Duration-Based Volatility Estimation

A Dual Approach to RV

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Motivation

- The prevailing approach to high frequency volatility estimation is to measure the **price change (return) per time unit**
- We put forward the **dual approach** of measuring the **time duration (passage time) per unit price change**
- Our analysis fills an existing gap in the IV estimation literature by:
  1. Developing a broad class of duration-based IV estimators
  2. Identifying situations in which the duration-based approach is advantageous
  3. Shedding new light on the microstructure properties of real data

Literature Review

- Magnitude-based approach to volatility estimation:
  1. Discrete time (parametric): rich **ARCH** literature
  2. Continuous time (non-parametric): rich **RV** literature
- Duration-based approach to volatility estimation:
  1. Discrete time (parametric): rich **ACD** literature
  2. Continuous time (non-parametric): lack of "**DV**" literature

     Cho and Frees (JF '88) consider non-parametric duration-based estimation under constant volatility but their procedure is not suitable for IV estimation

Our Contribution

- Develop duration-based analogues to RV, range-based RV (RRV), and other power/multipower variations
- Derive an asymptotic theory for our duration-based estimators showing consistency for IV and promising asymptotic efficiency
- Demonstrate excellent finite sample efficiency and robustness to both (finite activity) jumps and microstructure noise
- Document superior performance in comparison to subsampled BV both on simulated and real stock data

Main Idea

- Inherent duality between the **increment** h and corresponding **passage time** dt for a Brownian motion:

$$\mathbb{E}\left[h^2 \mid dt\right] \sim \sigma^2 dt \qquad \qquad \mathbb{E}\left[dt \mid h\right] \sim \frac{h^2}{\sigma^2}$$

- Passage time moment subject to **Jensen effect!**
  - Resolution: Use the *reciprocal* passage time

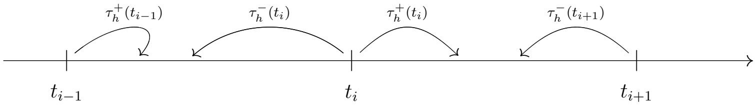
$$\mathbb{E}\left[\frac{1}{dt} \mid h\right] \sim \frac{\sigma^2}{h^2} \qquad \Rightarrow \qquad \hat{\sigma}_h^2 = const \times \frac{h^2}{dt}$$

- Passage time moment subject to **Censoring effect!**
  - Resolution: Exploit the time reversibility of the Brownian motion

- Passage time moment subject to **Discretization effect!**
  - Resolution: Apply discretization error theory for Brownian maxima

Integrated Variance Estimators Based On Passage Times

- Consider a fixed time grid $0 \equiv t_0 < t_1 < \ldots < t_N \equiv 1$ consisting of N intervals with mesh size $\Delta_i = t_{i+1} - t_i$



- From a sequence of unbiased local variance estimates $\hat{\sigma}_h^2(t_i)$ we can construct an IV estimate

- Define **DV** as a duration-based counterpart to RV based on a sequence of **reciprocal passage times** instead of squared returns:

$$\widehat{DV}_{N,h} = \sum_{i=0}^{N-1} \hat{\sigma}_h^2(t_i)\ \Delta_i$$

Brownian Passage Times - A Multitude of Definitions

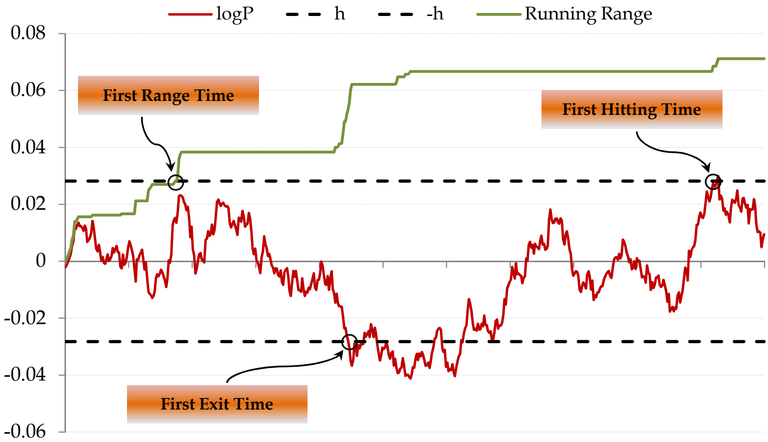
- Forward passage times for threshold h:

