}$$

Similarly, the price of a consol is given by

$$q^c \otimes P = [I_S - \beta\Pi]^{-1}\beta\Pi P. \tag{15}$$

We calculate the price of finite-maturity bonds in a recursive fashion. First, the price of the one-period bond in state $y$ is

$$q_y^1 = \frac{\beta\Pi_y.P}{u_1'(c_y^1)} = \frac{\beta \sum_{z=1}^{Y} \Pi_{yz} P_z}{u_1'(c_y^1)}, \tag{16}$$

where $\Pi_{y\cdot}$ denotes row $y$ of the matrix $\Pi$. Then the price of the bond of maturity $k$ is

$$q_y^k = \frac{\beta \Pi_{y\cdot}(P \otimes q^{k-1})}{u_1'(c_y^1)} = \frac{\beta \sum_{z=1}^{Y} \Pi_{yz} P_z q_z^{k-1}}{u_1'(c_y^1)} \ . \tag{17}$$

Repeated substitution yields the bond price formula

$$q_y^k = \frac{\beta^k (\Pi^k)_{y\cdot} P}{u_1'(c_y^1)} = \frac{\beta^k \sum_{z=1}^{Y} (\Pi^k)_{yz} P_z}{u_1'(c_y^1)}. \tag{18}$$

Given the consumption allocations and asset prices the budget equations (2) or (1) now determine the asset positions for economies with finite-maturity bonds or the consol, respectively.

## A.2 Linear Sharing Rules

We can easily calculate the linear sharing rules for the three HARA families of utility functions under consideration. Some straightforward algebra yields the following sharing rules (as a function of Negishi weights which are determined through the budget equations).

For power utility functions the linear sharing rule is

$$c_y^h = e_y \cdot \left( \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right) + \left( A^h - \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \sum_{i \in \mathcal{H}} A^i \right) = m^h e_y + b^h. \tag{19}$$

Note that for CRRA utility functions, $A^h = 0$ for all $h \in \mathcal{H}$, the sharing rule has zero intercept, $b^h = 0$, and household $h$ consumes a constant fraction

$$m^h = \left( \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right)$$

of the total endowment. For quadratic utility functions, we obtain

$$c_y^h = e_y \cdot \left( \frac{(\lambda^h)^{-1}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{-1}} \right) + \left( B^h - \frac{(\lambda^h)^{-1}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{-1}} \sum_{i \in \mathcal{H}} B^i \right). \tag{20}$$

For CARA utility functions the linear sharing rules are

$$c_y^h = e_y \cdot \frac{\tau^h}{\sum_{i \in \mathcal{H}} \tau^i} + \left( \tau^h \ln(\lambda^h) - \frac{\tau^h}{\sum_{i \in \mathcal{H}} \tau^i} \sum_{i \in \mathcal{H}} \tau^i \ln(\lambda^i) \right), \tag{21}$$

where $\tau^h = 1/a^h$ is the constant absolute risk tolerance of agent $h$.

## A.3  Allocations and Prices for the Example in Section 3.3

Consumption allocations are as follows.

$$
\begin{aligned}
c^1 &= (0.688, 0.666, 0.666, 0.644)^\top = \frac{6}{11}\left(d^1 + d^2\right) - 0.425,\\
c^2 &= (0.678, 0.667, 0.667, 0.656)^\top = \frac{3}{11}\left(d^1 + d^2\right) + 0.121,\\
c^3 &= (0.674, 0.667, 0.667, 0.660)^\top = \frac{2}{11}\left(d^1 + d^2\right) + 0.304.
\end{aligned}
$$

The fluctuations of agents' consumption allocations across the four states are fairly small. The reason for this small variance is the small dividend variance of the two stocks. The state-contingent stock prices are solutions to any agent's Euler equations and are

$$
\begin{aligned}
p^1 &= (19.43, 19.01, 18.98, 18.58)^\top,\\
p^2 &= (19.40, 18.98, 19.01, 18.60)^\top.
\end{aligned}
$$

