Solving Continuous Time Models in Financial Economics.

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The fundamental asset pricing relation is given by

$$0 = \Lambda(t)D(t)dt + E_t\left[d\left[\Lambda(t)P(t)\right]\right]. \qquad (1)$$

$E_t[\bullet]$ is the expectation conditional on current information.

1. $\Lambda(t)$ is the stochastic discount factor (SDF) for the valuation of an investment.
2. $D(t)$ is the dividend payment from the equity per unit of time.
3. $P(t)$ is the price of equity.

Generic Asset Pricing Models (Continue).

Asset pricing models are generally distinguished by the following stochastic process for the SDF

$$\frac{d\Lambda}{\Lambda} = \mu_\Lambda(x(t), t)dt + \sigma_\Lambda(x(t), t)d\theta_t.$$

where $x(t)$ is the state variable, This state variable is assumed to follow an Ito process.

$$\frac{dx}{x} = \mu_x(x(t), t)dt + \sigma_x(x(t), t)d\omega_t.$$

$\mu(\bullet)$ stands for mean and $\sigma(\bullet)$ refers to standard deviation. $d\theta_t$ and $d\omega_t$ are Brownian motion.

The Asset Pricing Model leads to an ODE

In the one-dimensional case, this equilibrium price function satisfies the solution to a second order linear ODE. As an initial value problem (IVP), it takes the form

$$P''(x) + a(x)P'(x) + b(x)P(x) = g(x),\ P(0) = p_0,\ P'(0) = p_1. \tag{2}$$

How can one solve such a problem?

We provide a general technique to quickly and accurately solve the ODE associated with such models.

Analytic functions.

Definition

Let $f(z)$ be a complex-valued function of $z = x + iy$ defined in an open set (domain) D of the complex plane $\mathbf{C}$. Suppose that this function is **analytic** such that it can be expressed as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \text{ for } |z - z_0| < r \text{ and } z_0 = x_0 + iy_0\,, \tag{3}$$

where, $f^{(k)}(z_0)$ denotes the k-th order complex derivative of $f(z)$ at $z = z_0$.

Campbell and Cochrane's External Habit Model

Their utility function is

$$\frac{[C_t - X_t]^{1-\gamma}}{1-\gamma}, \tag{4}$$

where C_t is the individual's consumption at time t, and X_t is their habitual level of consumption at time t.
They introduce $S_t^a = \frac{C_t^a - X_j}{C_t^a}$ is the surplus consumption ratio at time t, C_t^a is the average consumption of all individuals at time t.
The state variable $s = \ln(\frac{C^a - X}{C^a}) - \ln(\bar{S})$ measures the deviation of consumption from the external habit X relative to its steady state value $\ln(\bar{S})$.

$$ds = (\phi - 1)sdt + \lambda(s)\sigma d\omega, \tag{5}$$

where $\phi < 1$. The sensitivity function $\lambda(s)$ is given by

$$\lambda(s) = \begin{cases} \sqrt{1-2s}/\bar{S} - 1 & \text{if } s < \frac{1-(\bar{S})^2}{2}, \\ 0 & \text{if } s \geqslant \frac{1-(\bar{S})^2}{2} \end{cases} \text{ when } \bar{S} = \sigma\sqrt{\frac{\phi\gamma}{1-\phi-\frac{b}{\gamma}}}.$$

Ito's Lemma

Let $x \in \mathbb{R}$ follow the stochastic process,

$$dx = f(x)dt + S(x)d\omega,$$

where $f(x)$ is the instantaneous mean of the state variable $x \in \mathbb{R}$. $d\omega$ is Brownian motion, so that $S(x)$, provides the instantaneous impact of these random shocks on the state variable. Consequently, $\Sigma(x) = S(x)E_t\left(d\omega d\omega\right)S(x)$ is the instantaneous variance-covariance matrix for the state variables.

Lemma

Suppose $F(x,t)$ is C^2 for $x \in \mathbb{R}$ and C^1 in t. Then

$$dF = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}f(x) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x \partial x'}\Sigma(x)\right)\right)dt + \frac{\partial F}{\partial x}S(x)d\omega.$$

Also, the following multiplication rules are true: $d\omega d\omega = 1dt$, $d\omega dt = 0$ and $dtdt = 0$.