$$\tau_h^+(t) = \begin{cases} \inf\limits_{\theta>0} \{ W_{t+\theta} - W_t = h \} & \text{(first hitting time)} \\ \inf\limits_{\theta>0} \{ |W_{t+\theta} - W_t| = h \} & \text{(first exit time)} \\ \inf\limits_{\theta>0} \{ \sup\limits_{[t,t+\theta]} W_s - \inf\limits_{[t,t+\theta]} W_s = h \} & \text{(first range time)} \end{cases}$$

- Backward passage times for threshold h:

$$\tau_h^-(t) = \begin{cases} \inf\limits_{\theta>0} \{ W_{t-\theta} - W_t = h \} & \text{(first hitting time)} \\ \inf\limits_{\theta>0} \{ |W_{t-\theta} - W_t| = h \} & \text{(first exit time)} \\ \inf\limits_{\theta>0} \{ \sup\limits_{[t-\theta,t]} W_s - \inf\limits_{[t-\theta,t]} W_s = h \} & \text{(first range time)} \end{cases}$$

Brownian Passage Times - A Multitude of Definitions



A Multitude of DV Estimators

$$\widehat{DV}_{N,h} = \sum_{i=0}^{N-1} \hat{\sigma}_h^2(t_i)\ \Delta_i$$

- Local estimators of σ^2 given by the scaled reciprocal passage times:

$$\hat{\sigma}_h^2(t_i) = \mu_1^{-1} \frac{h^2}{\tau_h}$$

- **First exit time DV:** based on reciprocal first exit times
- **First range time DV:** based on reciprocal first range times

- More generally, local estimators of σ^p are given by the $p/2$ power of the reciprocal passage times:

$$\hat{\sigma}_h^p(t_i) = \mu_{p/2}^{-1} \frac{h^p}{\tau_h^{p/2}}$$

- Can define natural DV analogues to power and multipower variations

Duality

Theorem (Duality for magnitude and passage time functionals)

Define the following standard Brownian functionals

$$H_t = \sup_{\theta \in [0;t]} B_\theta, \quad M_t = \sup_{\theta \in [0;t]} |B_\theta|, \quad R_t = \sup_{\theta \in [0;t]} B_\theta - \inf_{\theta \in [0;t]} B_\theta$$

and let (for $h > 0$)

$$\tau_h^{HT} = \inf\{t | H_t = h\}, \ \tau_h^{ET} = \inf\{t | M_t = h\}, \ \tau_h^{RT} = \inf\{t | R_t = h\}$$

be the first range time, first exit time, and first hitting time respectively. Then we have the following identities in distribution:

$$H_1 \overset{\mathcal{D}}{=} \frac{1}{\left(\tau_1^{HT}\right)^{1/2}}, \quad M_1 \overset{\mathcal{D}}{=} \frac{1}{\left(\tau_1^{ET}\right)^{1/2}}, \quad R_1 \overset{\mathcal{D}}{=} \frac{1}{\left(\tau_1^{RT}\right)^{1/2}}$$

Integrated Variance Estimators Based On Passage Times

- We establish the following main asymptotic result:

$$\sqrt{N}\ \left(\widehat{DV}_{N,h} - IV\right) \sim \text{Mixed Normal}\left(0, \nu \int_0^1 \sigma_u^4\ du\right)\ ,$$

  where

$$\nu \approx \begin{cases} 0.7681 & \text{(first exit time)} \\ 0.4073 & \text{(first range time)} \\ 2.0000 & \text{(first hitting time)} \end{cases}$$

  is the variance factor of the individual passage time estimators at each grid point.