The price vector of the consol is

$$
q^c = (19.40, 18.99, 19.01, 18.61)^\top.
$$

Bond prices for bonds of various maturity are

$$
\begin{aligned}
q^1 &= (0.963, 0.946, 0.954, 0.938)^\top,\\
q^2 &= (0.918, 0.899, 0.906, 0.887)^\top,\\
q^5 &= (0.790, 0.773, 0.775, 0.758)^\top,\\
q^{10} &= (0.612, 0.599, 0.599, 0.586)^\top,\\
q^{25} &= (0.284, 0.277, 0.277, 0.271)^\top,\\
q^{50} &= (0.079, 0.077, 0.077, 0.075)^\top.
\end{aligned}
$$

# B  Technical Details

## B.1  Additional Proofs

**Proof of Lemma 1:** Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_Y$ be the eigenvalues of the matrix $\Pi$. Since $\Pi$ is a transition matrix $\lambda_1 = 1$. Since $\Pi$ is diagonalizable and nonsingular, $\Pi = C\Lambda C^{-1}$ where $C$ is invertible and $\Lambda$ is diagonal containing only, but all of, the eigenvalues $\lambda_i$. Furthermore, $C^{-1}\Pi^k C = \Lambda^k$ for any $k = 1, 2, \ldots$ (see, for example, Simon and Blume (1994, Theorem 23.7)).

Statement (1). Multiplying the statement's matrix equation by $C^{-1}$ from the left and by $C$ from the right leads to the equivalent system,

$$
\sum_{k=1}^{L} a_k\, \Lambda^k = -I_Y.
$$

$\Lambda$ is diagonal and has only $L$ distinct entries. As a result this last system is equivalent to the $L$-dimensional linear system

$$M \cdot (a_1, \ldots, a_L)^T = -(1_Y)^T,$$

where $1_Y$ is the $Y$-dimensional row vector of all ones and

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_2 & (\lambda_2)^2 & \cdots & (\lambda_2)^L \\ \lambda_3 & (\lambda_3)^2 & \cdots & (\lambda_3)^L \\ & & \ddots & \\ \lambda_L & (\lambda_L)^2 & \cdots & (\lambda_L)^L \end{bmatrix}.$$

where we assume w.l.o.g. that $\lambda_1 = 1, \lambda_2, \ldots, \lambda_L$ are the $L$ distinct eigenvalues of $\Pi$. Column $k$ contains the corresponding (distinct) eigenvalues of $\Pi^k$. The matrix $M$ has full rank $L$ since all eigenvalues are nonzero. Thus, the original matrix equation has a unique solution. Note that the first equation requires $\sum_{k=1}^L a_k = -1$.

Statement (2). Multiplying the statement's matrix equation by $C^{-1}$ from the left and by $C$ from the right implies,

$$\sum_{k=1}^L a_k \, \Lambda^k = 0.$$

The diagonal matrix $\Lambda$ has only $L - 1$ distinct nonzero entries. As a result this last system is equivalent to the $(L - 1)$-dimensional linear system

$$M' \cdot (a_1, \ldots, a_L)^T = 0,$$

where

$$M' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_2 & (\lambda_2)^2 & \cdots & (\lambda_2)^L \\ \lambda_3 & (\lambda_3)^2 & \cdots & (\lambda_3)^L \\ & & \ddots & \\ \lambda_{L-1} & (\lambda_{L-1})^2 & \cdots & (\lambda_{L-1})^L \end{bmatrix}.$$

where we assume w.l.o.g. that $\lambda_1 = 1, \lambda_2, \ldots, \lambda_{L-1}$ are the $L - 1$ distinct nonzero eigenvalues of $\Pi$. The matrix $M'$ has full row rank $L - 1$. Thus, the original matrix equation must have a nontrivial solution. (In fact, the system has a one-dimensional linear solution manifold.) Note that the first equation requires $\sum_{k=1}^L a_k = 0$. $\square$

