Tackling multiplicity of equilibria with Gröbner bases

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Motivation

- Multiplicity of competitive equilibria is often bad news
- When equilibrium can be characterized by system of polynomial equations, can find all solutions numerically. Gröbner bases one possible method

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- Multiplicity of competitive equilibria is often bad news
- When equilibrium can be characterized by system of polynomial equations, can find all solutions numerically. Gröbner bases one possible method
- Often one wants to make statements about class of models
  - How can one detect non-uniqueness?
  - Can one give good bounds on the maximal number of equilibria?
  - Can one estimate size of set of parameters for which there is uniqueness?

Polynomials

- Monomial in $x_1, x_2, \ldots, x_n$: $\quad x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$.
  Exponents $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^N$
- Polynomial f in the n variables $x_1, x_2, \ldots, x_n$ is a linear combination of finitely many monomials with coefficients in a field $\mathbb{K}$

$$f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{K}, \quad S \subset \mathbb{Z}_+^N \text{ finite}$$

Examples of $\mathbb{K}$: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
Write $f \in \mathbb{K}[x_1, ..., x_n]$

Polynomial Equation Solving

- Objective: find all (positive) real solutions to $f(x) = 0$
- Study of polynomial equations on algebraically closed fields
  Field $\mathbb{R}$ is not algebraically closed, so interpret $f : \mathbb{C}^n \to \mathbb{C}^n$
- Define solution set

$$V(f_1, f_2, \ldots, f_n) = \{x \in \mathbb{C}^n : f_1(x) = f_2(x) = \ldots = f_n(x) = 0\}$$

- Polynomial ideal is

$$I = \langle f_1, \ldots, f_n \rangle = \{\sum_{i=1}^{n} h_i f_i : h_i \in \mathbb{K}[x]\}$$

$f_1, \ldots, f_n$ is one basis of I

Polynomial Equation Solving (cont.)

- Obviously a given ideal can have different bases, i.e. for polynomials $g_1, ..., g_k$ and $f_1, ..., f_n$

$$\langle g_1, ..., g_k \rangle = \langle f_1, ..., f_n \rangle$$

- One basis with nice properties is the so called *lexicographic Gröbner basis*
- Why do we care ?

Polynomial Equation Solving (cont.)

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- One basis with nice properties is the so called *lexicographic Gröbner basis*
- Why do we care ?
  If

$$\langle g_1, ..., g_k \rangle = \langle f_1, ..., f_n \rangle$$

  then

$$V(f_1, ..., f_n) = V(g_1, ..., g_k)$$

- To solve a system of equations, find another basis which has nice properties, e.g. is triangular

Shape Lemma and Gröbner basis

- Let $I = \langle f_1, f_2, \ldots, f_n \rangle$ be a radical zero dimensional polynomial ideal. Let G denote the (lexicographic) Gröbner basis of I. Suppose all roots have distinct value for last coordinate x_n
- THEN

$$G = \{x_1 - v_1(x_n), x_2 - v_2(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

Polynomial r has degree d, polynomials v_i have degrees less than d

Buchberger's Algorithm

- **Buchberger's algorithm** allows calculation of Gröbner bases
- If all coefficients of $f_1, \ldots, f_n$ are rational then the polynomials r, $v_1, v_2, \ldots, v_{n-1}$ have rational coefficients and can be computed exactly!
- Solving $f_1(x) = f_2(x) = \ldots = f_n(x) = 0$ reduces to solving $r(x_n) = 0$
- Software SINGULAR computes Gröbner basis and numerically approximates all solutions
  Available free of charge at www.singular.uni-kl.de

Shape lemma in SINGULAR

Consider the following simple example:

$$x - yz^3 - 2z^3 + 1 = -x + yz - 3z + 4 = x + yz^9 = 0$$

ring $R = 0, (x, y, z), lp;$
ideal $I = ($
$x - y * z * *3 - 2 * z * *3 + 1,$
$-x + y * z - 3 * z + 4,$
$x + y * z * *9);$
groebner(I);