- Underlying assumptions:
  - No leverage
  - Lipschitz continuity of the volatility process
  - Mesh size $\Delta = O\left(N^{-1}\right)$, threshold $h = o(N^{-1/2})$

Subsampling DV Estimators

- In practice, the observation record is discrete and we only observe the value of the process at N grid points but *not* in between
- For a feasible version of DV consider coarser sub-grid $\{t_{i_1}, \ldots, t_{i_K}\}$ where $K = o(N),\ K \to \infty$:

$$\sqrt{K}\ \left(\widehat{DV}_{K,h} - IV\right) \sim \text{Mixed Normal}\left(0, \nu \int_0^1 \sigma_u^4\, du\right)$$

- Convergence rate is now the slower $K^{-1/2}$
- Efficiency loss can be mitigated by averaging the estimator over all possible sub-grids of mesh size $\Delta = K^{-1}$
- The outcome is **feasible DV akin to subsampled RV!**

Main Challenges to IV Estimation

Microstructure noise

Jumps

Robustness of DV to Microstructure Noise

- An intuitive advantage of the passage time approach is that the threshold h **can be chosen large enough** to achieve noise-robustness
- To formalize this intuition we adopt an AR(1) noise structure:

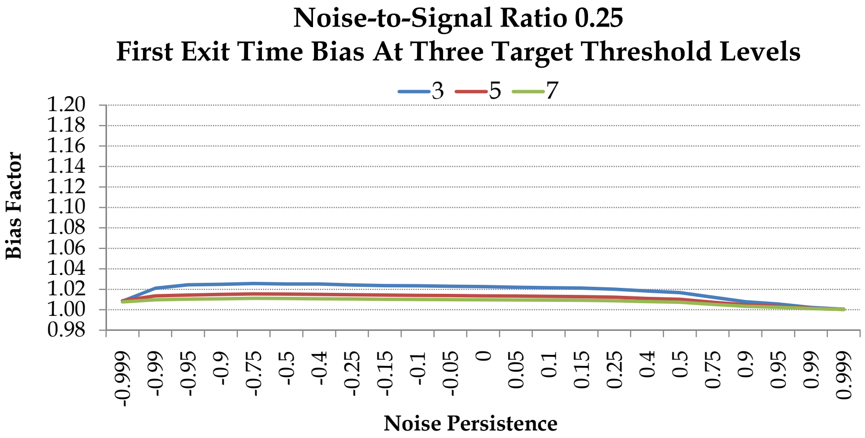
$$\begin{aligned} \widetilde{p}_i &= p_i + u_i \\ u_i &= \rho u_{i-1} + \varepsilon_i \ , \end{aligned}$$

  where $\varepsilon_i \sim N(0, (1-\rho^2)\omega^2)$, so that $\mathbb{E}[u_i] = 0$, $\mathbb{V}[u_i] = \omega^2$
- For $\rho = 0$ we obtain a Gaussian i.i.d. specification representative for transaction prices
- For $\rho >> 0$ we obtain a persistent autoregressive specification representative for quotes

Robustness of DV to Microstructure Noise Cont'd

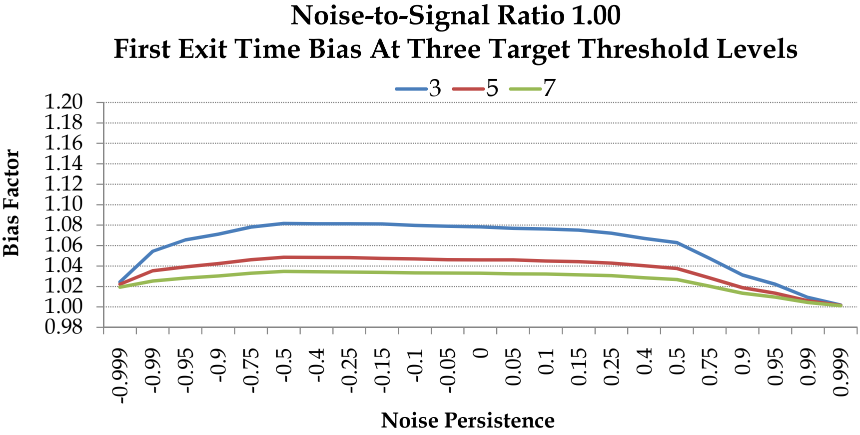
- Given a passage time transition on noisy data, it is possible to infer the expected magnitude of the latent transition
- AR(1) noise leads to an **upward bias** of our reciprocal passage time estimators
- We plot the upward bias factor as a function of the noise persistence ρ for two different noise-to-signal ratios:
  1. $\lambda = 0.25$ (moderate noise)
  2. $\lambda = 1.00$ (high noise)

