ML Estimation of a First-Price Auction Model Using the MPEC Approach

Pamela Cardozo (QED), Chris Conlon (Yale),
Brent Hickman (Iowa), Nicolas Roys (UCL)
ICE 2008

August 8, 2008

The Model:

- 2 risk-neutral bidders for one indivisible object

The Model:

- 2 risk-neutral bidders for one indivisible object
- Each has a private value v_i $i = 1, 2$, representing maximal willingness to pay for the object

The Model:

- 2 risk-neutral bidders for one indivisible object
- Each has a private value v_i $i = 1, 2$, representing maximal willingness to pay for the object
- Beliefs are summarized by a distribution $V \sim F_V(v|\theta)$
  - (note the iid assumption)

The Model:

- 2 risk-neutral bidders for one indivisible object
- Each has a private value v_i $i = 1, 2$, representing maximal willingness to pay for the object
- Beliefs are summarized by a distribution $V \sim F_V(v|\theta)$
  - (note the iid assumption)
- Bidders choose bids $b_i = \beta(v_i)$ to maximize their expected payoff:

$$(v_i - b_i)F_V(\beta^{-1}(b_i)|\theta)$$

The Model:

- 2 risk-neutral bidders for one indivisible object
- Each has a private value $v_i \;\; i = 1, 2$, representing maximal willingness to pay for the object
- Beliefs are summarized by a distribution $V \sim F_V(v|\theta)$
  - (note the iid assumption)
- Bidders choose bids $b_i = \beta(v_i)$ to maximize their expected payoff:

  $$(v_i - b_i)F_V(\beta^{-1}(b_i)|\theta)$$
- The solution to β is characterized by the following IVP:
  - $\beta^{'}(v) = \frac{(v-\beta(v))f_V(v)}{F_V(v)}$; $\beta(0) = 0$.

THE EMIRICAL GOAL: to estimate F_V from a sample of observed bids $\{b_t\}_{t=1}^{T}$, using our theoretical knowledge of bid formulation.

Structural Auction Econometrics 101

Parametric Methods:

- Maximum Likelihood:
  - Paarsch (1992)
  - Donald & Paarsch (1993)
  - Paarsch (1997)
- Others:
  - Laffont, Ossard & Vuong (1995)–Simulated NLS
  - Eliyakime, Laffont, Loisel and Vuong (1997)–NLS
  - Donald & Paarsch (2002)–GMM

Structural Auction Econometrics 101

Parametric Methods – DRAWBACKS

- **Computational Burden:**

  All of the above procedures rely on the existence of analytic solutions to the equilibrium bid function.

Structural Auction Econometrics 101

Parametric Methods – DRAWBACKS

- **Computational Burden:**

  All of the above procedures rely on the existence of analytic solutions to the equilibrium bid function.

  closed-form $\beta \Rightarrow$ closed-form likelihood function
  $L(\theta; \{b_t\}_{t=1}^{T}) = \prod_{t=1}^{T} f_V(\beta^{-1}(b_t))$

Structural Auction Econometrics 101

Parametric Methods – DRAWBACKS

- **Computational Burden:**

  All of the above procedures rely on the existence of analytic solutions to the equilibrium bid function.

  closed-form $\beta \Rightarrow$ closed-form likelihood function

  $L(\theta; \{b_t\}_{t=1}^{T}) = \prod_{t=1}^{T} f_V(\beta^{-1}(b_t))$

  - **PROBLEM:** explicit solutions for β are hard to come by!

  $\Rightarrow$ we are stuck with some nested solution algorithm

Structural Auction Econometrics 101

Parametric Methods – DRAWBACKS

- **Computational Burden:**

  All of the above procedures rely on the existence of analytic solutions to the equilibrium bid function.

  closed-form $\beta \Rightarrow$ closed-form likelihood function
  $L(\theta; \{b_t\}_{t=1}^{T}) = \prod_{t=1}^{T} f_V(\beta^{-1}(b_t))$

  - **PROBLEM:** explicit solutions for β are hard to come by!