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$[1] = 2z11 + 3z9 - 5z8 + 5z3 - 4z2 - 1$
$[2] = 2y+18z10+25z8-45z7-5z6+5z5-5z4+5z3+40z2-31z-6$
$[3] = 2x - 2z9 - 5z7 + 5z6 - 5z5 + 5z4 - 5z3 + 5z2 + 1$

Parameter Shape Lemma - Assumptions

What if last equation reads $ex + yz^9 = 0$ with e being a parameter? Let $E \subset \mathbb{R}^m$, be an open set of parameters, $(x_1, \ldots, x_n) \in \mathbb{C}^n$ a set of variables and let $f_1, \ldots, f_n \in \mathbb{K}[e_1, \ldots, e_m; x_1, \ldots, x_n]$, $\mathbb{K} \subset \mathbb{R}$. Assume that for each $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_m) \in E$, the ideal $I(\bar{e}) = \langle f_1(\bar{e}; \cdot), \ldots, f_n(\bar{e}; \cdot) \rangle$ is zero-dimensional and radical. Furthermore, assume that there exist $u_1, \ldots, u_n \in \mathbb{K}$, such that for all $\bar{e}$ any solutions $\bar{x} \neq \bar{x}'$ satisfying

$$f_1(\bar{e}; \bar{x}) = \ldots = f_n(\bar{e}; \bar{x}) = 0 = f_1(\bar{e}; \bar{x}') = \ldots = f_n(\bar{e}; \bar{x}')$$

also satisfy $\sum_i u_i \bar{x}_i \neq \sum_i u_i \bar{x}'_i$.

Parameter Shape Lemma - Result

There exist $r, v_1, \ldots, v_n \in \mathbb{K}[e; y]$ and $\rho_1, \ldots, \rho_n \in \mathbb{K}[e]$ such that for $y = \sum_i u_i x_i$ and for all $\bar{e}$ outside a closed lower-dimensional subset E_0 of E,

$$\{x \in \mathbb{C}^n : f_1(\bar{e}, x) = \ldots = f_n(\bar{e}, x) = 0\}$$
$$\{x \in \mathbb{C}^n : \rho_1(\bar{e})x_1 = v_1(\bar{e}; y), \ldots, \rho_n(\bar{e})x_n = v_n(\bar{e}; y) \text{ for } \; r(\bar{e}; y) = 0\}.$$

Shape lemma with parameters in SINGULAR

ring R= $(0, e), (x, y, z), lp$;
ideal $I = ($
$x - y * z * *3 - 2 * z * *3 + 1,$
$-x + y * z - 3 * z + 4,$
$e * x + y * z * *9)$;
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groebner(I);
$[1] = 2 * z11 + 3 * z9 - 5 * z8 + (5e) * z3 + (-4e) * z2 + (-e)$
$[2] = (-e2 - e) * y + (-8e - 10) * z10 + (-10e - 15) * z8 + (20e + 25) * z7 + (5e) * z6 + (-5e) * z5 + (5e) * z4 + (-5e) * z3 + (-20e2 - 20e) * z2 + (16e2 + 15e) * z + (3e2 + 3e)$
$[3] = (-e-1)*x+2*z9+5*z7-5*z6+5*z5-5*z4+5*z3-5*z2-1$

Bounding the number of positive real solutions

- Given a Gröbner basis

$$\{\rho_1(\bar{e})x_1 - v_1(\bar{e};y), \dots, \rho_n(\bar{e})x_n - v_n(\bar{e};y), r(\bar{e};y)\}$$

  number of real positive solutions of $r(\bar{e};y) = 0$ bounds number of positive solutions for $f_1, \dots, f_n$
- Descartes' bound and Sturm's algorithm
  - Univariate polynomial $r(x_n) = \sum_{i=0}^{d} a_i\ x_n^i$ of degree d
  - Descartes's Rule of Signs
    The number of positive real roots of a polynomial is at most the number of sign changes in its coefficient sequence
  - Sturm's algorithm gives exact number of real zeros in a given interval