Robustness of DV to Microstructure Noise Cont'd



Negligible bias for moderate noise ($\lambda = 0.25$) at all persistence levels ρ

Robustness of DV to Microstructure Noise Cont'd



Negligible bias for high noise ($\lambda = 1.0$) if sufficiently persistent ($\rho > 0.9$)

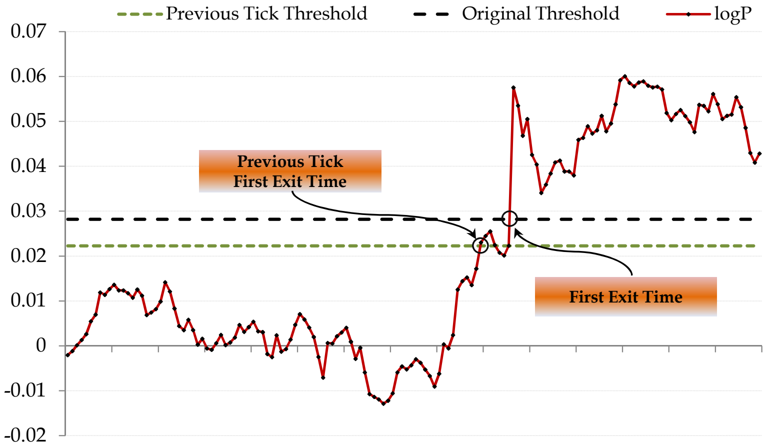
Robustness of DV to Jumps

- Another advantage of the passage time estimators is their inherent robustness to finite jumps
- Jumps above the threshold level h are **effectively truncated!**

- The lower the chosen threshold, the higher the degree of jump-robustness but ... the lower the degree of noise-robustness

- To avoid lowering the threshold too much, we propose an asymptotically equivalent "previous tick" passage time estimator
- It utilizes the lower threshold level h^- corresponding to the level at one tick prior to the crossing of the target threshold h

“Previous Tick” Passage Times



Better jump-robustness in finite samples than the standard passage times!

Empirical Analysis of The “Robust” DV Estimators

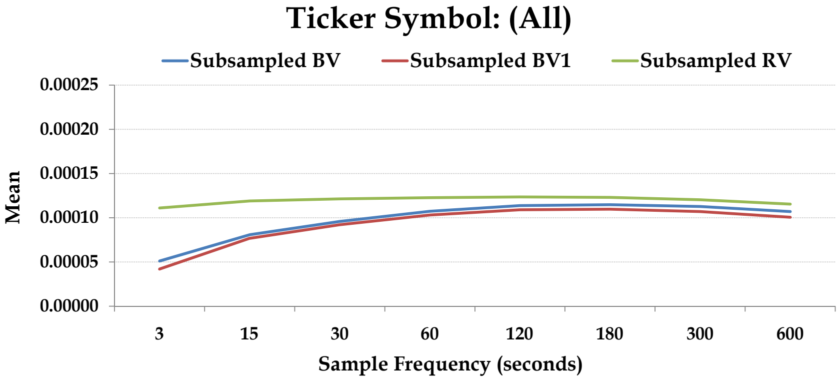
Real Stock Data

Simulated Stock Data

DV Analysis of The Dow Jones 30

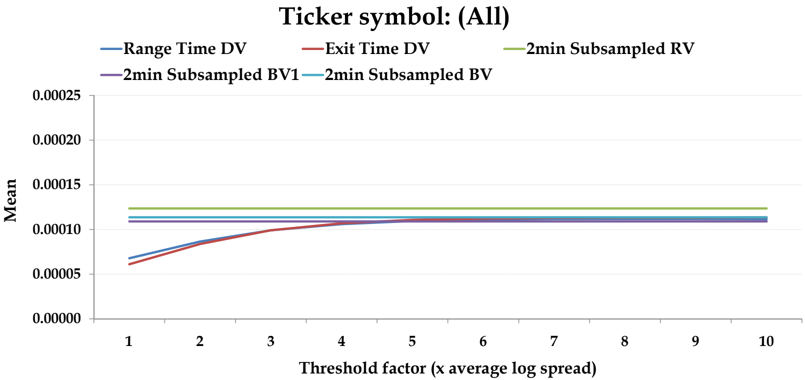
- We analyze the performance of DV on the Dow Jones 30 stocks for 601 trading days in the period January 1, 2005 to May 31, 2007
- We work with mid-quotes known to have relatively low and persistent noise, so DV should be unbiased for moderate threshold levels
- DV is jump-robust, so we choose 2min subsampled BV as benchmark
- We produce DV signature plots for the mean, standard deviation, and correlation of DV with 2min subsampled BV
- Focus on **first exit time DV** and **first range time DV** for thresholds from 1 to 10 log-spreads