An Example by Campbell and Cochrane

The stochastic discount factor is

$$\Lambda = e^{-\beta t}\left[S^a C\right]^{-\gamma}. \tag{6}$$

By Ito's Lemma

$$\frac{d\Lambda}{\Lambda} = \left[\gamma(1-\phi)s - \beta - \gamma\bar{x} + \frac{\gamma^2\sigma^2}{2}\left(1+\lambda(s)\right)^2\right] dt - \gamma\sigma\left(1+\lambda(s)\right)^2 \, d\omega \tag{7}$$

γ is coefficient of risk aversion. $\bar{x}$ is expected consumption growth, and σ is its standard deviation with Brownian motion $d\omega$.

$$\frac{dC}{C} = \frac{dD}{D} = \bar{x}dt + \sigma d\omega. \tag{8}$$

$$ds = (\phi - 1)sdt + \lambda(s)\sigma d\omega, \tag{9}$$

where $\phi < 1$.

Finding the ODE for Campbell-Cochrane Model

1. Use Ito's Lemma to find the asset pricing relation (1) in terms of the price-dividend function $p = \frac{P}{D}$, dividend growth and the SDF.
2. Substitute in the stochastic processes for SDF (7) and dividend (consumption) growth (8) into the asset pricing relation.
3. Guess that the price-dividend function is a function of the surplus consumption ratio, $p(s)$, and use Ito's Lemma to find stochastic process for the price-dividend ratio.

$$dp = \left[p'(s)(\phi - 1)s + \frac{1}{2} p''(s)\lambda(s)^2\sigma^2 \right] dt + p'(s)\lambda(s)\sigma d\omega. \tag{10}$$

4. Substitute the stochastic process for the price-dividend ratio (10) into the asset pricing relation from step 2.
5. Use Ito's rules and apply conditional expectation to Brownian motion to yield an ODE for the equilibrium price-dividend function, $p(s)$.

The IVP Campbell-Cochrane Model

The Cambell-Cochrane Model

Solve the following initial-value problem for the price-dividend function $p(s)$:

$$c_2(s)p'' = c_1(s)p' + c_0(s)p - 1; \quad p(0) = p_0, \quad p'(0) = p_1,$$

where

(a) $c_0(s) = [\beta + (\gamma - 1)\bar{x}] - \gamma(1-\phi)s - \frac{\sigma^2}{2}\left[\gamma\lambda(s) + \gamma - 1\right]^2$

(b) $c_1(s) = (1-\phi)s + \sigma^2(\gamma - 1)\lambda(s) + \sigma^2\gamma\lambda(s)^2$

(c) $c_2(s) = \frac{\sigma^2}{2}\lambda(s)^2$

(d) Sensitivity Function:

$$\lambda(s) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1-2s} - 1 & \text{if } s \leq \frac{1-\bar{S}^2}{2} \\ 0 & \text{if } s > \frac{1-\bar{S}^2}{2} \end{cases}$$

Cauchy-Kovalevsky Theorem

Theorem

If the coefficients and output function in the initial value problem:

$$y'' = a(x)y' + b(x)y' + g(x); \quad y(0) = y_0, \quad y'(0) = y_1 \qquad (11)$$

can be represented by the power series:

$$a(x) = \sum_{k=0}^{\infty} a_k x^k, \ b(x) = \sum_{k=0}^{\infty} b_k x^k, \ g(x) = \sum_{k=0}^{\infty} g_k x^k \ \text{ for } |x| < r,$$

then the initial value problem (11) has a unique power series solution $y(x) = \sum_{k=0}^{\infty} y_k x^k$ *for* $|x| < r$.

