Generalizing examples in computational experiments

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Intuition and theorems

- In applied theory, ideally
  - Start with economic question
  - Work out examples to gain intuition, form conjecture
  - Prove theorems for classes of economies
- Often last step is difficult. For example, in general equilibrium analysis comparative static statements are rarely possible

Classical computational experiments

- Want to investigate a model economy
- Calibrate the model economy so that it mimics the world along certain dimensions, given parametric classes of utility and production functions
- Compute equilibrium to explore quantitative and qualitative implications of the model economy
- Often there is no generally accepted strategy to pick the ‘right parameters’, but it is not possible to prove anything for all parameters or even all reasonable ones

Between examples and theorems: Modern computational experiment

- Repeat experiment for many different values of the parameters
- Infer that the set of parameters for which conjecture is false is 'small'

Formally, suppose the parameters lie in some compact set E.
Define $\Phi \subseteq E$ as the set of parameters for which conjecture is true.
Given a finite set $F \subseteq E$ such that $e \in \Phi$ whenever $e \in F$, can one obtain lower bounds on the 'size' of Φ ?

An example

- 2 agents, 2 goods in a pure exchange economy with CES utility
- Multiplicity is possible, but conjecture is that it occurs for small set of parameter values

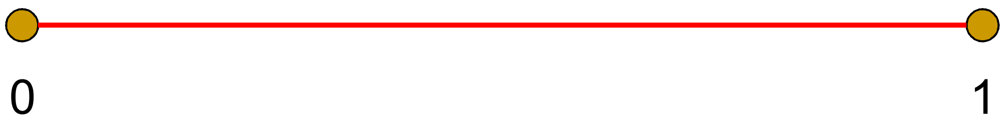
$$u_1(x_1,x_2) = -\alpha_1 x_1^{1-\sigma_1} - (1-\alpha_1)x_2^{1-\sigma_1}, \quad u^2(x_1,x_2) = -\alpha_2 x_1^{1-\sigma_2} - (1-\alpha_2)x_2^{1-\sigma_2}$$

$$\text{Parameters are } (e^1,e^2) \in [0,1]^4, \quad (\alpha_1,\alpha_2) \in [0,1]^2, \; (\sigma_1,\sigma_2)$$

Example continued

- Fix elasticities of substitution, how `likely’ is multiplicity?
- Suppose we can determine (fast) whether there are multiple equilibria for a given economy, if parameters are integers and not too large (tomorrow….)
- How can one say anything about the volume of parameters that yield multiple equilibria?