**Proof of Corollary 1:** In an economy with bonds of maturities $k = 1, 2, \ldots, L$, budget constraint (5) becomes

$$m^h \cdot e + b^h 1_Y = \eta^h \cdot e + \theta_1^h(1_Y - q^1) + \sum_{k=2}^L \theta_k^h(q^{k-1} - q^k). \tag{22}$$

A sufficient condition for two-fund separation is $m^h = \eta^h$ for all agents $h \in \mathcal{H}$. (This condition is only sufficient but not necessary since there could be other stock weights $\tilde{\eta}^h \neq m^h$.) For this condition to hold agent $h$'s bond portfolio must satisfy

$$b^h 1_Y = \theta_1^h(1_Y - q^1) + \sum_{k=2}^{L} \theta_k^h(q^{k-1} - q^k), \tag{23}$$

that is, the $L$ bonds must span the consol. That fact follows from Theorem 2.

In the proof of Theorem 2 we showed that a sufficient condition for the previous system of equations to have a solution is that the matrix equation

$$(b^h - \theta_1^h)I_Y + \sum_{k=1}^{L-1}(\theta_k^h - \theta_{k+1}^h)(\beta\Pi)^k + \theta_L^h(\beta\Pi)^L = 0. \tag{24}$$

has a solution. Note that the coefficients satisfy $(b^h - \theta_1^h) + \sum_{k=1}^{L-1}(\theta_k^h - \theta_{k+1}^h) + \theta_L^h = b^h$.

Case (a). Suppose the transition matrix $\Pi$ has only nonzero eigenvalues. Multiply equations $I_Y + \sum_{k=1}^{L} a_k \Pi^k = 0$ (Lemma 1, Part (1)) by $\beta^L$ to obtain $\beta^L I_Y + \sum_{k=1}^{L} \beta^{L-k} a_k^* (\beta\Pi)^k = 0$ and define the sum of the (new) coefficients to be $M_a = \beta^L + \sum_{k=1}^{L} \beta^{L-k} a_k^*$. Then multiplying through by $\frac{b^h}{M_a}$ yields the expression

$$(\frac{b^h}{M_a}\beta^L)I_Y + \sum_{k=1}^{L}(\frac{b^h}{M_a}\beta^{L-k}a_k^*)\,(\beta\Pi)^k = 0, \tag{25}$$

where the sum of the coefficients $\frac{b^h}{M_a}(\beta^L + \sum_{k=1}^{L}\beta^{L-k}a_k^*)$ equals $b^h$. Matching the coefficients in equations (24) and (25) gives the expressions of the corollary.

Case (b). Suppose the transition matrix $\Pi$ has a zero eigenvalue. Multiply equations $\sum_{k=1}^{L} a_k \Pi^k = 0$ (Lemma 1, Part (2)) by $\beta^L$ to obtain $\sum_{k=1}^{L} \beta^{L-k} a_k^* (\beta\Pi)^k = 0$ and define the sum of the (new) coefficients to be $M_b = \sum_{k=1}^{L} \beta^{L-k} a_k^*$. Then multiplying through by $\frac{b^h}{M_b}$ yields the expression

$$\sum_{k=1}^{L}(\frac{b^h}{M_b}\beta^{L-k}a_k^*)\,(\beta\Pi)^k = 0, \tag{26}$$

where the sum of the coefficients $\sum_{k=1}^{L}\frac{b^h}{M_b}\beta^{L-k}a_k^*$ equals $b^h$. Matching the coefficients in equations (24) and (26) yields $\theta_1^h = b^h$ and the other expressions of the corollary. $\square$

## B.2 Kronecker Products

Let $A$ be an $n \times p$ matrix and $B$ be an $m \times q$ matrix. Then the Kronecker or direct product $A \bigotimes B$ is defined as the $nm \times pq$ matrix

$$A\bigotimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1p}B \\ a_{21}B & a_{22}B & \cdots & a_{2p}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{np}B \end{bmatrix}.$$

Langville and Stewart (2004) list many useful properties of the Kronecker product. For our purposes we need the following properties.