Simplifying the coefficients

Definition

$$r(s) = \sqrt{1 - 2s} \quad \text{for } |s| < r_0 = (1 - \bar{S}^2)/2.$$

(a) $c_0(s) = \frac{[2\beta + 2(\gamma-1)\bar{x} - \sigma^2]\bar{S}^2 - \sigma^2\gamma^2}{\bar{S}^2} + \frac{\sigma^2\gamma^2 - \gamma(1-\phi)\bar{S}^2}{\bar{S}^2} s + \frac{\sigma^2\gamma}{\bar{S}} r(s)$

(b) $c_1(s) = \frac{\sigma^2(\bar{S}^2 + \gamma)}{\bar{S}^2} + \frac{(1-\phi)\bar{S}^2 - 2\sigma^2\gamma}{\bar{S}^2} s - \frac{\sigma^2(1+\gamma)}{\bar{S}} r(s)$

(c) $c_2(s) = \frac{\sigma^2(1+\bar{S}^2)}{2\bar{S}^2} - \frac{\sigma^2}{\bar{S}^2} s - \frac{\sigma^2}{\bar{S}} r(s)$

(d) Sensitivity Function: $\lambda(s) = \frac{1}{\bar{S}} r(s) - 1$

Summary

The coefficients $c_0(s)$, $c_1(s)$, $c_2(s)$ can be written in the form:

$$c(a_0, a_1, a_2; s) = a_0 + a_1 s + a_2 r(s) \quad \text{for } |s| < r_0\,.$$

Derivatives of $c(a_0, a_1, a_2; s)$ at $s = 0$

By mathematical induction, one can get

$$r^{(k)}(s) = \begin{cases} r(s) & \text{if } k = 0, \\ -\frac{1}{1-2s} r(s) & \text{if } k = 1, \\ -\frac{(2k-3)!!}{(1-2s)^k} r(s) & \text{if } k \geq 2. \end{cases}$$

Since $c(a_0, a_1, a_2; s) = a_0 + a_1 s + a_2 r(s)$, we get

$$c^{(n)}(a_0, a_1, a_2; 0) = \begin{cases} a_0 + a_2 & \text{if } k = 0, \\ a_1 - a_2 & \text{if } k = 1, \\ -a_2(2k-3)!! & \text{if } k \geq 2 \end{cases}$$

and

$$\frac{1}{k!} c^{(k)}(a_0, a_1, a_2; 0) = \begin{cases} a_0 + a_2 & \text{if } k = 0, \\ a_1 - a_2 & \text{if } k = 1, \\ -\frac{(2k-3)!!}{k!} a_2 & \text{if } k \geq 2 \end{cases}$$

The Power Series of Coefficients

$c_0(s)$, $c_1(s)$, and $c_2(s)$ can be represented by the power series

$$c_j(s) = \sum_{k=0}^{\infty} c_j^{(k)} s^k \qquad \text{for } |s| < r_0 = (1 - \bar{S}^2)/2,$$

where $c_j^{(k)} = \frac{1}{k!} c_j^{(k)}(0)$ is the k^{th} Taylor coefficient of $c_j(s)$ at $s = 0$ and $j = 0, 1, 2 \cdots$. Note that $c_2(s) \neq 0$ for $|s| < r_0$.

Corollary

The Campbell-Cochrane model has a unique power series solution:

$$p(s) = \sum_{k=0}^{\infty} p_k s^k \qquad \textit{for } |s| < r_0.$$

Recurrence Relation for the p_k

Substitute

$$p = \sum_{k=0}^{\infty} p_k s^k, \; p' = \sum_{k=0}^{\infty} (k+1) p_{k+1} s^k, \; p'' = \sum_{k=0}^{\infty} (k+1)(k+2) p_{k+2} s^k$$

into the differential equation $c_2(s)p'' = c_1(s)p' + c_0(s)p - 1$. In addition, substitute in the power series for the coefficients $c_0(s)$, $c_1(s)$, and $c_2(s)$. After calculating the product of power series we end up with

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} c_2^{(k-j)} (j+1)(j+2) p_{j+2} \right) s^k$$
$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \left(c_1^{(k-j)} (j+1) p_{j+1} + c_0^{(k-j)} p_j \right) \right] s^k - 1.$$

Recurrence Relation for the p_k (Continue)

Match the coefficients.

$$\sum_{j=0}^{k} c_2^{(k-j)}(j+1)(j+2)p_{j+2} = \sum_{j=0}^{k}\left(c_1^{(k-j)}(j+1)p_{j+1} + c_0^{(k-j)}p_j\right) - \delta_{k,0}.$$