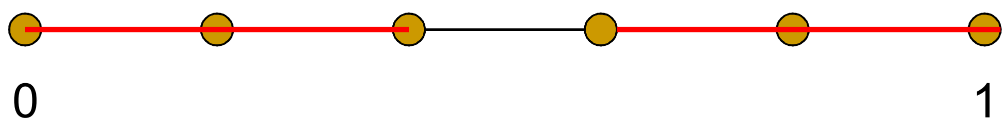
Connected components



If Φ has 1 connected component, if $0 \in \Phi$ and $1 \in \Phi$,

then $[0,1] \subseteq \Phi$

Connected components

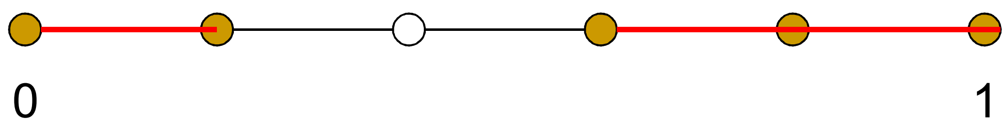


If Φ has k connected components, if

$$i/h \in \Phi, \text{ for all } i = 0, 1, \ldots, h$$

then the Lebesgue measure of Φ is at least 1-(k-1)/h

Connected components



If Φ has 2 connected components, if

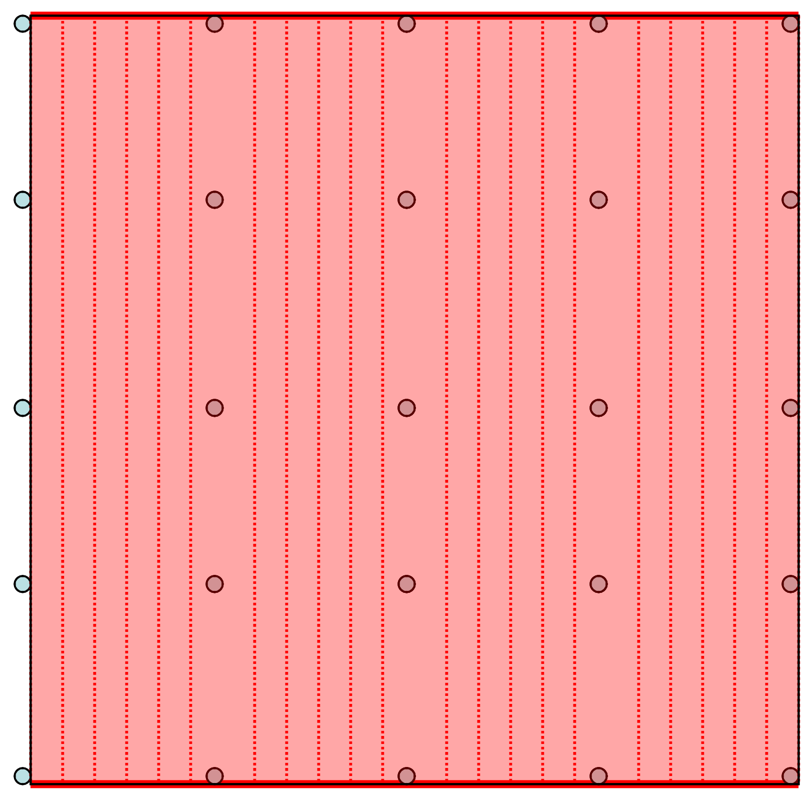
$$i/h \in \Phi, \text{ for all } i = 0,1 \text{ and } 3,\ldots,h$$

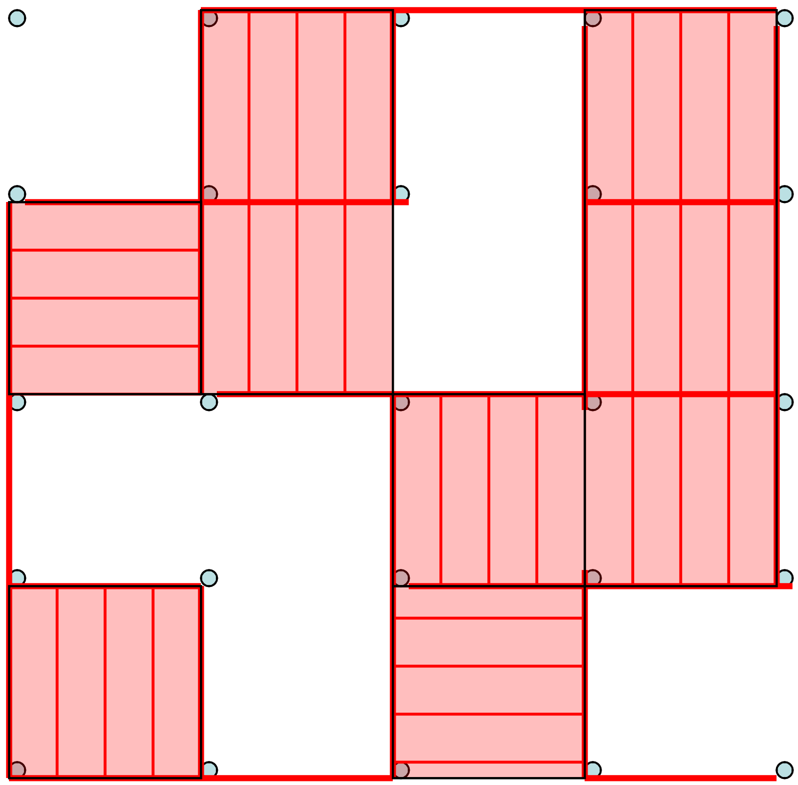
then the Lebesgue measure of Φ is at least 3/5

Will not work in higher dimensions….