1. If $A$ and $B$ are stochastic (Markov matrices) then $A \otimes B$ is stochastic.

2. $\text{rank}(A \otimes B) = \text{rank}(A)\text{rank}(B)$.

3. Let $A$ and $B$ be two square matrices. Let $\lambda$ ($\mu$) be an eigenvalue of $A$ ($B$) and $x_A$ ($x_B$) be the corresponding eigenvector. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ and $x_A \otimes x_B$ is the corresponding eigenvector. Every eigenvalue of $A \otimes B$ arises as a product of eigenvalues of $A$ and $B$.

4. If $A$ and $B$ are diagonalizable then $A \otimes B$ is diagonalizable.

5. $(PDP^{-1}) \otimes (PDP^{-1}) = (P \otimes P)(D \otimes D)(P^{-1} \otimes P^{-1})$

In Sections 4 and 5.3 we defined economies with special transition matrices that are $J$-fold Kronecker products of $D \times D$ transition matrices $\Xi$, so $\Pi = \Xi \otimes \Xi \otimes \cdots \otimes \Xi = \bigotimes_{j=1}^{J} \Xi$. Property 1 of Kronecker products implies that $\Pi$ is a stochastic matrix (Markov transition matrix). The following properties of $\Pi$ follow from the characteristics of $\Xi$ and the listed properties of Kronecker products.

**Lemma 2** *Let the transition matrix $\Pi$ be a $J$-fold Kronecker product of the matrix $\Xi$, which has only real nonzero eigenvalues, is diagonalizable, and has $l$ distinct eigenvalues. Then $\Pi$ has the following properties.*

1. *$\text{rank}(\Pi) = (\text{rank}(\Xi))^J$.*

2. *The matrix $\Pi$ has $D^J$ real nonzero eigenvalues, $\binom{J+l+1}{l-1}$ of which are distinct.*

3. *The matrix $\Pi$ is diagonalizable, that is, the eigenvector matrix $C$ of $\Pi$ has full rank $D^J$.*

In Sections 4 and 5.3 we encountered the special case of the $2 \times 2$ transition matrix by

$$\Xi = \begin{bmatrix} \frac{1}{2}(1+\xi_H) & \frac{1}{2}(1-\xi_H) \\ \frac{1}{2}(1-\xi_L) & \frac{1}{2}(1+\xi_L) \end{bmatrix}$$

with $\xi_H, \xi_L \in (0,1)$. This matrix $\Xi$ has $D = 2$ distinct eigenvalues, 1 and $\xi = (\xi_H + \xi_L)/2 < 1$. For the computation of bond portfolios we need to find $(a_1^*, a_2^*, \ldots, a_{J+1}^*)$ where

$$(a_1^*, a_2^*, \ldots, a_{J+1}^*)^T = -M^{-1} \cdot 1_{J+1}$$

and

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi & \xi^2 & \cdots & \xi^{J+1} \\ \xi^2 & \xi^4 & \cdots & \xi^{2(J+1)} \\ & & \ddots & \\ \xi^J & \xi^{2J} & \cdots & \xi^{J(J+1)} \end{bmatrix}.$$

Closed-form solutions to these equations do exist. We leave the calculation to the reader and Mathematica.

# C   Additional Results

## C.1   More Results for Section 5.3

For the examples in Section 5.3, see Table VIII, we report the analog results for $\beta = 0.99$ and when the persistence parameter is $\xi = 0.5$. These parameter changes do not result in any qualitatively different results.

[Tables XI, XII, XIII HERE]

## C.2   More Results for Section 6.2

For the examples in Section 6.2, Tables XIV and XIV report the analog results for Tables IX and X when the persistence parameter is $\xi = 0.5$. This parameter change does not result in any qualitatively different results. (Again numbers that are too close to computer machine precision to be meaningful are not reported and instead replaced by "$\approx 0$".)