Recurrence Relation for the p_k

$$2c_2^{(0)}p_2 = c_1^{(0)}p_1 + c_0^{(0)}p_0 - 1,$$

$$\begin{aligned}
&(k+1)(k+2)c_2^{(0)}p_{k+2} \\
&= (k+1)\left(c_1^{(0)} - kc_2^{(1)}\right)p_{k+1} + \left(c_0^{(k-1)} + c_1^{(k)}\right)p_1 + c_0^{(k)}p_0 \\
&+ \sum_{j=2}^{k}\left[c_0^{(k-j)} + jc_1^{(k-j+1)} - j(j-1)c_2^{(k-j+2)}\right]p_j\,.
\end{aligned}$$

Error Analysis

Normal Form

The price-dividend function $p(s)$ satisfies

$$p'' = a(s)p' + b(s)p + g(s) \quad \text{for } |s| < r_0 = (1 - \bar{S}^2)/2,$$

where $a(s) = \frac{c_1(s)}{c_2(s)}$, $b(s) = \frac{c_0(s)}{c_2(s)}$, and $g(s) = -\frac{1}{c_2(s)}$.

Definition

The error of the approximation $p(s) \approx p_n(s) = \sum_{k=0}^{n} p_k s^k$ for $|s| < r_0$ is

$$R_n(s) = p(s) - p_n(s) = \sum_{k=n+1}^{\infty} p_k s^k.$$

Error Analysis (Continue)

Theorem

Let M_a, M_b, M_g be the maximal values of $|a(z)|$, $|b(z)|$, $|g(z)|$ on the circle of radius $r < (1 - \bar{S}^2)/2$ in the complex plane:

$$C_r = \{z = s + yi : |z| = \sqrt{s^2 + y^2} = r\}.$$

For $n \geq 2$ and $|s| < r$, we have

$$|R_n(s)| \leq \frac{M_g + [(r+1)|p_1| + |p_0|]\, M}{2} \cdot \sum_{k=n+1}^{\infty} \prod_{l=2}^{k-1} \left[\frac{l-1}{(l+1)r} + \frac{(l+r)M}{(l+1)l} \right] |\mu r|^k$$

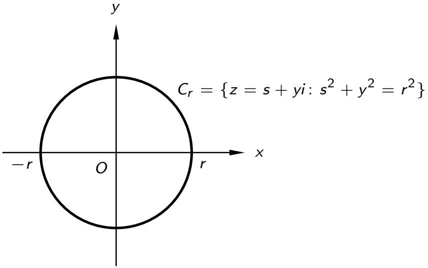
for $|s| < \mu r$ with $\mu \in (0,1)$, where $M = \max\{M_a, M_b\}$.

Cauchy Integral Formula

Cauchy Integral Formula

$$\frac{a^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{C_r} \frac{a(z)}{z^{k+1}}\, dz \quad \text{for } k = 0, 1, 2, \ldots .$$

The Complex Plane $\mathbb{C} = \{z = s + yi : s, y \text{ are real numbers}\}$



Why we have to worry about Complex plane?

Suppose we look at

$$y(x) = \frac{1}{1+x^2},$$

which is well defined on real line. However, when we complexify the function we get

$$y(z) = \frac{1}{1+z^2},$$

with $z = x + yi$. In the complex plane, the function has a singularity at $z = i$, since $i^2 = -1$. As a result, the function has a radius $r < 1$.

Finding M_a, M_b, and M_g

Use $a(z) = \frac{c_1(z)}{c_2(z)}$, $b(z) = \frac{c_0(z)}{c_2(z)}$, $g(z) = -\frac{1}{c_2(z)}$, and

$$0 < \lambda(r) \le \lambda(z) \le \sqrt{\lambda(-r)^2 + 2r/\bar{S}^2} \quad \text{for } z \in C_r\,.$$

One can get

$$M_a = \frac{2(1-\phi)r}{\sigma^2\lambda(r)^2} + \frac{2|\gamma-1|}{\lambda(r)} + 2\gamma,$$
$$M_b = \frac{2\left[\gamma(1-\phi)r + \beta + (\gamma-1)\bar{x}\right]}{\sigma^2\lambda(r)^2} + \left(\gamma + \frac{|\gamma-1|}{\lambda(r)}\right)^2,$$
$$M_g = \frac{2}{\sigma^2\lambda(r)^2}.$$