Detecting multiplicity

- If along a path in parameter-space number of solutions increases, there must be a critical point
- Search for critical points by solving $r(\bar{e}; y) = 0$ together with $\partial_y r(\bar{e}; y) = 0$
- Easy for case of one parameter

Finding one point in each connected component

Let $V \subset \mathbb{C}^n$ be a variety of dimension d with $I(V) = \langle f_1, \ldots, f_s \rangle$, $f_i \in \mathbb{Q}[x]$ for all $i = 1, \ldots, s$. Given a point $a \in \mathbb{Q}^n$, $a \notin V$, let

$$\mathcal{C}(V, a) = \{x \in V : \text{rank} \begin{pmatrix} \partial_x f(x) \\ a - x \end{pmatrix} \leq n - d\}.$$

The set $\mathcal{C}(V, a)$ meets every semi-algebraically connected component of $V \cap \mathbb{R}^n$, moreover, for generic $a \in \mathbb{Q}^n$, the dimension of $\mathcal{C}(V, a)$ is smaller than d.

Approximate volume of semi-algebraic sets

- Suppose the set of exogenous parameters is $[0,1]^l$. Let $\Phi \subset [0,1]^l$ be the set of parameters for which there exists a unique Walrasian equilibrium.
- Let G be a grid of points in $[0,1]^l$, $G = \{1/n, \ldots, 1\}^l$. For $x \in [0,1]^l$, let $\Im(x) = 1$ if there is a unique equilibrium for the parametrization and zero otherwise.
- We want to estimate the volume of Φ which is given by $\int_{[0,1]^l} \Im(x) dx$.

Approximate volume of semi-algebraic sets (cont.)

- Suppose it is known that the fraction of points in G for which there are multiple equilibria is not larger than some $\gamma \in (0,1)$.
- Let λ denote a bound on the maximal number of connected components of Φ intersected with any axes-parallel line. Then Koiran (1995) shows that

$$\left| \int_{[0,1]^l} \Im(x) dx - \gamma \right| < \frac{l}{n} \lambda$$

CES economies

- Simple Arrow-Debreu exchange economy with H agents and L goods
- Suppose that each agent has CES utility, with marginal utility being of the form

$$u^h(c) = \sum_{l=1}^{L} v_{hl}(c)$$

with

$$v'_{hl}(c) = (\alpha_l^h)^{-\sigma_h}(c_l)^{-\sigma_h}$$

Equilibrium equations for CES economies

- Normalize $p_1 = 1$ and eliminate all Lagrange multipliers, $\lambda^h = 1/(\alpha_{h1} c_1^h)^{N/M^h}$.
- Defining $q_l = p_l^{1/N}$, $l = 2, \ldots, L$, we obtain as P.E.E.

$$
\begin{aligned}
\alpha_{h1} c_1^h - \alpha_{hl} c_l^h q_l^{M^h} &= 0, \quad h \in \mathcal{H}, l = 2, \ldots, L, \\
c_1^h - e_1^h + \sum_{l=2}^{L} q_l^N (c_l^h - e_l^h) &= 0, \quad h = 1, \ldots, H, \\
\sum_{h=1}^{H} c_l^h - e_l^h &= 0, \quad l = 1, \ldots, L-1.
\end{aligned}
$$

Maximal number of equilibria

Suppose $H = L = 2$ and $\sigma^h = \sigma \in \mathbb{N}$. All equilibrium q's must satisfy

$$\left(\alpha^1 e_2^2 + \alpha^2 e_2^1 - \alpha^1\alpha^2(e_2^1 + e_2^2)\right) q^\sigma - \alpha^1\alpha^2(e_1^1 + e_1^2)q^{\sigma-1} + (1-\alpha^1)(1-\alpha^2)(e_2^1 + e_2^2)\ q + \left(\alpha^1\alpha^2(e_1^1 + e_1^2) - \alpha^1 e_1^1 - \alpha^2 e_1^2\right) = 0$$