[Tables XIV, XV HERE]

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| $J$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $\theta_1$ | 6.354 | 0.278 | 1.046 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_2$ | $-33.099$ | 11.671 | $-0.441$ | 1.091 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_3$ | 49.961 | $-44.292$ | 15.203 | $-0.919$ | 1.121 | 0.996 | 1.000 | 1.000 | 1.000 |
| $\theta_4$ | | 56.693 | $-50.666$ | 17.219 | $-1.193$ | 1.139 | 0.996 | 1.000 | 1.000 |
| $\theta_5$ | | | 60.272 | $-54.060$ | 18.293 | $-1.338$ | 1.148 | 0.995 | 1.000 |
| $\theta_6$ | | | | 62.116 | $-55.811$ | 18.847 | $-1.413$ | 1.153 | 0.995 |
| $\theta_7$ | | | | | 63.052 | $-56.700$ | 19.128 | $-1.452$ | 1.155 |
| $\theta_8$ | | | | | | 63.523 | $-57.147$ | 19.270 | $-1.471$ |
| $\theta_9$ | | | | | | | 63.760 | $-57.372$ | 19.341 |
| $\theta_{10}$ | | | | | | | | 63.879 | $-57.485$ |
| $\theta_{11}$ | | | | | | | | | 63.938 |

Table XI: Bond Portfolio Spanning one Unit of the Consol, $\xi = 0.5$, $\beta = 0.95$

| $J$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $\theta_1$ | 2.008 | 0.992 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_2$ | $-29.549$ | 2.260 | 0.990 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_3$ | 129.830 | $-30.821$ | 2.312 | 0.990 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_4$ | | 130.866 | $-31.078$ | 2.323 | 0.989 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_5$ | | | 131.074 | $-31.129$ | 2.325 | 0.989 | 1.000 | 1.000 | 1.000 |
| $\theta_6$ | | | | 131.115 | $-31.140$ | 2.325 | 0.989 | 1.000 | 1.000 |
| $\theta_7$ | | | | | 131.123 | $-31.142$ | 2.325 | 0.989 | 1.000 |
| $\theta_8$ | | | | | | 131.125 | $-31.142$ | 2.325 | 0.989 |
| $\theta_9$ | | | | | | | 131.125 | $-31.142$ | 2.325 |
| $\theta_{10}$ | | | | | | | | 131.126 | $-31.142$ |
| $\theta_{11}$ | | | | | | | | | 131.126 |

Table XII: Bond Portfolio Spanning one Unit of the Consol, $\xi = 0.2$, $\beta = 0.99$

| $J$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $Y$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| $\theta_1$ | 32.917 | $-3.507$ | 1.297 | 0.991 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_2$ | $-192.757$ | 64.788 | $-8.012$ | 1.594 | 0.981 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\theta_3$ | 263.149 | $-257.143$ | 86.021 | $-11.014$ | 1.793 | 0.975 | 1.000 | 1.000 | 1.000 |
| $\theta_4$ | | 300.313 | $-293.911$ | 98.150 | $-12.729$ | 1.906 | 0.971 | 1.000 | 1.000 |
| $\theta_5$ | | | 320.120 | $-313.514$ | 104.618 | $-13.644$ | 1.966 | 0.969 | 1.001 |
| $\theta_6$ | | | | 330.340 | $-323.630$ | 107.957 | $-14.116$ | 1.997 | 0.967 |
| $\theta_7$ | | | | | 335.531 | $-328.768$ | 109.652 | $-14.356$ | 2.029 |
| $\theta_8$ | | | | | | 338.146 | $-331.357$ | 110.506 | $-14.598$ |
| $\theta_9$ | | | | | | | 339.459 | $-332.656$ | 111.365 |
| $\theta_{10}$ | | | | | | | | 340.116 | $-333.960$ |
| $\theta_{11}$ | | | | | | | | | 340.775 |