The Values of Parameters

$\sigma =$ standard deviation of consumption growth ≈ 0.003233

$\bar{S} =$ average surplus consumption ratio ≈ 0.07737

$\gamma =$ coefficient of risk aversion ≈ 2

$\phi =$ persistence of surplus consumption ratio ≈ 0.9896

$\bar{x} =$ steady-state consumption growth ≈ 0.001567

$\beta =$ discount factor ≈ 0.005738

Initial Conditions.

1. p_0 = price-dividend at steady-state ≈ 219.6, where $219.6 = 18.3 * 12$ is the average historic price-dividend ratio in Campbell and Cochrane.
2. p_1 = rate of change of price-dividend ≈ 111.76

$$p'(s) = \frac{\{E_t[R^e(s)] - R^b(s) - \sigma^2\gamma(1+\lambda(s))\}p(s)}{\gamma\sigma^2\lambda(s)(1+\lambda(s))} . \qquad (12)$$

Then evaluating (12) at $s = 0$ determines the second initial condition

$$p_1 = p'(0) = \frac{\left\{E_t[R^e(0)] - r^b - \frac{\gamma\sigma^2}{\bar{S}}\right\} p_0}{\frac{\gamma\sigma^2}{\bar{S}}\left(\frac{1}{\bar{S}} - 1\right)} . \qquad (13)$$

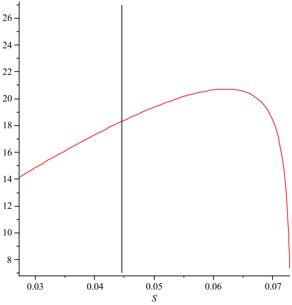
The value of p_1 is found by replacing $E_t\left[R^e(0)\right] - r^b$ with the average equity premium. In the simulation this initial condition is used to set $p_1 = 111.76$.

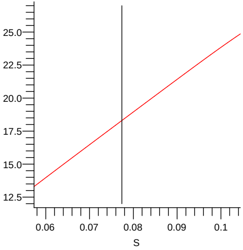
Price-dividend function in the CC model.

The radius of convergence for CC model is at least

$$r_0 = \frac{1 - \bar{S}^2}{2} \approx 0.4990.$$

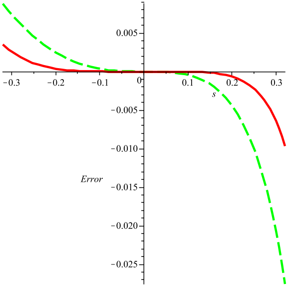
The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}] = [0.032, 0.061]$. The y-axis records the price-dividend ratio.





Approximation Error.

The error analysis for the Campbell and Cochrane model. The parameter values are $r^b = 0.00078$, $\bar{x} = 0.00157$, $\phi = 0.9896$, $\gamma = 2$, $\sigma = 0.00323$, $b = 0$, $p_0 = 219.60$, $p_1 = 111.76$, $\bar{S} = 0.0448$ and $\mu r = 0.32$. The y-axis for the dotted line compares the 475^{th} order Taylor polynomial for the price-dividend ratio with the first order Taylor polynomial. In addition, the solid line compares the 475^{th} order Taylor polynomial for the price-dividend ratio to it's fourth order Taylor polynomial.



Equity Premium and Standard Deviation of Stock Returns.

The x-axis gives the surplus consumption ratio on the support of the distribution $S = [\bar{S}e^{-0.32}, \bar{S}e^{0.32}]$. The y-axis records the equity premium and standard deviation. The equity premium line is the bottom curve, while the top curve represents the standard deviation.

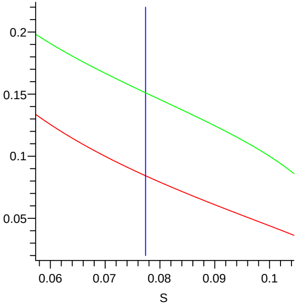


Figure 5

Two dimensional asset pricing model.