  $\Rightarrow$ we are stuck with some nested solution algorithm

- **Restrictiveness**
  - Parametric assumptions have the potential to introduce mis-specification errors
  - The only known explicit solutions come from simple F_Vs

Review of Generic ML Approaches

Procedure Without Explicitly Defined $\beta(\cdot)$: For an unknown parameter vector θ and a sample of bids $\{b_t\}_{t=1}^{T}$, we

1. Supply an initial guess θ_0

Review of Generic ML Approaches

Procedure Without Explicitly Defined $\beta(\cdot)$: For an unknown parameter vector θ and a sample of bids $\{b_t\}_{t=1}^{T}$, we

1. Supply an initial guess θ_0
2. solve the equilibrium equation $\beta^{'}(v) = h\left(F_V(v|\theta), \beta(v), v\right)$ to obtain a vector of private values $\beta^{-1}(b_t; \theta_k) = v_{\theta_k,t} \;\; \forall t$.

Review of Generic ML Approaches

Procedure Without Explicitly Defined $\beta(\cdot)$: For an unknown parameter vector θ and a sample of bids $\{b_t\}_{t=1}^{T}$, we

1. Supply an initial guess θ_0
2. solve the equilibrium equation $\beta^{'}(v) = h\left(F_V(v|\theta), \beta(v), v\right)$ to obtain a vector of private values $\beta^{-1}(b_t; \theta_k) = v_{\theta_k,t} \ \ \forall t$.
3. Plug these values into the score vector and Hessian of the Likelihood function.

Review of Generic ML Approaches

Procedure Without Explicitly Defined $\beta(\cdot)$: For an unknown parameter vector θ and a sample of bids $\{b_t\}_{t=1}^{T}$, we

1. Supply an initial guess θ_0
2. solve the equilibrium equation $\beta^{'}(v) = h\left(F_V(v|\theta), \beta(v), v\right)$ to obtain a vector of private values $\beta^{-1}(b_t; \theta_k) = v_{\theta_k,t} \;\; \forall t$.
3. Plug these values into the score vector and Hessian of the Likelihood function.
4. Update the guess to θ_{k+1} via some adaptation of Newton's method

Review of Generic ML Approaches

Procedure Without Explicitly Defined $\beta(\cdot)$: For an unknown parameter vector θ and a sample of bids $\{b_t\}_{t=1}^{T}$, we

1. Supply an initial guess θ_0
2. solve the equilibrium equation $\beta^{'}(v) = h\left(F_V(v|\theta), \beta(v), v\right)$ to obtain a vector of private values $\beta^{-1}(b_t; \theta_k) = v_{\theta_k,t} \ \ \forall t$.
3. Plug these values into the score vector and Hessian of the Likelihood function.
4. Update the guess to θ_{k+1} via some adaptation of Newton's method
5. Return to step 2 and continue iterating until the stopping criterion is satisfied.

Structural Auction Econometrics 101

Nonparametric Estimation: Guerre, Perrigne & Vuong (2000)

- The FOCs can be reworked to express private values as a function of bids and the distribution over bids:

Structural Auction Econometrics 101

Nonparametric Estimation: Guerre, Perrigne & Vuong (2000)

- The FOCs can be reworked to express private values as a function of bids and the distribution over bids:
  - $\beta^{-1}(b) = b + \frac{F_B(b)}{(N-1)f_B(b)} \quad \Rightarrow \quad v = \beta^{-1}(b, F_B)$

Structural Auction Econometrics 101

Nonparametric Estimation: Guerre, Perrigne & Vuong (2000)

- The FOCs can be reworked to express private values as a function of bids and the distribution over bids:
  - $\beta^{-1}(b) = b + \frac{F_B(b)}{(N-1)f_B(b)} \quad \Rightarrow \quad v = \beta^{-1}(b, F_B)$
  - **ADVANTAGE:** F_B is directly observable from the bid data!

Structural Auction Econometrics 101

Nonparametric Estimation: Guerre, Perrigne & Vuong (2000)

- The FOCs can be reworked to express private values as a function of bids and the distribution over bids:
  - $\beta^{-1}(b) = b + \frac{F_B(b)}{(N-1)f_B(b)} \quad \Rightarrow \quad v = \beta^{-1}(b, F_B)$
  - **ADVANTAGE:** F_B is directly observable from the bid data!

GPV PROCESS:

1. Estimate $F_B(b)$ and $f_B(b)$ via kernel-smoothing and compute a set of pseudo-private values.
2. Use the sample of pseudo-private values to estimate $F_V(v)$ and $f_V(v)$, nonparametrically.