BV Signature Plot



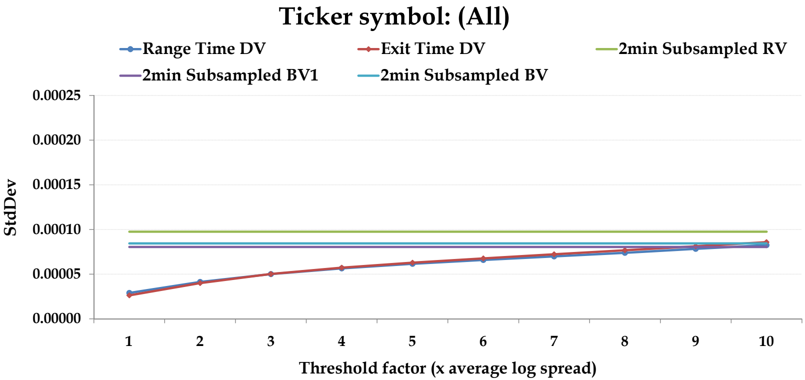
BV is downward biased at higher frequencies, so 2min is close to optimal!

DV Signature Plots: Mean



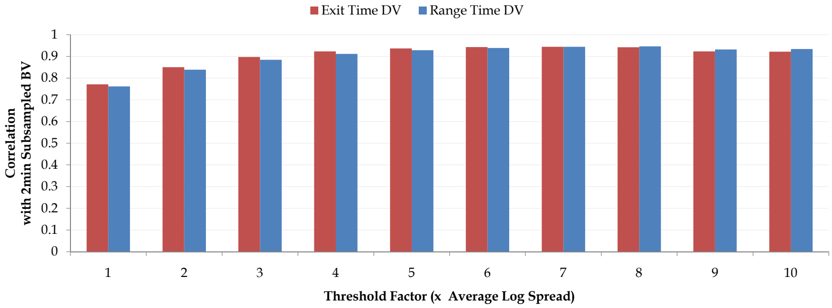
DV for thresholds above 4 log-spreads has **the same mean as BV**!