Consider the initial value problem (IVP) for second-order linear PDE of the form

$$\frac{\partial^2 p}{\partial t^2} = A(x,t)\frac{\partial^2 p}{\partial x^2} + B(x,t)\frac{\partial^2 p}{\partial x \partial t} \tag{14}$$

$$+C(x,t)\frac{\partial p}{\partial x} + D(x,t)\frac{\partial p}{\partial t} + E(x,t)p + g(x,t),$$

$$p(x,0) = p_0(x), \quad \text{and} \quad \frac{\partial p}{\partial t}(x,0) = p_1(x), \tag{15}$$

where the coefficients are analytic functions near $(x,t) = (0,0)$.

Two dimensional Cauchy-Kovalevsky Theorem.

Theorem

If all the coefficients and the output function in the differential equation (14) are analytic in a neighborhood U of $(0,0)$ in $\mathbf{R}^2$ and if $p_0(x)$ and $p_1(x)$ are analytic in a neighborhood V of (0) in $\mathbf{R}$, then the initial value problem (14), and (15) has a unique solution in a neighborhood W of $(0,0)$ in $\mathbf{R}^2$

Region of convergence.

Theorem

If the coefficients A, B, C, D, E and the force term g in (14) are analytic in the set $\{(x,t) \in \mathbf{R}^2 : |x| < r_1,\ |t| < r_2\}$, and furthermore the coefficients are bounded in absolute value by M and the force term g is bounded in absolute value by L, then the region of analyticity of the solution contains the set

$$\left\{(x,t) \in \mathbf{R}^2 : \left|\frac{\rho_1 x}{r_1} + \frac{\rho_2 t}{r_2}\right| < \left(1 - \frac{M(\rho_1/r_1)}{(\rho_2/r_2)} - \frac{M(\rho_1/r_1)^2}{(\rho_2/r_2)^2}\right)\right\}, \tag{16}$$

where $\rho_1 > 1$ and $\rho_2 > 1$ large enough so that

$$\frac{\rho_1}{\rho_2} < \frac{r_1}{r_2}\left(\sqrt{\frac{1}{4} + \frac{1}{M}} - \frac{1}{2}\right).$$

Wachter's (2002) extension of Campbell and Cochrane.

Consumption growth is made more realistic.

$$dc = xdt + \sigma_1 d\omega_1, \tag{17}$$

and

$$dx = (\psi - 1)(x - \bar{x})dt + \sigma_2 d\omega_2, \tag{18}$$

where $E_t[d\omega_1 d\omega_2] = \rho$. Every thing else the same as in Campbell and Cochrane.
The stochastic process for the SDF is now

$$\frac{d\Lambda}{\Lambda} = \left[\gamma(\phi - 1)s - \beta - \gamma\bar{x} - \gamma x + \frac{\gamma^2\sigma_1^2}{2}(1 + \lambda(s))^2\right] dt \tag{19}$$

$$-\gamma\sigma_1(1 + \lambda(s))\, d\omega_1\,,$$

so that it is a function of both the surplus consumption ratio and consumption growth.

IVP in Wachter

Using the same procedures as in Campbell and Cochrane's model the price dividend function $p(s, x)$ now satisfies a second order linear PDE

$$\frac{\sigma_2^2}{2}\frac{\partial^2 p}{\partial x^2} = c_{11}(s)\frac{\partial^2 p}{\partial s^2} + c_{12}(s)\frac{\partial^2 p}{\partial s \partial x} \tag{20}$$

$$+c_1(s)\frac{\partial p}{\partial s} + (c_2(s) - (\psi - 1)x)\frac{\partial p}{\partial x} + (c_0(s) + (\gamma - 1)x)p - 1\,.$$

The coefficients $c_0(s)$, $c_1(s)$, and $c_2(s)$ are the same as in the Campbell and Cochrane model. There are two additional coefficients

$$c_{11}(s) = c\left(-\frac{\sigma_1^2(1+\bar{S}^2)}{2\bar{S}^2}, \frac{\sigma_1^2}{\bar{S}^2}, \frac{\sigma_1^2}{\bar{S}}; s\right), \text{ and}$$

$$c_{12}(s) = c\left(\rho\sigma_1\sigma_2, 0, -\frac{\rho\sigma_1\sigma_2}{\bar{S}}; s\right),$$

Initial Conditions.