Structural Auction Econometrics 101

Extensions of the 2-step GPV Estimator:

- Li, Perrigne & Vuong (2000)-symmetric CIPI
- Li, Perrigne & Vuong (2002)-symmetric APV
- Flambard & Perrigne (2006)-asymmetric IPV
- Brendstrup & Paarsch (2003)-asymmetric Dutch IPV
- **Campo, Perrigne & Vuong (2003)-asymmetric APV**
- Athey, Levin & Seira (2004)-asymmetric APV w/unobserved heterogeneity
- Lu (2004); Campo, Guerre, Perrigne & Vuong (2006)-risk averse bidders

Nonparametric Methods – DRAWBACKS

Nonparametric Methods – DRAWBACKS

- **Curse of Dimensionality**

Structural Auction Econometrics 101

Nonparametric Methods – DRAWBACKS

- **Curse of Dimensionality**
  - Slower (statistical) convergence (especially for multiple dimensions)

Structural Auction Econometrics 101

Nonparametric Methods – DRAWBACKS

- **Curse of Dimensionality**
  - Slower (statistical) convergence (especially for multiple dimensions)
  - Memory requirements/runtime increase exponentially in the number of variates
    - Asymmetric bidders
    - Auction-specific covariates

Quote from Campo, Perrigne and Vuong (2003)

"On the other hand, [nonparametric estimation] requires a large number of data. Thus, parametric estimation methods need to be developed if more than two types of bidders are entertained and if some observed heterogeneity of the auctioned objects needs to be introduced."

MPEC APPROACH:

THE GOAL:

To perform Maximum Likelihood estimation on a parametric empirical auction model which was previously thought to be computationally intractable.

We will accomplish this by reformulating the unconstrained likelihood maximization as a constrained optimization problem, letting the solvers do the work.

MPEC APPROACH:

Rather than choosing θ_k and $\{v_{\theta_k,t}\}_{t=1}^{T}$ sequentially, we will choose them *simultaneously*, subject to the constraint that the final solution must be consistent with the equilibrium: $\beta^{-1}(b_t;\theta_k) = v_{\theta_k,t} \;\; \forall t$:

MPEC APPROACH:

Rather than choosing θ_k and $\{v_{\theta_k,t}\}_{t=1}^{T}$ sequentially, we will choose them *simultaneously*, subject to the constraint that the final solution must be consistent with the equilibrium: $\beta^{-1}(b_t;\theta_k) = v_{\theta_k,t} \;\; \forall t$:

- $\max_{\{\theta,\mathbf{v}_T\}} \prod_{t=1}^{T} f_V\left(v_t|\theta\right)$

  s.t. $v_t = \beta^{-1}(b_t) \quad \forall t,$

  $\frac{d\beta^{-1}(b)}{db} = \frac{F_V(\beta^{-1}(b)|\theta)}{(\beta^{-1}(b)-b)f_V(\beta^{-1}(b)|\theta)}$

MPEC APPROACH:

Rather than choosing θ_k and $\{v_{\theta_k,t}\}_{t=1}^{T}$ sequentially, we will choose them *simultaneously*, subject to the constraint that the final solution must be consistent with the equilibrium: $\beta^{-1}(b_t;\theta_k) = v_{\theta_k,t} \;\; \forall t$:

- $\max\limits_{\{\theta,\mathbf{v}_T\}} \prod_{t=1}^{T} f_V\left(v_t|\theta\right)$

  s.t. $v_t = \beta^{-1}(b_t) \quad \forall t,$

  $$\frac{d\beta^{-1}(b)}{db} = \frac{F_V(\beta^{-1}(b)|\theta)}{(\beta^{-1}(b)-b)f_V(\beta^{-1}(b)|\theta)}$$

PROBLEM: no explicit solution to $\beta^{-1}(\cdot)$

MPEC APPROACH:

Rather than choosing θ_k and $\{v_{\theta_k,t}\}_{t=1}^{T}$ sequentially, we will choose them *simultaneously*, subject to the constraint that the final solution must be consistent with the equilibrium: $\beta^{-1}(b_t;\theta_k) = v_{\theta_k,t} \;\; \forall t$:

- $\max\limits_{\{\theta,\mathbf{v}_T\}} \prod_{t=1}^{T} f_V\left(v_t|\theta\right)$

  s.t. $v_t = \beta^{-1}(b_t) \quad \forall t,$

  $$\frac{d\beta^{-1}(b)}{db} = \frac{F_V(\beta^{-1}(b)|\theta)}{(\beta^{-1}(b)-b)f_V(\beta^{-1}(b)|\theta)}$$