DV Signature Plots: StdDev



DV has markedly **lower standard deviation than BV!**

DV Signature Plots: Correlation with BV



DV for thresholds above 4 log-spreads is **highly correlated with BV!**

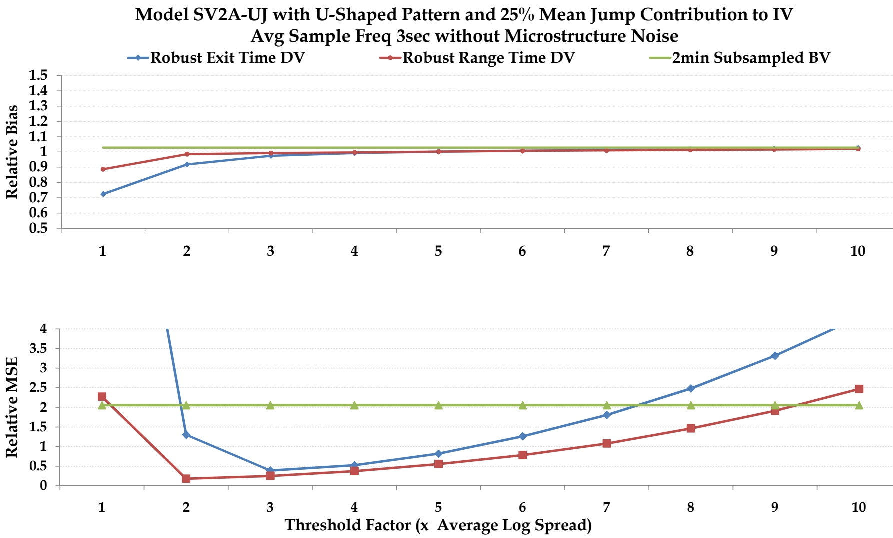
Monte Carlo Study of DV

- Very similar results across different SV models
- Scenario of main interest: "jump" days with 25% mean jump contribution to IV and U-shape volatility pattern
- The adopted AR(1) noise structure gives rise to five distinct cases:
  1. No noise
  2. Non-persistent noise ($\rho = 0$) at moderate level ($\lambda = 0.25$)
  3. Persistent noise ($\rho = 0.99$) at moderate level ($\lambda = 0.25$)
  4. Non-persistent noise ($\rho = 0$) at high level ($\lambda = 1.00$)
  5. Persistent noise ($\rho = 0.99$) at high level ($\lambda = 1.00$)
- Compare the performance of DV vis-a-vis 2min subsampled BV by:

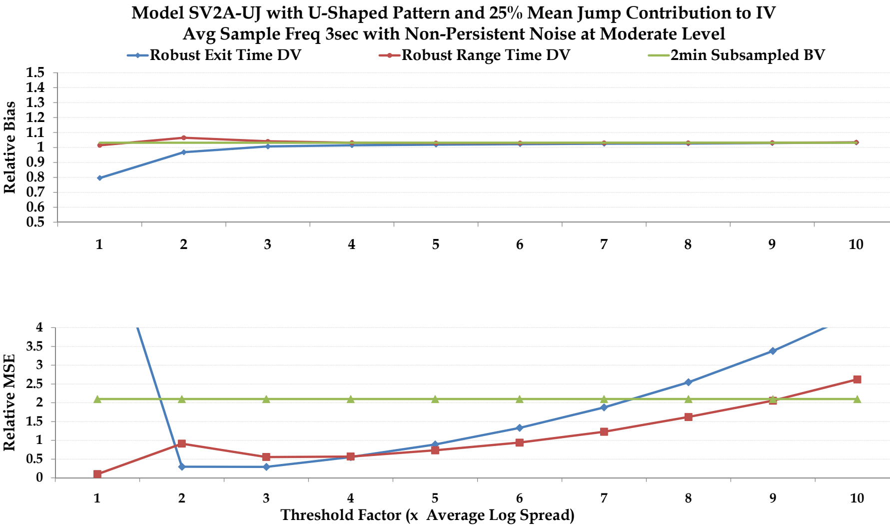
  (i) A relative bias measure: mean of $\widehat{IV}\,/\,IV$