Let κ denote the maximal number of connected components of Φ along any axis-parallel line. Want to use this to bound epsilon-entropy of $\eth\Phi$

One connected component





Two connected components

Main result (Koiran)

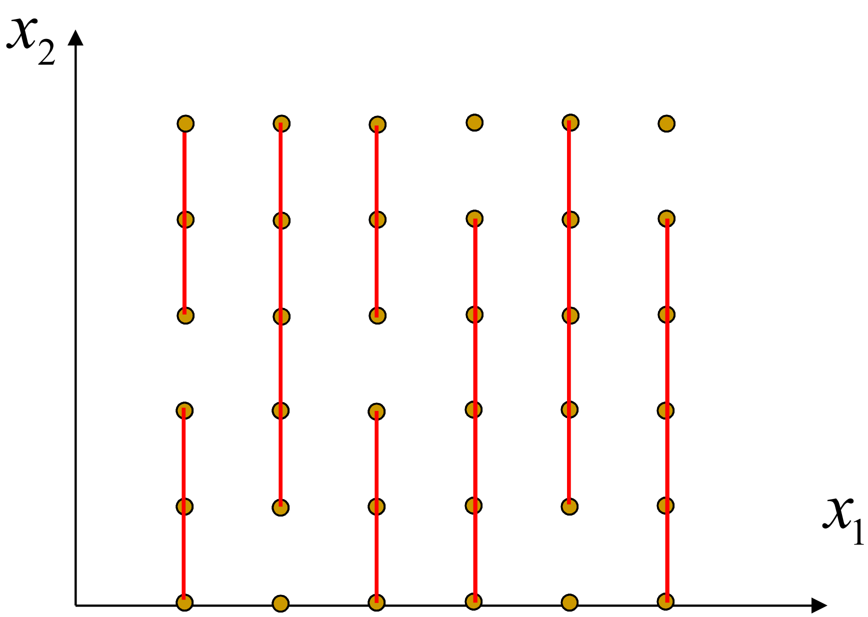
- Let $\Im$ denote the generalized indicator function with

$$\Im(x) = \begin{cases} 1 \text{ if x} \in \Phi \\ 0 \text{ otherwise.} \end{cases}$$

- Prove by induction that

$$\left| \int_{[0,1]^L} \Im(x)dx - \frac{1}{h^L} \sum_{i_1,\ldots,i_L} \Im(i_1/h,\ldots.,i_L/h) \right| \le \kappa L \frac{1}{h}$$

Intuition for L=2



$$\int_{[0,1]^2} \Im(x)\,dx = \int_0^1 \int_0^1 \Im(x_1, x_2)\,dx_1\,dx_2$$

For arbitrary $x_2 \in [0,1]$, cannot say much about $\int_0^1 \Im(x_1, x_2)\,dx_1$.

But $\left| \int_0^1 \Im(x_1, x_2)\,dx_1 - \frac{1}{h}\sum_{i=1}^{h} \Im\left(\frac{i}{h}, x_2\right) \right| \leq \kappa \frac{1}{h}$

Connected Components: Polynomials

- Given a polynomial equation in one unknown, $p(x) = \sum_{i=0}^{d} \alpha_i x^i$ the number of zeros is bounded by d
- Let $p_1 = \ldots = p_n = 0$ be a system of polynomial equations in n unknowns of degrees $d_1, \ldots, d_n$ Bezout's theorem says that the number of non-degenerate real solutions is bounded by $\prod_{i=1}^{n} d_i$

Number of connected components of semi-algebraic sets (Milnor)

Consider a 'semi-algebraic' set $A = \bigcap_{j=1}^{m} \{x : p_j(x) > 0\} \subseteq \mathbb{R}^n$

The number of connected components is at most $\frac{1}{2} d(d-1)^{n-1}, d = \sum_{j=1}^{m} d_j$.

To see why, define $p = \prod_{j=1}^{m} p_j$. By Sard's theorem, there is a small η such that the number of connected components of A can be bounded by the number of solutions to $p - \eta = 0,\ \partial p / \partial x_i = 0,\ i = 2, \ldots, n$.

More useful bounds (same intuition)

Consider a semi-algebraic set $A = \bigcap_{j=1}^{m} \{x : p_j(x) = 0\} \subseteq \mathbb{R}^n$,

where each p_j is the sum of monomials $\xi_{ij} x_1^{\alpha_{ij}(1)} \ldots x_n^{\alpha_{ij}(n)}$.