PROBLEM: no explicit solution to $\beta^{-1}(\cdot)$
...SOLUTION: l_∞ Polynomial Projection as in Michelangeli (2008)

l_∞ Polynomial Projection:

Let $\alpha(b) \equiv \beta^{-1}(b)$ denote the Inverse bid function, defined by $\alpha'(b) = \frac{F_V(\alpha(b)|\theta)}{(\alpha(b)-b)f_V(\alpha(b)|\theta)}$

- Replace $\alpha(\cdot) : [0, \overline{b}] \to \mathbb{R}$ with $\widehat{\alpha}(b) = \sum_{n=0}^{N} a_n C_n(b)$
  - $C_n(b)$ is the n^{th} basis polynomial with weight a_n.

l_∞ Polynomial Projection:

Let $\alpha(b) \equiv \beta^{-1}(b)$ denote the Inverse bid function, defined by
$$\alpha'(b) = \frac{F_V(\alpha(b)|\theta)}{(\alpha(b)-b)f_V(\alpha(b)|\theta)}$$

- Replace $\alpha(\cdot) : [0, \overline{b}] \to \mathbb{R}$ with $\widehat{\alpha}(b) = \sum_{n=0}^{N} a_n C_n(b)$
  - $C_n(b)$ is the n^{th} basis polynomial with weight a_n.
  - $\mathbf{b}_K$ is a grid of $K \geq N$ Chebyshev nodes on $[0, \overline{b}]$
  - Define a set of errors $\{\lambda_k\}_{k=1}^{K} \geq 0$, one for each of the following K inequalities:

    $-\lambda_k \leq \widehat{\alpha}'(b_k) - \frac{F_V(\widehat{\alpha}(b_k)|\theta)}{(\widehat{\alpha}(b_k)-b_k)f_V(\widehat{\alpha}(b_k)|\theta)} \leq \lambda_k \quad \forall k = 1 \ldots K$ and

    $\widehat{\alpha}(0) = 0,$

    where and the last equation comes from a boundary condition.

MPEC APPROACH:

FINAL OPTIMIZATION PROBLEM:

MPEC APPROACH:

FINAL OPTIMIZATION PROBLEM:

- $$\max_{\left\{\theta, \mathbf{v}_T, \{a_n\}_{n=0}^{N}, \{\lambda_k\}_{k=1}^{K}\right\}} \prod_{t=1}^{T} f_V\left(v_t|\theta\right) - \Lambda \sum_{k=1}^{K} \lambda_k$$

  s.t. $v_t = \sum_{n=0}^{N} a_n C_n(b_t) = \widehat{\alpha}(b_t) \quad t = 1 \ldots T,$

  $-\lambda_k \leq \widehat{\alpha}'(b_k) - \frac{F_V(\widehat{\alpha}(b_k)|\theta)}{(\widehat{\alpha}(b_k) - b_k) f_V(\widehat{\alpha}(b_k)|\theta)} \leq \lambda_k \; k = 1 \ldots K,$

  $\sum_{n=0}^{N} a_n C_n(0) = 0,$

  $\mathbf{b}_k \in \left[0, \max_t \{b_t\}\right]^K$ (Cheb nodes), $\mathbf{b}_T$ (data), Λ (penalty) given.

Optional Constraints (For Numerical Stability)

- FIRST MOMENT: $\frac{1}{T}\sum_{t=1}^{T} v_t = \mathrm{E}[V|\theta]$
- RATIONALITY: $b_t < \widehat{\alpha}(b_t)$
- NON-DECREASING: $\widehat{\alpha}(b_{k+1}) - \widehat{\alpha}(b_k) \geq 0$
- NON-DECREASING: $\widehat{\alpha}'(b_k) \geq 0$
- FEASIBILITY: $\max_t \{b_t\} \leq \rho \mathrm{E}[V|\theta]$, $\rho < 1$
  - l_∞ polynomial projection seems to work best when $\rho \leq .95$