  (ii) A relative MSE measure: mean of $195(\widehat{IV} - IV)^2\,/\,IQ$

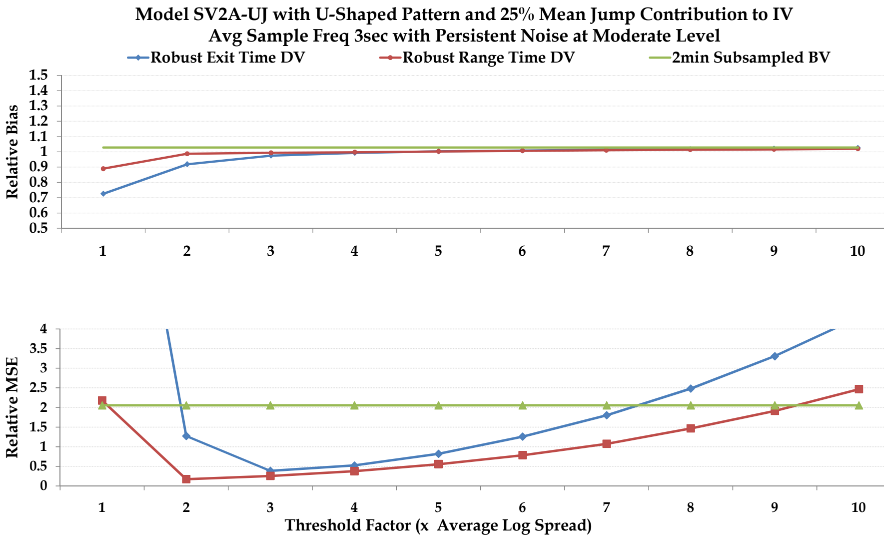
Monte Carlo Study of DV 1/5: No Noise



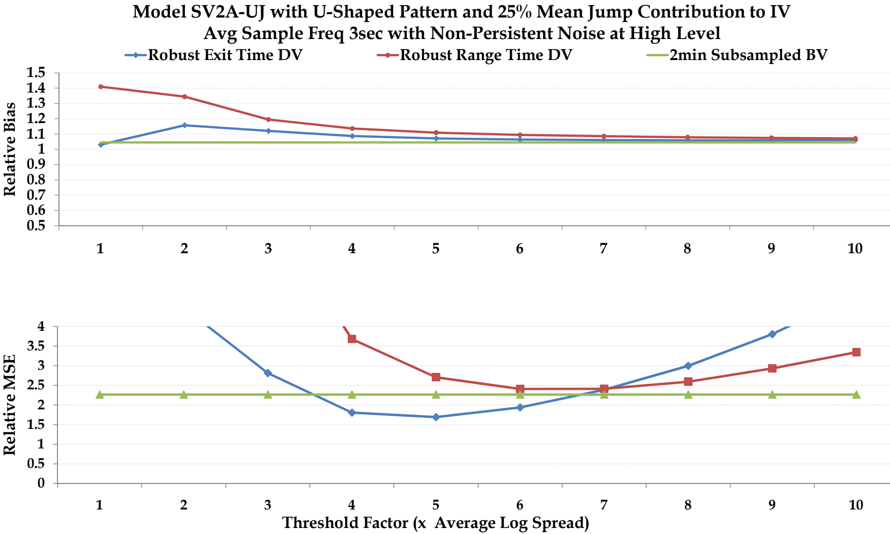
Monte Carlo Study 2/5: Moderate Nonpersistent Noise



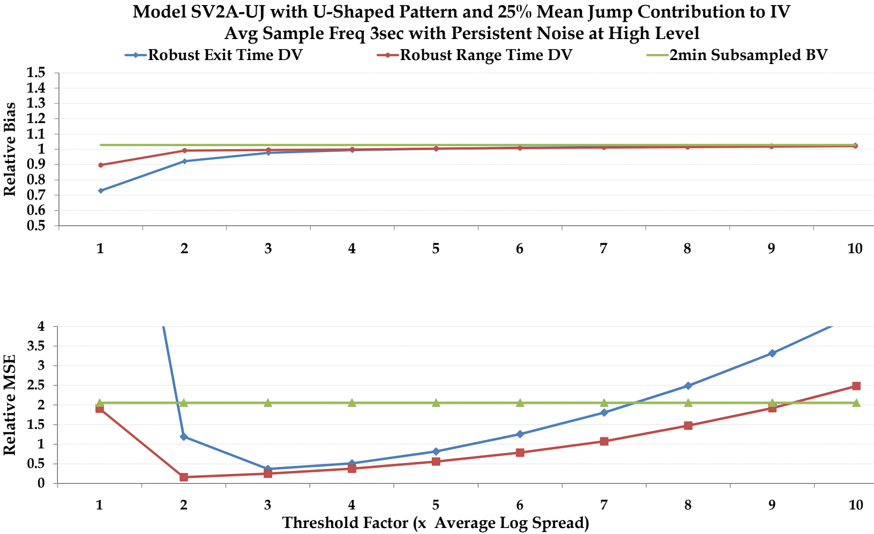
Monte Carlo Study of DV 3/5: Moderate Persistent Noise



Monte Carlo Study of DV 4/5: High Nonpersistent Noise



Monte Carlo Study of DV 5/5: High Persistent Noise



Summary and Conclusions

- Novel dual approach to realized return variation measurement based on reciprocal passage times
- Duration-based counterparts to RV, range-based RV (RRV), and other power/multipower variations
- Asymptotic theory showing consistency for IV and promising asymptotic efficiency
- Natural robustness to both jumps and microstructure noise
- Promising finite sample efficiency in comparison to subsampled BV both on simulated and real stock data

- Exciting "to do" list!