Detecting multiple equilibria

- Suppose $\sigma_1 = \sigma_2 = 3$, $\alpha_1 = 1/5, \alpha_2 = 4/5$ and $e_2^1 = e_1^2 = 1$, $e_1^1 = e_2^2 = f$
- With these parameters the univariate representation above becomes

$$r(q) = (f + 16)q^3 - (4f + 4)q^2 + (4f + 4)q - f - 16$$

- Critical points must satisfy

$$(f + 16)q^3 - (4f + 4)q^2 + (4f + 4)q - f - 16 = 0$$
$$3(f + 16)q^2 - 2(4f + 4)q + (4f + 4) = 0$$

- If $f > 44$ the economy has three equilibria

How likely is multiplicity for $H = L = 2$

- Estimate size of set of individual endowments and α for which equilibria are unique
- For simplicity take $\alpha^1, \alpha^2 \in [0,1]^2$ and $(e_1^1, e_2^1, e_1^2, e_2^2) \in [0,1]^4$
- To get fraction of economies with unique equilibrium, use probabilistic approach...

Estimates for γ

- Suppose the $(X_i)_{i=1}^m$ are Bernoulli random variables with success probability $p \in (0,1)$ and $\hat{p}$ denotes the empirical frequency of success.
- Denoting $S = n\hat{p}$, Hoeffding' inequality can be written as

$$P\left(p - \hat{p} > t\right) \leq \exp(-2mt^2).$$

- In the experiments below, we use 200000 draws and want to bound γ from below with precision $t = 0.005$. Hence we obtain that

$$P\left(\gamma - \hat{\gamma} > 0.005\right) \leq e^{-10}.$$

Number of connected components

- Along axes parallel lines zones of multiplicity must be seperated by critical economies
- Can use Gröbner bases to find bounds on number of critical economies
- Take for example identical $\sigma = 9$, want to see how many q, e_1^1 solve

$$\left(\alpha^1 e_2^2 + \alpha^2 e_2^1 - \alpha^1\alpha^2(e_2^1 + e_2^2)\right) q^9 - \alpha^1\alpha^2(e_1^1 + e_1^2)q^8 + (1-\alpha^1)(1-\alpha^2)(e_2^1 + e_2^2)\ q + \left(\alpha^1\alpha^2(e_1^1 + e_1^2) - \alpha^1 e_1^1 - \alpha^2 e_1^2\right) = 0$$

$$9\left(\alpha^1 e_2^2 + \alpha^2 e_2^1 - \alpha^1\alpha^2(e_2^1 + e_2^2)\right) q^8 - 8\alpha^1\alpha^2(e_1^1 + e_1^2)q^7 + (1-\alpha^1)(1-\alpha^2)(e_2^1 + e_2^2) = 0$$

- All q must satisfy $K_1 q^{16} + K_2 q^8 + K_3 q^7 + K_3 = 0$

Some results

- Take identical σ
- Results:

σ	λ	$\hat{\gamma}$	VOL
3	2	0	0.001
5	2	$13/100000$	0.006
7	2	$36/100000$	0.006
9	2	$71/100000$	0.006
25	2	$338/100000$	0.009
51	2	$629/100000$	0.01

Table: Fraction of Economies with Multiple Equilibria

Conclusion

- Several methods to compute all complex solutions to polynomial equations
- Gröbner bases provide one (not extremely efficient?) way
- Advantages of Gröbner bases:
    - Bounding maximal number of real solutions over set of parameters
    - Detecting non-uniqueness
    - Estimating probability of hitting multiplicity

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