1. Let $p_0(s) = p(s)$ which is the analytic solution of the Campbell and Cochrane model
2. Now the partial derivative $\frac{\partial p}{\partial s}$ at $x = 0$ is identical to $p'(s)$. As a result, the standard deviation of stock returns at $x = 0$ can be written as

$$\Sigma(s,0) = \sqrt{\Sigma(s)^2 + 2\sigma_2\rho\frac{\Sigma(s)}{p(s)}\frac{\partial p}{\partial x}(s,0) + \sigma_2^2\left[\frac{1}{p(s)}\frac{\partial p}{\partial x}(s,0)\right]^2}.$$

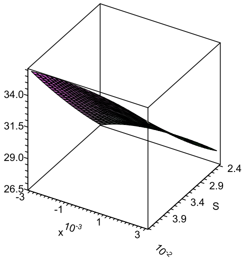
Here $\Sigma(s)$ is the standard deviation of stock returns in the Campbell and Cochrane model. This equation is a quadratic function in the initial condition $\frac{\partial P}{\partial x}(s,0) = p(s)$ which can be solved for its positive root.

$$\frac{\partial p}{\partial x}(s,0) = \left\{-\rho\Sigma(s) + \sqrt{\Sigma(s,0)^2 - (1-\rho^2)\Sigma(s)^2}\right\}\frac{p(s)}{\sigma_2}. \quad (21)$$

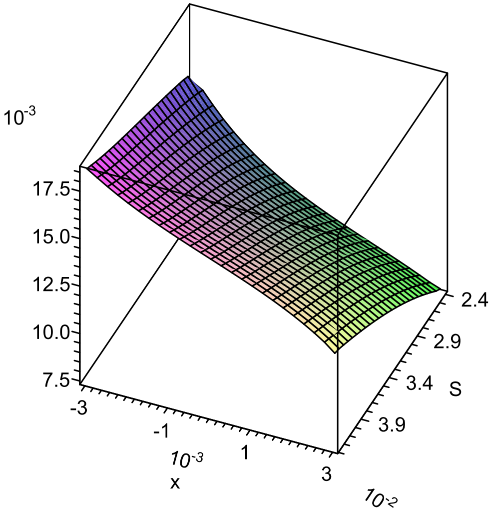
The positive root is used so that the price-dividend ratio increases when dividend growth increases. Suppose $\Sigma(s,0) = \Sigma(s)\sqrt{\kappa^2 + 1}$.

Price-dividend function in Wacter's model.

The price-dividend function in the Wachter model when the initial condition is $\frac{\partial p}{\partial x}(s,0) = \left\{-\rho + \sqrt{\rho^2+\kappa^2}\right\} \frac{p(s)\Sigma(s)}{2\sigma_2}$ with $\kappa = 1.29$. The parameters for Wachter's model are $r^b = 0.00016$, $\bar{x} = 0.00163$, $\phi = 0.9851$, $\gamma = 1.1$, $b = 0.0067$, $\rho = 0.35$, $\sigma_1 = 0.00289$, $\sigma_2 = 0.00075$, $\psi = 0.9669$, $\bar{S} = 0.0302$ and $r = 0.4995$. The x-axis gives the surplus consumption ratio, $[\bar{S}e^{-124\sigma_1}, \bar{S}e^{124\sigma_1}] = [0.0219, 0.0463]$, the y-axis is the consumption growth $x \in [-4\sigma_2, 4\sigma_2] = [-0.0031, 0.0031]$.



Equity Premium function in Wacter's model.



Conclusion

★ Asset pricing models can be represented by either an ODE or PDE. Most applied models assume analytic functions for the mean and standard deviation of the stochastic discount factor and state variables. As a result, the Cauchy-Kovalevsky Theorem may be used to prove that the solutions for stock price or returns are also an analytic function. Thus, Taylor polynomial approximations provide quick and accurate representation of the solution to most applied asset pricing problems.