Let Q $\subseteq \mathbb{R}^n$ be the convex hull of all $(\alpha_{ij}(1), \ldots, \alpha_{ij}(n))$ together with the n unit vectors in $\mathbb{R}^n$. The number of connected components of A is then bounded by $2^{n-1} \mathrm{vol}(Q)$.

Polynomial problems in economics ?

- For normal form games, Nash equilibria can be characterized by polynomial system of equations (e.g. McKelvey and McLennan (1997))
- In general equilibrium, most interesting utility functions do not seem polynomial, but often tricks can be applied to characterize equilibrium by a polynomial system

Back to the CES example

- Suppose elasticities are identical and integer-valued. Then equilibrium is characterized by the following system of equations:

$$(\alpha_1 c_1^1)^\sigma - ((1-\alpha_1)c_2^1)^\sigma p_2 = 0$$

$$(\alpha_2 c_1^2)^\sigma - ((1-\alpha_2)c_2^2)^\sigma p_2 = 0$$

$$(c_1^1 - e_1^1) + p_2(c_2^1 - e_2^1) = 0$$

$$c_1^1 - e_1^1 + c_1^2 - e_1^2 = 0$$

$$c_2^1 - e_2^1 + c_2^2 - e_2^2 = 0$$

Back to the CES example

Or…

$$(\alpha_1 c_1^1) - ((1-\alpha_1)c_2^1)q = 0$$

$$(\alpha_2 c_1^2) - ((1-\alpha_2)c_2^2)q = 0$$

$$(c_1^1 - e_1^1) + q^{\sigma}(c_2^1 - e_2^1) = 0$$

$$c_1^1 - e_1^1 + c_1^2 - e_1^2 = 0$$

$$c_2^1 - e_2^1 + c_2^2 - e_2^2 = 0$$

In this example, kappa=2 !!!!

Tractability

- Randomization over E
  - If dimension of E is large, the methods are hardly applicable. However, if one is content with probabilistic statements, there is no curse of dimensionality. Suppose one can verify conjecture for N draws of random reals from E

The previous formula $\int_{[0,1]^L} \Im(x)dx \le \kappa L \frac{1}{h}$

implies that $\text{Prob}\left[\int_{[0,1]^L} \Im(x)dx \le \delta + \kappa L / h \right] \ge 1-(1-\delta)^N$

Tractability and randomization

- What happens if at some points we find multiplicity?
- Suppose we have m Bernoulli rv with success probability p and denote by $\hat{p}$ the empirical frequency of success.
- Then Hoeffdings inequality implies

$$\Pr(p - \hat{p} > t) \leq \exp(-2mt^2)$$

- Want to use m=200000 to get t around 0.005

Example – results…

- 200000 draws in parameter space, elasticity of substitution of 5, results hold with probability 1-exp(-10)
- Relative frequency of multiplicity is 0.00011
- Bound on volume is 0.0064

- Can we vary sigma? Not polynomial anymore ☹

Beyond polynomials: Pfaffian functions

A Pfaffian chain of order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subseteq \mathbb{R}^n$ is a sequence of analytic functions $f_1, \ldots, f_r$ satisfying differential equations

$$df_j(x) = \sum_{1 \leq i \leq n} g_{ij}(x, f_1(x), \ldots, f_j(x))dx_i \text{ for } 1 \leq j \leq r.$$

The g_{ij} are polynomial in $x = (x_1, \ldots, x_n), y_1, \ldots, y_j$ of degree not exceeding α.

A function $f(x) = p(x, f_1(x), \ldots, f_r(x)),$ with p being polynomial of degree β is called a Pfaffian function of order r and degree (α, β).

Bounds for Pfaffian sets

Let $f_i,\ i=1,\ldots,n$ be Pfaffian functions on $\mathbb{R}^L$ or $\mathbb{R}^L_{++}$ having common Pfaffian chain of order r and degrees (α,β_i) respectively. Let $\beta=\max_i \beta_i$. Then the number of connected components of $\{x: f_1(x)=\ldots=f_n(x)=0\}$ does not exceed

$$2^{\frac{r(r-1)}{2}+1}\beta(\alpha+2\beta-1)^{L-1}((2L-1)(\alpha+\beta)-2L+2)^r$$

References

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