Table XIII: Bond Portfolio Spanning one Unit of the Consol, $\xi = 0.5$, $\beta = 0.99$

| $\gamma$ | 1 | | 3 | | 5 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| $B\backslash b$ | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 |
| 1 | 7.8 $(-4)$ | 7.9 $(-4)$ | 5.3 $(-6)$ | 5.3 $(-6)$ | 9.0 $(-4)$ | 9.0 $(-4)$ | 1.2 $(-2)$ | 1.3 $(-2)$ |
| 2 | 1.9 $(-4)$ | 1.9 $(-4)$ | 8.6 $(-4)$ | 8.6 $(-4)$ | 5.6 $(-3)$ | 5.6 $(-3)$ | 3.6 $(-2)$ | 3.9 $(-2)$ |
| 5 | 2.2 $(-6)$ | 2.3 $(-6)$ | 2.8 $(-3)$ | 2.8 $(-3)$ | 1.3 $(-2)$ | 1.3 $(-2)$ | 6.6 $(-2)$ | 7.4 $(-2)$ |
| 10 | 1.3 $(-9)$ | 1.4 $(-9)$ | 2.6 $(-3)$ | 2.6 $(-3)$ | 1.2 $(-2)$ | 1.2 $(-2)$ | 6.7 $(-2)$ | 7.5 $(-2)$ |
| 30 | 1.2 $(-11)$ | 8.8 $(-14)$ | 7.7 $(-4)$ | 7.7 $(-4)$ | 5.1 $(-3)$ | 5.1 $(-3)$ | 4.2 $(-2)$ | 4.7 $(-2)$ |
| 50 | $\approx 0$ | $\approx 0$ | 1.6 $(-4)$ | 1.6 $(-4)$ | 1.4 $(-3)$ | 1.4 $(-3)$ | 1.7 $(-2)$ | 1.9 $(-2)$ |
| 100 | $\approx 0$ | $\approx 0$ | 1.2 $(-6)$ | 1.2 $(-6)$ | 1.3 $(-5)$ | 1.3 $(-5)$ | 2.7 $(-4)$ | 2.8 $(-4)$ |

Table XIV: Welfare Loss from Bond Ladder ($\xi = 0.5$)

| $\gamma$ | 1 | | 3 | | 5 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| $B\backslash b$ | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 | 0.05 | 0.3 |
| 1 | (.496, .095) | (.472, .570) | (.499, .034) | (.492, .203) | (.500, .205) | (.498, .123) | (.501, .010) | (.506, .060) |
| 2 | (.494, .064) | (.463, .387) | (.498, .024) | (.489, .141) | (.499, .014) | (.496, .086) | (.501, .007) | (.505, .041) |
| 5 | (.489, .051) | (.431, .307) | (.496, .020) | (.475, .121) | (.498, .012) | (.487, .075) | (.500, .006) | (.500, .035) |
| 10 | (.480, .050) | (.380, .300) | (.491, .023) | (.447, .137) | (.495, .015) | (.469, .088) | (.498, .007) | (.491, .042) |
| 30 | (.461, .050) | (.265, .300) | (.473, .035) | (.336, .210) | (.479, .027) | (.376, .161) | (.488, .016) | (.431, .095) |
| 50 | (.454, .050) | (.223, .300) | (.460, .043) | (.260, .260) | (.465, .038) | (.290, .229) | (.474, .029) | (.345, .172) |
| 100 | (.450, .050) | (.202, .300) | (.451, .049) | (.205, .297) | (.451, .049) | (.209, .293) | (.453, .047) | (.218, .284) |
| $(m^1, b^1)$ | (.45 , .05 ) | (.2 , .3 ) | (.45 , .05 ) | (.2 , .3 ) | (.45 , .05 ) | (.2 , .3 ) | (.45 , .05 ) | (.2 , .3 ) |

Table XV: $(\hat{m}^1, \hat{b}^1)$ for Table XIV