Experiment: Exponential Case with 2 Bidders

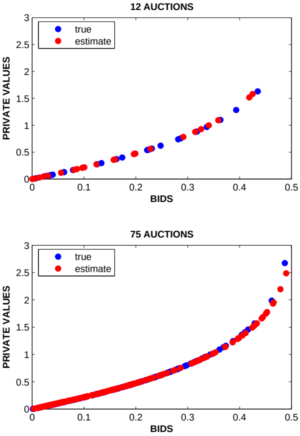
1. Generate random sample $\{v_{it}\}_{t=1}^{T}$, $i = 1, 2$ (private values) from $F_V(v) = 1 - e^{-v\theta}$, with $\theta = 2$
2. Compute the associated bids $b_{it} = \beta(v_{it})$
3. Using $\{b_{1t}, b_{2t}\}_{t=1}^{T}$, estimate θ in AMPL via the MPEC approach
   - Sample sizes: $T = 12, 25, 75, 225$.
4. Results:
   - Parameter estimates
   - Compare approximated private values to actual private values

Results: Exponential Case with 2 Bidders, $\theta = 2$

Table: Parameter Estimates

Sample Size	$\widehat{\theta}$	Std Err	Poly-Order	# Nodes	Fit: $\max_k \{\lambda_k\}$
12 (24)	2.038	0.0180	10	200	0.0010
25 (50)	1.986	0.0805	10	200	0.0074
75 (150)	1.968	0.0260	10	200	0.0847
225 (450)	1.967	0.0086	10	200	0.0653

Private Value Estimates



Results: Campo, Perrigne and Vuong (2003) Data

- CPV estimate a first-price auction for oil drilling rights in the Gulf of Mexico
  - two-step non-parametric GPV approach

Results: Campo, Perrigne and Vuong (2003) Data

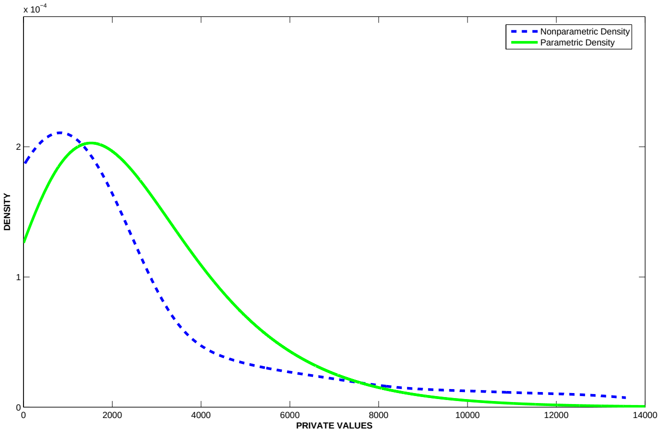
- CPV estimate a first-price auction for oil drilling rights in the Gulf of Mexico
  - two-step non-parametric GPV approach
- We parametrically estimate the model via Maximum Likelihood
  - Assume private values follow a Fisher-Tippett Extreme Value distribution:
    $V \sim F_V(v; \mu, \beta) = e^{-e^{-\frac{(v-\mu)}{\beta}}}$
    Scale bids by a factor of $\frac{1}{1000}$: helps out numerically
- **Results:**
  - Parameter estimates
  - Compare density estimate to non-parametric estimate of CPV

Results: Campo, Perrigne and Vuong (2003) Data

Table: Parameter Estimates

Sample Size	58 (116)
$\widehat{\mu}$	1.523 (???)
$\widehat{\beta}$	1.813 (???)
Polynomial Order	22
# Nodes	50
Fit: $\max_k \{\lambda_k\}$	.0975
# of Iterations	3727
Run Time	97.326 seconds

Results: Campo, Perrigne and Vuong (2003) Data



Results: Campo, Perrigne and Vuong (2003) Data

- TWO SOURCES OF MIS-SPECIFICATION:
  1. Distributional choice: a more flexible distribution would be preferable
     - 3,4 or 5 parameter distributions
  2. Assumption of *independent* private values
     - Use a multivariate distribution which allows for correlation

Results: Campo, Perrigne and Vuong (2003) Data

- TWO SOURCES OF MIS-SPECIFICATION:
  1. Distributional choice: a more flexible distribution would be preferable
     - 3,4 or 5 parameter distributions
  2. Assumption of *independent* private values
     - Use a multivariate distribution which allows for correlation

- Our largest hinderance to fixing these problems:

  AMPL does not include a built-in gamma function!!! ($#!%!!!)

Potential Future Uses for ML-MPEC

1. Efficient estimation with small sample sizes
2. Inclusion of covariates
   - Condition density estimates on all available information
3. Control for unobserved auction-specific heterogeneity